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Dr. Word Problem - Solving Word Problems with the Four Operations Using Singapore Bar Models

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Introduction

Dr. Word Problem and the Four Operations is a unit focused on the four operations of arithmetic. This unit explores the use of Singapore bar models as a tool for solving word problems. The unit teaches students how to represent word problems with Singapore bar models beginning with a basic addition fact. The lessons are scaffold to help the students to understand how to draw models to represent subtraction, multiplication, and division. The final phase addresses representing multi-step problems with bar models.

While the students at my school are high achievers in mathematics, word problems seem to be the weakest area of the math curriculum for them. My fourth graders typically have a difficult time deciding which operation a word problem requires. Often I find my students using some of the coping strategies and limited strategies that Sowder (1995) describes. One coping strategy that they use is adding the numbers that they find in the problem, regardless of what the word problem states. Another poor strategy choice is "Limited strategy 5, looking for isolated key words to tell which operation to use". This strategy is taught over and over at each grade level. Since each word problem is slightly different, children cannot figure out a strategy that automatically works for each one. Therefore, students try to develop coping strategies. Some teachers on the other hand are either teaching some of these limited strategies that are described in their teacher's manual, or they are teaching their students the way they were taught in school. This unit tries to look at word problems in a different way, so that students can develop a greater mathematical understanding that will enable them to be more successful problem solvers.

Through my work on this project, after analyzing word problems and looking for similarities and differences I have come to the following conclusions. When working with the simpler problems, students can learn and become more familiar with the basic structure of word problems. However, beyond the basic structure students will encounter differences within each problem. The subtle differences among very similar problems further complicate this subject matter. As students deal with multi-step problems there definitely is not a set approach to take. The students must truly understand what is going on mathematically in each situation.

So in this unit I have examined problems and grouped them into suites. The words and situations are different within the suites, but the basic structure is the same. By examining the different dimensions of the suites of problems, I have learned to approach problem solving in a clearer, more systematic way than I did in the past.

This unit not only looks at the suites of problems, but also introduces the students to the Singapore bar models as a strategy for solving word problems.

The Singapore bar models provide the students with a visual component to help them understand what is going on mathematically in the problem. These models can be used to represent the four operations pictorially. The Singapore bar models are easy to draw and can be used to solve algebra problems without writing an equation. Hopefully, the combination of bar models and analyzing suites of problems will lead students and teachers to more successful problem solving and help to avoid some of the pitfalls caused by coping and limited strategies.

Attaching symbolic meaning to words is a difficult task for students at all levels. In fourth grade the students have just learned the four operations. This is really the first time that they have four operations to choose from when solving a problem. Their level of understanding the operations of multiplication and division, in particular, is in the beginning stages. So, word problems further complicate the issue because they have to figure out the words, the operation needed, and then they have to attach symbols to the words. This unit is intended to help students learn which mathematical operations to use when certain actions are presented in a problem. Upon completion of the unit I expect the students will have developed a greater understanding of the concepts of all four operations, and especially multiplication and division.

This unit is designed to be taught over a three-week period for approximately forty-five minutes per lesson. Beginning with addition, the students will experience suites of addition problems and the related subtraction problems. Next, the unit will move into multiplication and the related division problems. At the start, the suite of problems will involve one diagram focused around one scenario. These problems will have the same story line, but will be slightly different. The students will later face individual problems, not connected to their related counterparts, and the problems will involve one of the four operations. This is a more typical representation of the real world, where a word problem stands alone. After the students have learned the four operations and how to represent problems from all of the operations using a Singapore bar model, they will study two-step problems involving the four operations in any combination. The culminating activity will be for the students to put together a collection of word problems that include all four operations focused around a theme. My students will write their own word problems around one scenario of their choice, and also illustrate their scenarios with Singapore bar models.

Rationale

A key feature of this unit is that it requires students to spend a significant portion of time working on word problems. Through this intensive exposure to word problems the students should become more comfortable with them and less afraid of them. Problem solving is a very important aspect of many of the high stakes tests that have become so important to school districts throughout the United States. On high stakes tests, the concepts are often embedded within a word problem. Some students automatically give up when they see the words. Others that know the concept have a difficult time understanding the task and are not able show what they know. By developing the confidence and the skills to tackle word problems, students will not only improve their math skills with the four basic operations, but also will hopefully transfer their knowledge to solving problems involving decimals, fractions, and all other areas of math.

The Singapore bar models were chosen for several reasons. First, Singapore leads the world in math achievement so the strategies that the Singapore approach teaches are proven to be successful. Secondly, the bar pictures are a visual representation of the words. They may help children translate from words to symbols, particularly if the words and language are difficult for them.

Another reason why this unit is appropriate and necessary for fourth graders is because the four operations are a major component of elementary math. Students need to understand the connectedness of the four operations and this unit ties the different operations together. According to Liping Ma, (Ma, 1999), there are two relationships that connect the four basic operations. These two relationships are the "derived operation" and the "inverse operation". An example of the "derived operation" is, "Multiplication is an operation derived from the operation of addition. It solves certain kinds of addition problems in an easier way" (Ma, 1999, p. 113). The inverse relationship is the other connection that Ma refers to in her explanation of the connections between the four operations. Subtraction is the inverse of addition because subtraction of a fixed number from a variable number is the inverse of addition of (the same) fixed number to a variable number. Multiplication and division are also inverse operations. Ma's (1999) analogy describes the importance of connecting the operations with this analogy:

> "These two kinds of relationships tightly connect the four operations. Because all of the topics of elementary mathematics are related to the four operations, understanding of the relationships among the four operations, then, becomes a road system that connects all of elementary mathematics. With this road system, one can go anywhere in the domain" (Ma, 1999, p.113).

