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Curriculum Units by Fellows of the National Initiative
2007 Volume VI: Keeping the Meaning in Mathematics: The Craft of Word Problems

Introduction

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Symbolic notation is an important source of power for mathematics. This specialized writing system compresses large amounts of information into compact, easily manipulated form. For the knowledgeable user, this notation is a versatile problem solving tool. However, from the educational point of view, this power does not come for free: teachers must work hard to help students keep the symbolism invested with meaning. If a student cannot interpret and work with the symbols in meaningful ways, his only option is to try to manipulate them according to formal rules. Frequently the rules become too complex to cope with, and the student completely loses touch with mathematics.

Word problems can help students stay in touch. Dealing with a word problem requires translating its verbal statements into symbolic ones, manipulating the symbols to solve equations, then translating back to answer the question posed in the problem. This process requires students to assign meaning to symbols, thereby keeping students tied to mathematics. Thus, word problems should not be thought of as a separate topic in the curriculum. They have a central role to play in mathematics education.

From this point of view, the avoidance of word problems so common in our math instruction is a prime symptom of the deficits of mathematics education in the U.S.

The seminar on *The Art and Craft of Word Problems* was devoted to remedying this deficit, at least in the classrooms of the seminar Fellows. Each fellow has written a curriculum unit centered around word problems. Word problems are not simply a prominent feature in these units, they are dealt with in a systematic way intended to give students an overall understanding of how to approach word problems in the relevant subject area. More broadly, the goal is to instill a habit of careful reading and interpretation.

George Polya, in his often cited writings on problem solving, listed four key steps for dealing with any problem:

1. Understand the problem.
2. Make a plan.
3. Carry out the plan.
4. Look back.

Of these steps, the first is by far the most important. Students who learn to read mathematics words problems and interpret them carefully develop skills that will help them in all mathematics courses, and far beyond mathematics.

The most frequently proffered advice for understanding word problems is to learn the vocabulary. Each

mathematical operation has a variety of words or phrases which invoke it; addition is suggested by "in all", "all together', "sum", "more than", "added to'" and so forth. The sound advice to know the words involved in problem statements is unfortunately frequently taken to extremes, resulting in the "key word" approach: decide what operation to perform by identifying a word associated to one of the operations. However, this approach has serious shortcomings. It is easy to write word problems with a phrase such as "more than," or "the sum of", but which require subtraction for their solution. There is no substitute for careful reading and understanding. Particularly with multistep problems, ability to read and understand and translate into mathematics is essential, since the proliferation of possible problem types easily outstrips efforts at classification. Unfortunately, the national aversion to word problems severely curtails the extent and variety of multistep problems our students see, thereby limiting their experience and ability in applying mathematics.

The units prepared for this seminar promote understanding word problems, not only one by one, but through comparison. Because they will see and compare a variety of word problems, students will get a feel for the kinds of issues that can arise, and for some possible responses. The phrase used in the seminar for this comparative study was "exploring the problem territory". As students work on the problems assembled for these units, they should also gain some familiarity with the problem territory, and, hopefully, will feel somewhat more at home in it.

The grade levels taught by seminar fellows ranged from primary to high school. Accordingly, their seminar units cover a range of topics. At the primary level, the units by Tanya Shannon, Huwerl Thornton and Nancy Wasser deal with addition and subtraction. All these units ensure variety in their problems by drawing on the taxonomy developed by T. Carpenter et al, in the book {\it Children's Mathematics} This book makes the point that, although addition problems tend to be relatively easy for children to conceptualize, the operation of subtraction has many faces, some of which are much harder for children to deal with than others. The standard "take away" scenario is the simplest to deal with. If students are presented only with take away problems, they will not develop a robust concept of how subtraction gets used. These units will ensure a wider experience with the subtraction operation.

In addition to their common reliance on the Carpenter et al. taxonomy, the units have further individual foci. Tanya Shannon has varied the complexity of the numbers in her problems, and sometimes expresses them in numerals, and sometimes with words. Huwerl Thornton has adapted strategies originally devised to teach reading for use with his word problems. Nancy Wasser has created a bilingual set of problems, and has incorporated problems to help students grasp the structure of the decimal place value system for writing numbers.

At a slightly higher grade level, Valerie Schwartz has developed a suite of problems designed to take students from single step problems to multistep problems incorporating both addition/subtraction and multiplication/division. Her unit also undertakes to teach the use of Singapore bar models. This highly effective method for representing problems graphically enables elementary students to solve many problems that in the U.S. are considered part of algebra.

Karlene McGowen takes a language arts perspective in her unit. She seeks to approach mathematics word problems in a multidisciplinary way, by using story books and a "problem of the week" theme to get students to analyze the language in word problems in greater depth than standard instruction affords.

Moving beyond elementary school, Jill Smith's unit treats the issue of proportion in several important settings, including the analysis of unit rates, the geometric topic of scaling, and the use of percents. Moses Jackson focuses on percents, and investigates a range of problems using the "three piece percent formula". He shows

how to apply this formula in a wide variety of settings. Angel Johnson has created a unit to introduce her students to algebraic notation. She develops parallel lessons, first treating numerical expressions of several levels of complexity, and then progressing through the same stages with symbolic expressions. Part of the work of this unit is for students to translate words into expressions and expressions into words - to be able to read expressions. This makes explicit the task that is implicit in standard word problems, and provides extra practice with this important skill.

The topic featured in the units of Patricia Marasco, Paula Shaffer-Roche and Tyler Willoughby is linear equations, in one or two variables. Each of these units deals with several types of problems and works to help students see the similarities and differences between the types. The problems vary in complexity, from problems that could be dealt with readily by standard arithmetic methods to problems that require the full apparatus of elimination. Paula Shaffer-Roche has paid particular attention to this progression of complexity. She discusses a technique, known as *false position*, that enables students to approach many varieties of problems, including mixture problems, interest rate problems and combined sales problems, using arithmetic reasoning. Each writer has an individual approach to the subject. This gives the reader a three-fold opportunity to enrich his/her own ideas about motivating this topic and the possibility of further enrichment through comparing the units.

The most advanced topic treated in these units is quadratic equations, the theme of Nancy Rudolph's contribution. She has assembled a substantial collection of problems based on quadratic equations. They deal with projectile motion and several aspects of geometry. She includes problems to advance her laudable "personal mission" of having students understand the important topic of the behavior of area under scaling. Her problems deal both with direct issues - evaluating, finding roots and finding extrema of various quadratic functions - and with inverse issues - finding a quadratic function satisfying specified conditions. The variety of the problems she presents should help her readers explore the problem territory of this key topic in algebra.

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