By developing a connected road system, I seek to help my students to develop a better understanding of arithmetic.

As students solve math and word problems, they need to be flexible and know different ways to approach the problems. While this unit is focused on one strategy, the Singapore bar models, the goal is to teach the students this technique so that they can increase their repertoire of strategies. Students will be required to use the bar models as I teach this unit. However, once this unit is completed, the students may or may not select the bar models as their approach to solving a given problem. As with any other strategy, some students will find that it works well for them and others may prefer a different method.

Why did I choose to use the Singapore bar models? Singapore students were first in the 1995, 1999 and 2003 Trends in International Mathematics and Science Study (TIMSS), which is designed to measure trends in students' mathematics and science achievement in four-year cycles. The TIMSS 2003 results were released on December 14, 2004. (New AIR Study Compares the Quality of U.S. Math Instruction with Singapore Recognized World Leader U.S. Trails, But Both Nations Could Learn from Each Other 2/7/05 http://www.air.org/news/documents/Singapore.htm.)

Based on these findings I thought it would be interesting to expose my students and myself to Singapore models. Word problems seem to be particularly difficult for students. These models seem to simplify the problems and provide a visual representation to help the students to think about the problem. According to the Cognitively Guided Instruction model, "It is not children's manipulations of materials that is important; it is their understanding of the principles involved in the manipulations" (Carpenter, et al 1999, p. 68).

Based on a study conducted by the American Institute for Research (Leinwand et al., 2005) it is important to look to Singapore and to learn from how this country teaches math to its students. This report states:

> "It is unreasonable to assume that Singaporean students have mathematical abilities inherently superior to those of U.S. students; rather, there must be something about the system that Singapore has developed to teach mathematics that is better than the system we use in the United States. That's why it's important to take a closer look, and see how the U.S can learn and how the U.S can improve."

Objectives

Teacher Objectives

This unit has objectives for the teacher and for the students. First, I will discuss objectives for the teachers. In Ma's book (1999), the author compares math teachers in China and in the United States. There are striking differences in the knowledge base of the teachers. The Chinese teachers are more knowledgeable than American teachers in their own math skills and methods for solving equations. The book then analyzes reasons why. One main reason is summarized well with the analogy of the taxi driver. She compares the way the Chinese teachers understand math to a proficient taxi driver who knows all of the roads and many ways to get to the same place. Ma associates the American teachers to a newcomer in an area. A newcomer may know a few roads and one way to get to a location. Ma states, "A teacher's map of school mathematics must be more complicated and flexible" (Ma, 1999, p. 123).

A major objective of the seminar that led to the creation of this unit is for teachers to analyze and categorize word problems based on similarities and differences. It is this in depth analysis that really begins to help the teacher become a more proficient "taxi driver". At first glance, categorizing the word problems seems to be an easy task, but comparing similarities and differences goes beyond grouping the problems based on the operation involved. It looks more closely at the components and the structure of the problems the students are solving. There are several categories within each operation. Below the categories are defined and an example of each type of problem is given. These categories are from *Children's Mathematics Cognitively Guided Instruction* (Carpenter, et al., 1999). Additional insight has been added based on the article (Sowder, 1995).

Student Objectives

This unit clearly meets national and state standards since the four operations are such an integral part of elementary mathematics. In addition to simply learning the four operations, this unit looks at the relationships shared by the various operations. The inverse relationship of addition and subtraction, the inverse relationship of multiplication and division, and the relationship that addition and multiplication share are all explored through problem solving. By making connections between related word problems and examining different ways to solve the same word problems, students should develop a better understanding of the relationships between the four operations. As stated earlier, developing an understanding of the connectedness that the operations share is crucial to student learning.

Going beyond the basic calculations, students need to think critically and to develop problem-solving skills. This unit not only provides opportunities to develop these skills, but also teaches new strategies and approaches for dealing with word problems. The suites of problems should help make students aware of the connections that closely related problems share, and thus help them to develop a greater mathematical understanding. The expectation is that if students have a better understanding of the four operations they can use this knowledge to closely examine word problems and become better problem solvers.

As the strategies and activities of this unit unfold, the students will learn the Singapore bar model approach. This is just one strategy to help students solve word problems. It is visual and a more abstract "manipulative" to help students solve problems. This method will provide the students with one more approach or option when they face a word problem.

Addition and Subtraction Categories

Join

Join problems involve the operation of addition. Within this category there are three different types. The most basic kind is when two quantities are joined and the result is unknown. Here is an example:

Daquan has 3 toy cars. His grandma buys him 2 more for his birthday. How many cars does he have now?

This problem would be solved with addition. Daquan starts with 3 toy cars and then 2 more toy cars that his grandma gives him are added to his collection. The answer is unknown, but by adding 3 and 2 the answer can be determined.

3 + 2 = []

start unknown change

The answer is 5 toy cars.

This same scenario could be written slightly differently so that the change is unknown. In this situation,

Daquan has 3 toy cars. His grandma gives him some for his birthday. Now Daquan has 5 toy cars. How many toy cars did Daquan's grandma give him?

The representation of this problem is:

3 + [] = 5

The [] represents 2 toy cars in this problem.

Even though this is an addition problem, the subtraction equation 5 - 3 = 2 is used to solve it. Larry Sowder (1995) refers to this type of problem as a *missing addend* problem. Missing addend problems are confusing for children because the real world operation is addition, as the equation shows. However, mathematically it is solved with subtraction.

The third type of join problem is when the start is unknown. This is another missing addend problem.

Daquan has some toy cars. His grandma gives him 2 more for his birthday. Now he has 5 toy cars. How many toy cars did Daquan have before his birthday?

The problem looks like this:

[] +2 = 5

The answer is 3 toy cars. Once again subtraction 5 - 2 = 3 is used to solve for the unknown start.

Separate

This dimension is composed of the same three subcategories: result unknown, change unknown, and start unknown.

Sydney has 7 pieces of gum. She gives 2 pieces to her friends. How many pieces of gum does she have left?

This is a result unknown because you know how many Sydney started with and you know the change. The part that is unknown is the result. Subtraction is used to separate the 2 pieces of gum that Sydney gave away from the 7 pieces that she had in the beginning.

7 - 2 = []

start result unknown change

The [] is 5 pieces of gum in this problem.

In the next problem, Sydney has an unknown change and once again subtraction is used to solve this type of problem.

Sydney had 7 pieces of gum. She gave some to her friends. Now she has 5 pieces left.

The number sentence below is used to solve for the answer, which is 2.

7 ñ [] = 5

She gave her friends 2 pieces of gum.

The last type in this suite of problems is when the start is unknown. This is another type of problem, *missing start*, that Sowder claims especially causes trouble for children. It seems to be a subtraction problem, but requires addition to solve it.

Sydney had some pieces of gum. She gave 2 pieces to her friends. She has 5 pieces left. How many pieces did she have before she gave some to her friends?

The number sentence for this problem can be written

[] - 2 = 5

The addition equation 2+ 5 = [] is used to solve this problem. She had 7 pieces of gum.

In the join and separate suites, three very similar problems can be created. Two of the problems deal with subtraction and one problem deals with addition. It is by revealing this kind of relationship between the problems and the operations that fosters a deeper understanding for the students.

Part-Part-Whole Problems

This dimension only has two different subcategories. As the name suggests either a part will be unknown or a whole will be unknown. There is no preferred order for a missing part. This is in contrast to the first two categories of problems, in which time provided a basis for ordering the numbers in the problem. An example of an unknown whole is:

There are 12 boys in the class and there are 10 girls in the class. How many students are in the class?

Addition is used to solve this problem. The major difference between the part-part-whole problems and the join problems is that there is no change over time: an action is not being performed and a quantity is not being changed or moved. The boy part and the girl part coexist and together equal the whole part. Students are not being added or removed.

The schema for this problem can be written

12 + 10 = []

part whole

The whole equals 22 students.

An example with an unknown part is:

There are 22 students in the class. There are 12 boys. How many are girls?

The problem is looking at the part of the class that is unknown. Despite the addition equation, subtraction is used to solve this one. A number sentence for this one could be

12 + [] = 22

The missing part is found by subtracting. The answer is 10 girls.

22 - 12 = []

Compare Problems

These problems are similar to the part-part-whole problems because there is no action involved. There is a set of objects or people and the problem is comparing the quantities. Carpenter (1999) classifies these problems based on three parts: the referent, the compared set, and the difference.

Mark has 6 baseball cards. Tom has 2. How many more baseball cards does Mark have?

6 - 2 = []

referent compared difference unknown

In this problem, the difference is what is missing. Mark's baseball cards are the referent and Tom's cards are the compared set. These two variables are known. When difference is unknown subtraction is used to solve the problem.

Now, if

Tom has 2 baseball cards and this is 4 less than Mark has, how many baseball cards does Mark have?

Here the unknown is the referent. The compared set and the difference are known. In order to solve this type of compare problem, addition would be needed. The appropriate number sentence is

4 + 2 = [].

A similar problem with an unknown compared set would look like this:

Mark has 6 baseball cards. Tom has less baseball cards than Mark. If the difference between their baseball card collections is 4, then how many baseball cards does Tom have?

In this problem subtraction is used.

6 - 4 = [].

Multiplication and Division Categories

Grouping and Partitioning Problems

This category shares some similarities with the join category for addition and subtraction. Again, there can be three different unknowns depending how the problem is set up. When the compare category was analyzed three problems could be created. Two were subtraction and one was addition. Similarly with this category, three problems can be created, two for division and one for multiplication.

In interpreting these problems, it is important to take into account that there are two interpretations of division: measurement and partitive. Measurement division is when the number of the groups is unknown, and partitive is when the size of the group is unknown. It is important for students to learn that, although these problems may seem quite different, they are both solved by the same operation, division.

Let's say Zachary makes bags of lollipops for his friends. He puts 4 lollipops in each bag and makes 5 bags. How many lollipops did he use altogether?

This problem asks for the total, which indicates that it is a multiplication problem. So the equation would be:

4 x 5 = []

In the next problem, the number of bags is unknown. This is measurement division.

Zachary has 20 lollipops. He makes some bags of lollipops for his friends. He puts four lollipops in each bag. How many bags can he make?

Equations representing this problem are

20 / 4 = [] or [] x 4 = 20.

As the second equation suggests, this problem could also be written as a missing factor problem using the multiplication.

A third type of problem calls for partitive division because the number in each group is unknown.

Zachary has 20 lollipops. He makes 5 bags. How many lollipops will he put in each bag?

Equations representing this problem are

20 / 5 = [] or [] x 5 = 20.

Rate Problems

There is also a group of problems known as rate problems. These are different from the previous group because they do not deal with objects that can be counted. They are a little more abstract, especially for children. The rate problems can also be divided into the same types: multiplication, measurement division, and partitive division.

A monkey eats 3 pounds of bananas a day. How many pounds of bananas will the monkey eat in 5 days?

This is a multiplication problem. We can represent it by

3 x 5 = [].

The next two problems will involve division.

A monkey eats 3 pounds of bananas a day. At this rate, how many days will 15 pounds of bananas feed the monkey?

This is a measurement division problem because the measurement or the number of days is missing. It can be schematized by the equations

15 / 3 = [] or 3x [] = 15.

A monkey eats 15 pounds of bananas in 5 days. If the monkey eats the same amount each day, how many pounds does the monkey eat in one day?

This problem is partitive division because the unknown is the amount or the part that the monkey eats in each day. The equation below will solve this problem.

15 /5 = [] or 5x [] = 15

Price Problems

These are similar to the previous group, but the rate involves a price. Oranges sell for 6 cents each. How much would 7 oranges cost? This is a multiplication problem and the equation to solve it is:

6 x 7 = []

In the measurement division problem, the number of oranges is unknown.

Oranges sell for 6 cents each. How many can you buy for 42 cents?

Equations that capture this equations are

42/ 6 = [] or 6 x [] = 42.

The last one in this trio is the partitive division example.

You have 42 cents and you buy 7 oranges. How much does each orange cost?

Equations for this one are

42 /7 = [] or 7 x [] = 42

Multiplicative Comparisons

In these problems one quantity is described in relation to the other. For example:

The bowling ball weighs 5 times as much as the basketball. If the basketball weighs 4 pounds, how much does the bowling ball weigh?

Multiplication is used to solve this type of problem. It is represented by the equation

4x 5 = [].

The equation looks like the other multiplication problems, but it is different from all of the other problems presented so far. The key concept is that one quantity is presented in relation to the other. One quantity is described in terms of how many times larger it is compared to the other.

Multi-step

This is by far the most daunting category. Based on the four operations, considering order, there are 16 possible types of problems. However, when all of the categories mentioned above are considered, there are a much greater number of combinations. My collection of multi-step problems, (see appendix) are by no means exhaustive of all of the different combinations. After collecting the problems, solving them, and classifying them, I soon realized the challenge of grouping them. I decided to group them based on the first step. Now I will discuss some of them from each category in more detail.

Here is problem number twenty-nine:

8 students sold 272 concert tickets at $3 each. Each student sold the same number of tickets. How much did each student collect?

The first step is to divide, since this is a grouping/partition problem. I must divide the number of tickets by the number of students. That means that I am doing partitive division. 272 divided by 8 equals 34. The next step is to multiply. This is a price problem with multiplication. 34 times $3 equals $102. Each student collected 102 dollars.

Interestingly enough, this problem could also be solved a different way. The number of tickets could have been multiplied by the ticket price first, to find the total amount of money collected. Then this would be a price problem with multiplication. Then the next step would be to divide the total amount of money, $816 by the 8 students. This would be a partitive division problem since I am dividing into a certain number of groups. The solution is $816 divided by 8 yields the same answer of $102. These alternative solution methods reflect the fact that multiplication is commutative (and that division by 8 is multiplication by 1/8).

Regardless of the order, the same operations and classification for those operations was used in the two methods of solving this problem. Certainly, my classifications of these multi-step problems are not the only way they can be grouped. However, it is a very reasonable first pass. Remember, Ma urged teachers to be more flexible. So, if a different way is chosen, then that is showing flexibility, which is a good thing. For the purpose of brevity, I am not going to continue to explore multiple ways of solving each problem as I go through the multi-step domain.

Here is problem 30:

Mr. Stone bought some books for $301. Each book cost $7. If his wife carried 18 books and he carried the rest. How many books did Mr. Stone carry?

This problem falls into the price category and uses measurement division in the first step because the total number of books has to be found. To do this, the total cost $301 is divided by the cost of one book. After performing this computation, we know that Mr. Stone had 43 books in total. The next step is to subtract, but since I am analyzing the problem, just knowing to subtract is not enough. I know that Mr. Stone and his wife are carrying them some place, presumably to the same place, so the books are not being separated. So, I considered this to be a part-part-whole problem involving a missing part. Now, I can complete the calculation by subtracting 18 from 43. The difference is 25, so Mr. Stone carried 25 books.

In number 33,

Rebecca bought four boxes of doughnuts. There were 6 doughnuts in each box. She divided the doughnuts equally among eight people. How many doughnuts did each person receive?

Based on my classification by first step, this is a grouping/partitioning problem involving multiplication. Six times four equals twenty-four. The next step is a grouping/partition problem that uses partitive division. 24 doughnuts divided by 8 people. So the answer is 3. Each person received 3 doughnuts. Interestingly all of the numbers in this problem are factors of 24.

Number 37 states:

A toy car costs $5. A toy airplane costs 4 times as much as the toy car. How much more does the toy airplane cost than the toy car?

The first step for me is multiplicative comparison. The cost of the car is multiplied by four since the airplane costs four times as much as the car. Therefore, $5 times 4 equals $20. The next step is to compare the prices using subtraction to find the unknown difference. The difference is $15 since $20 minus $5 equals $15.

In problem 40,

Rani had $47. After paying for 3kg of shrimp, she had $20 left. Find the cost of 1 kg of shrimp.

The first step is subtraction. It is a separate problem with an unknown change. Rani had $47. She spent some unknown amount and now is left with $20. After performing the calculation ($47-$20), the unknown change is determined to be $27. The next step is to divide $27 by 3. This is a price problem using partitive division. The cost of 1 kg would be $9.

A final problem from my collection of multi-step problems is from the part-part-whole domain. Number forty-three states:

A farmer had 2,000 chickens and ducks. After selling some of them, he had 650 chickens and 520 ducks left. How many chickens and ducks did he sell altogether?

First I chose to perform addition to find how many chickens and ducks a farmer has left after selling some of them. 650 chickens plus 520 ducks equals 1,170 chicken and ducks. Now I am left with a separate problem to find the unknown difference. I know that it is a separate problem because the chickens and ducks are being sold and separated from the group. 2,000 take away 1,170 equals 830 chicken and ducks that were sold.

Strategies for Teaching Students to Use the Singapore Bar Models

I plan to begin with basic addition facts that are less than 20. I will teach the students how to draw Singapore bar models to illustrate addition and subtraction facts. The problem will be presented in a very straightforward manner such as, the sum of 7 and 8, or the difference of 15 and 8. Once the students can create these, I plan to progress to two digit numbers followed by three digit numbers. This step will involve a slightly different model with the use of brackets to represent the number for which each part stands. The next step is for the students, when presented with a word problem, to make sense of the words and draw a bar model to represent it. From this point, I will show the students how one model can represent different problems within a given scenario. Typically, there are several scenarios for the same problem. All parts of the problem are represented in the diagram and different problems can be written based on the parts that are known and unknown. The problems that are generated include problems related by inverse relationships. This strategy strengthens the students' understanding of the relationships shared by the four operations. I plan to have the students draw one illustration for each scenario. As the students become proficient the task will change. I expect the students to move into writing several related word problems for a given model.

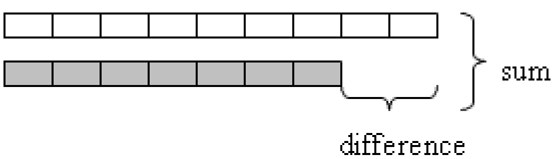
When the students have mastered addition and subtraction word problems by modeling with bar diagrams, I will teach them how to construct a similar model for multiplication and division. Initially I plan to provide a word problem that involves a basic fact such as 3 x7 and demonstrate and discuss the bar model representation. After sharing a few models, I will ask the students to construct the models given a basic fact. From this stage, I will teach the students how to name several problems with the same bar model. This process again involves the inverse relationships and the fact that multiplication is derived form addition. The final step is to have the students write several different scenarios for the same bar model involving multiplication and division.

The final stage is to tackle multi-step problems with a bar picture that combines different operations. These are significantly more complicated, but by this stage the students should be comfortable with the models. I will begin by guiding the students through models of each dimension. Then the students will work in groups to categorize problems based on the two operations needed to solve the problem. Next the students will solve the problems. I expect this stage will take more practice than the previous ones.

A teaching strategy that I will use for instruction and for having the children model their problems is found on the interactive Thinking Blocks website. Using this website as a teaching tool, I will initially show the students the video demonstrations through the computer onto the projection screen. After the class views the video, we will create our own problem. The students will participate in the process by manipulating the online models to create and solve the problem. Then the students could use this tool and work with partners to model and solve a problem given to them. This site can also be used for remediation, centers, for class work and homework, for enrichment, and as a teaching tool/resource.

Activity 1

The initial introduction to the Singapore bar models will involve simple straightforward addition and subtraction problems. An example of each kind follows and both of the problems come from the Primary Mathematics 3A Workbook (1992).



(a) 9+7=

The sum of 9 and 7 is ____.

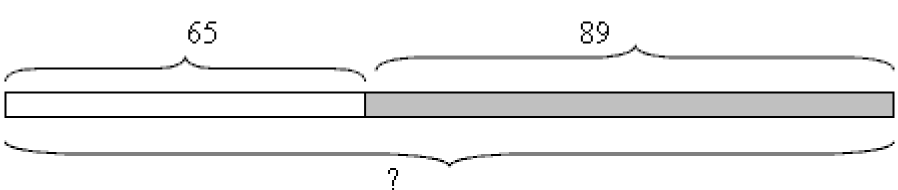
(b) 9-7=

The difference between 9 and 7 is ____.

Using a basic fact less than 20 will introduce the students to how the models work. I will give the students several examples using basic facts less than 20 until they grasp the concept. Then I will introduce them to the slightly more abstract diagram where the units are no longer represented individually. For example:

Find the sum of 65 and 89.

65+89=

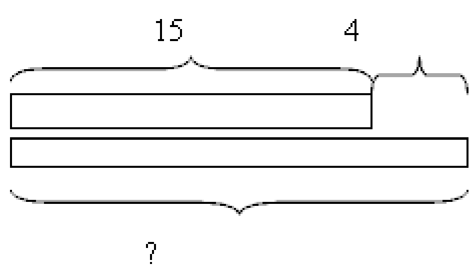


The sum of 65 and 89 is ______.

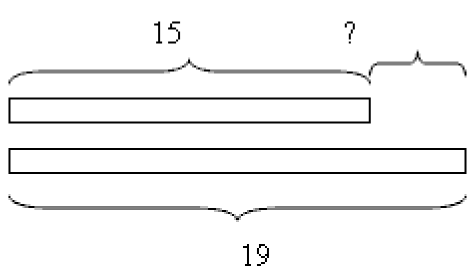
This model above is representative of the way the models appear once the students have been introduced to them. The end-to-end model is preferable since it is linear and makes a connection to measurement. I will have the students solve several problems that are just numerical, using the end-to end model. Then the students will use construction paper cut-outs to create models on their paper. This will provide a hands-on way to create the models. Once the students have made some with pre-cut models, they will draw their own representations. In making all models, one requirement will be that the related equation should also be written.

Following the work with straightforward equations, the next step will be to move the students into using the models to represent word problems. Initially we will work with addition and subtraction. The same model can often represent several word problems related by the commutative property or the inverse relationship shared by addition and subtraction. Developing these concepts through problem solving will help to make these connections more meaningful for the students and will lead to greater motivation.

Ryan's sister is 15 years old. How old is Ryan if he is 4 years older than his sister?

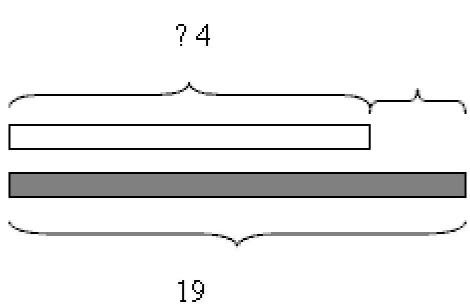


The equation that goes with this problem is 15+4=19 years or 4+15=19 years. A few other problems that could be asked using this same model, but changing where the unknown is, are as follows:



Ryan is 19 years old. His sister is 15 years old. How many years older is Ryan?

The equation for this problem is: 15+4=19 years.



Ryan is 19 years old. He is 4 years older than his sister. How old is his sister?

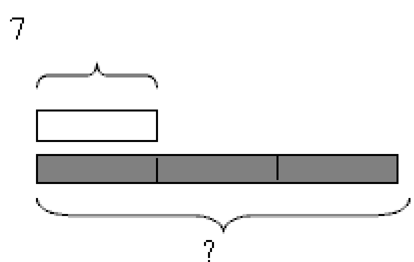
The equation for this problem is: 19 - 4 = 15 years old

Once the students are solving word problems, they will be required to write the equation and also to label the answers. The focus will be on being able to solve the problem with a bar model while also showing a corresponding equation. The numbers within the word problem will get more challenging. Similar problems with three digit numbers would be the next step to take with the students. The students will work with the same model for several related scenarios where each time the unknown is different. These related problems will continue to show inverse relationships and the commutative property. Once the students are comfortable with the models and with writing related problems, I will be able to move quickly to more difficult numbers since addition and subtraction through the millions has already been covered.

Activity 2

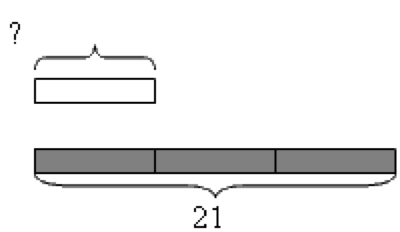
In the next activity I will teach the students to use the bar models to show multiplication and division. I will once again begin on the computer with the projection screen. I will use the Thinking Blocks website. This time I will model a multiplication problem. The Singapore bar models would be cumbersome to draw with a large multiplier, as 7 x 36. These examples come from the Primary Mathematics Workbook 3A (1992).

Roy had 3 times as many comic books as Samy. If Samy had 7 comic books, how many comic books did Roy have?



Roy had ____ comic books? The equation for this problem is 7x3=21. Once again by changing the unknown a new problem is created.

In this next problem the inverse relationship of division is represented. Roy has 21 comic books. Roy has 3 times as many comic books as Samy has. How many comic books does Samy have?



21 / 3 = 7

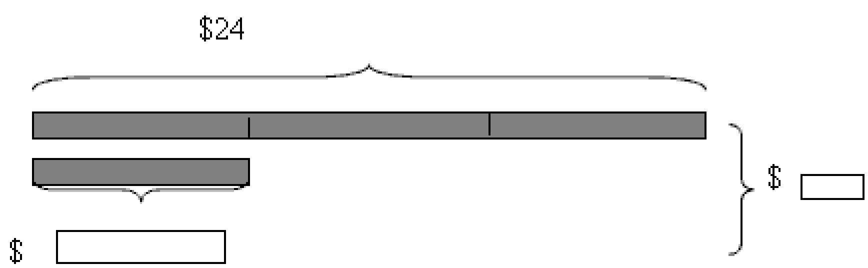
Much as I did with addition and subtraction, I am pairing up the inverse operations of multiplication and division to help the students to build a critical piece of understanding. By developing this understanding of the relationship between multiplication and division, the students will not only learn their division facts more easily, but also become better problem solvers. Carpenter, Franke, & Levi (2003) assert, "understanding the relation between multiplication and division is critical for learning division number facts and for dealing flexibly with problem situations involving multiplication and division."

Again, once the students learn how to use the model to represent division, the difficulty can increase to the level of their division skills. More difficult division problems can be solved later in the year, once long division has been taught.

Activity 3

The multi-step domain will be considerably more complicated to teach. I am focusing only on two-step problems. Mathematically there are 16 different combinations of the four operations since the order must be taken into account. Initially, I will group these problems together based on their types, or their similarities and differences. I will share examples of problems with the corresponding bar model. After taking the students through examples of each dimension, I will have the students work in cooperative groups to read word problems, look at the bar model and identify the two operations. They will also have to figure out the proper order in which to complete the operations. Once the students are proficient with this skill, I will move them into actually solving these two-step problems. This will begin as group work then it will progress to partner work and independent work.

Meihua saved $24. She saved 3 times as much as Weilin.



Write the missing numbers in the boxes as you answer each question below.

(a) How much did Weilin save?

(b) How much did they save together?

I will provide the students with many opportunities to solve the multi-step problems since these vary so much. The students will participate in discussions about the problems because my students can learn so much from each other. While practicing these problems it would be fun to give each student a word problem written on an index card. We could go outside and the students could draw the Singapore bar models and solve the problem on the blacktop using sidewalk chalk. The key to conquering the multi-step problems will be keeping it fun. This will be more frustrating for some students, but by mixing up the activities and keeping it fun the students will be more motivated.

Appendix A: Collection of Problems

Addition and Subtraction

Starter Problems for Singapore Bar Models

1) The sum of 6 and 8 is _____.

2) The sum of 9 and 7 is _____.

3) Find the sum of 65 and 89.

4) The difference between 9 and 7 is _____.

5) Find the difference between 96 and 68.

6) Find the difference between 387 and 512.

7) When 376 is subtracted from a number the answer is 825. Find the number.

Join R - Result Unknown C - Change Unknown S - Start Unknown

8R) Nancy has 5 M&M's. Jermaine adds 3 M&M's to her pile. How many do they have altogether?

8C) Nancy has 5 M&M's. Jermaine adds his to her collection. Now there are 8 M & M's. How many did Jermaine have?

8S) Nancy has some M&M's. If Jermaine adds 3 to her collection they will have 8. How many does Nancy have at the start?

Separate R - Result Unknown C - Change Unknown S - Start Unknown

9C) Mrs. Rogers made 150 pineapple tarts for a party. After the party, there were 16 tarts left. How many tarts were eaten at the party?

9R) Mrs. Rogers made 150 pineapple tarts for a party. The guests ate 134 tarts. How many tarts were left?

9S) Mrs. Rogers made some tarts for the party. The guests ate 134 tarts. She had 16 left. How many tarts did Mrs. Rogers make?

Part Part Whole P- Part Unknown W - Whole Unknown

10P) Sharon is blowing up 52 balloons for a party. She has 24 left to blow up. How many has she already blown up?

10P) Sharon is blowing up 52 balloons for a party. She has blown up 28. How many balloons does she have left to blow up?

10W) Sharon is blowing up balloons for a party. She has blown up 28. She has 24 left to blow up. How many balloons will she have blown up when she is finished?

Compare D - Difference Unknown C - Compare Unknown R - Referent Unknown

11R) Bob's sister 23 years old. She is 9 years older than Bob. How old is Bob?

11C) Bob is 14 years old. His sister is 9 years older. How old is his sister?

11D) Bob is 14 years old. His sister is 23 years old. How many years is the age difference between them?

12C) Mr. Jones traveled 386 km on Monday. He traveled 147 km more on Sunday than Monday. How many kilometers did he travel on Sunday?

13D) Randy collected 124 matchboxes. Josiah collected 283 matchboxes. Who collected more matchboxes? How many more?

14R) There are 3,402 boys at the parade. There are 987 more boys than girls. How many girls are there? How many pupils are there altogether?

Multiplication and Division

Starter Problems for Singapore Bar Models

15) Find the product of 41 x 5.

16) Find the product of 3 x 52.

Grouping/Partitioning Problems

M - Multiplication DM - Division-Measurement DP - Division-Partitive

17DP) Tim printed 900 pamphlets. He packed them equally into 8 boxes. How many pamphlets were there in each box? How many pamphlets were left over?

18M) Jessica put some cookies equally into 3 jars. If each jar had 24 cookies, how many cookies were there altogether?

19DM) Molly sold 70 flowers. They were in bunches of 5. How many bunches of flowers did she sell?

Rate Problems

M - Multiplication D - Division-Measurement DP - Division-Partitive

20DP) Molly saved $32 in 4 weeks. She saved the same amount of money each week. How much did she save in a week?

21M) Kayla used 4 kg of flour to make cakes. She used 18 eggs for each kilogram of flour. How many eggs did she use?

22DM) A bucket holds 7 liters of water. A tank holds 637 liters of water. How many buckets of water can the tank hold?

Price M - Multiplication DM - Division-Measurement D - Division-Partitive

23DM) Mrs. Gray paid $98 for the basket of fish. 1kg sells for $7. How many kg did she buy?

24DP) A fruit seller sold 12 kg of grapes. He collected $96. How much does 1kg cost?

25M) Mr. Smith sells fruit for $8 per kg. He sells 37 kg in one day. How much money did his customers spend on fruit at his fruit stand?

Multiplicative Comparison

26) Roy has 3 times as many comic books as Samy. If Samy had 7 comic books, how many comic books did Roy have?

27) There are 20 monkeys in a zoo. There are 4 times as many monkeys as tigers. How many tigers are there?

28) A florist has 145 yellow roses. She has 8 times as many red roses as yellow roses. How many more red roses than yellow roses does she have?

Multistep (annotated to show components)

1st step -Partitive Division

29) 8 students sold 272 concert tickets at $3 each. Each student sold the same number of tickets. How much money did each student collect? (partitive division/ price -muliplication)

1st step - Measurement Division

30) Mr. Stone bought some books for $301. Each book cost $7. His wife helped him carry 18 books and he carried the rest. How many books did he carry? (price- measurement division/ part-part-whole subtraction missing part)

31) Mr. Lee sold 462 flowers in bunches of 6 each. Each bunch sold for $5. How much money did he receive? (grouping/partition-division-measurement/ price-multiplication)

32) Miss Brown baked 600 pineapple tarts for sale. She packed them into packets of 6. She sold 45 packets. How many packets were not sold? (grouping/partition -measurement division/ separate- result unknown)

1st step Multiplication (grouping partition)

33) Rebecca bought four boxes of doughnuts. There were 6 doughnuts in each box. She divided the doughnuts equally among eight people. How many doughnuts did each person receive? (grouping/partition-multiplication/grouping/partition partitive division)

34) Jordan had 9 bags of paper plates. There were 10 plates in each bag. 25 plates were used for a party. How many plates were left? (grouping/partition- multiplication/ separate- result unknown)

35) Steven bought 7 planks, each 3 m long. He bought another plank 2 m long. Find the total length of the 8 planks. (rate - multiplication/ join - addition - result unknown)

36) Marty reads 8 pages of a book a day. After reading the book for 4 days, she still has 5 pages to read. How many pages are in the book? (rate -multiplication/part-part-whole-addition)

1st step Multiplicative Comparison

37) A toy car costs $5. A toy airplane costs 4 times as much as the toy car. How much more does the toy airplane cost than the toy car? (multiplicative comparison/ compare-subtract- result unknown)

38) A pole is 3m long. A rope is 8 times as long as the pole. If the rope is divided into 4 pieces, what is the length of each piece of rope? (multiplicative comparison/ grouping/partition - partitive division)

39) There are 650 men and twice as many women working in a factory. How many workers are there in the factory? (multiplicative comparison/ part-part-whole - addition- missing whole)

1st step Subtract

40) Rani had $47. After paying for 3kg of shrimp, she had $20 left. Find the cost of 1kg of shrimp. (price- subtract -unknown change/ price - partitive division)

41) A computer cost $1,900. A television set was $650 cheaper than the computer. Mr. Ray bought both the computer and the television set. How much did he pay? (compare - subtract - difference unknown/ join- addition- result unknown)

42) Pablo earns $3,000 a month. He spends $2,200 a month and saves the rest. How much money can he save in 6 months? (price - separate -difference unknown/ price - multiplication)

1st step Part-Part-Whole

43) A farmer had 2,000 chickens and ducks. After selling some of them, he had 650 chickens and 520 ducks left. How many chickens and ducks did he sell altogether? (part-part-whole - addition - missing whole/ separate- difference unknown)

44) 4 men and 5 women shared the cost of a present equally. The cost of the present was $117. How much did each of them pay? (part-part-whole - addition - whole unknown/ price - partitive division)

Appendix B: Resources for Classroom Use

The following resources are used in these lessons: an internet connection, a computer, an LCD projector, a collection of word problems, construction paper, glue, sidewalk chalk, paper and pencils.

Appendix C: Annotated Bibliography

Carpenter, T., Fennema, E., Franke, M., Levi, L., & Empson, S. (1999). *Children's*

mathematics: Cognitively guided instruction. Portsmouth, NH: Heinemann. This was a great book that clearly identifies different types of word problems.

Carpenter, T., Franke, M., & Levi, L. (2003) *Thinking mathematically: Integrating*

arithmetic & algebra in elementary school. Portsmouth, NH: Heinemann. This was not very useful for this unit, but is important for developing algebra concepts early.

Leinwand, Steven, (2005). *New AIR study compares the quality of U.S. math instruction*

with Singapore recognized world leader U.S. trails, but both nations could learn from

each other. 2/7/05 http://www.air.org/news/documents/ Singapore.htm. This website shares the results of a study that ranks Singapore the world leader in math achievement.

Ma, Liping. (1999). *Knowing and teaching elementary mathematics.* Mahwah, NJ:

Lawrence Erlbaum Associates, Publishers. This book compares Chinese teachers to American teachers and reveals some striking differences.

Singapore primary math texts, U.S. Edition Curriculum Planning and Development Division, Ministry of Education, Singapore: Federal Publications. These textbooks and workbooks have a more simplistic appearance, but are more challenging than the corresponding U.S. grade level. These books show bar models and present place value in a slightly different way.

Sowder, Larry, (1995). *Addressing the story-problem problem.* In J. Sowder & B.

Schappelle (Eds.), *Providing a foundation for teaching mathematics in the middle*

grades (123-131). New York: State University of New York Press. An interesting article that looks at limited and desired strategies for problem solving.

Thinking Blocks, (2006) http://www.thinkingblocks.com. Date last accessed 7/3/07. This is a great website and offers many uses for teachers and students.

Appendix D: Implementing District Standards:

Virginia Standards of Learning Grade 4: The essential knowledge and skills for each of these objectives lists problem solving.

4.6 The student will add and subtract whole numbers written in vertical and horizontal form, choosing appropriately between paper and pencil methods and calculators.

4.7 The student will find the product of two whole numbers when one factor has two digits or fewer and the other factor has three digits or fewer.

4.8 The student will estimate and find the quotient of two whole numbers, given a one- digit divisor.