Curriculum Units by Fellows of the National Initiative
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Introduction

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"Estimation" is a word that is heard often in mathematics education, but what the word means tends to be shadowy. It has something to do with rounding, but when you use it and what you use it for are not usually elaborated in much detail. The aim of this seminar was to take estimation out of the shadows, and to connect it with the main ideas of arithmetic.

Most specifically, the goal was to bring out the estimation capabilities of our base ten place value (a.k.a. decimal) system for writing numbers. This marvelously efficient notational scheme can express any whole number using only ten symbols, organized in carefully arranged sequences. A given sequence, such as 123, stands for a sum: 123 = 100 + 20 + 3. Each of the summands has a special form: it is a digit (i.e., 0, 1, 2, 3, 4, 5, 6, 7, 8, 9) times a power of 10. The place of the digit in the sequence tells what power of 10 it should be multiplied by. The 3 is multiplied by 1, which is 10^0 ; the 2 is multiplied by 10, which is 10^1 , and the 1 is multiplied by 100, which is 10^2 . The key to making this work is to insert a 0 wherever the corresponding power of ten is not needed to express the number. This principle is most dramatically used in the summands: because the 2 in 20 appears in the second place, with a zero to its right, we know that it means twenty, not two; because the 1 in 100 appears in the third place, with two zeros to its right, we know that it stands for one hundred, not ten or one. This is the principle of place value.

Most children learn place value in terms of vocabulary: ones, tens hundreds, thousands, etc. What is important for estimation is relative place value. This amounts to the observation that the relative value of two places depends only on their separation. Thus, 10 is ten times 1, 100 is ten times 10, 1000 is ten times 100, and so on: any place is worth ten times as much as the place immediately to its right. Similarly, any place is worth 100 times the place two places to the right, and 1/10 of the place just to its left, and 1/100 of the place two to the left; and so forth. The relative place value of two places depends only on their separation, not on their specific locations.

This means that in any base ten number, most of the value of the number is in the leftmost few digits. In fact, one can show that the leading (i.e., leftmost) digit of a number always accounts for at least half the size of the number; the leading two digits always account for at least 90% of the number; the leading three digits account for at least 99% of the number; the leading four digits account for at least 99.9% of the number, and so forth.

Since just the leading digit of the number accounts for most of the number, the place value of this leading digit already indicates the approximate size of the number. This leads to the idea of order of magnitude. The order of magnitude can be described as the exponent of the power of ten represented by the leading digit. An operationally simpler way to describe order of magnitude of a whole number is, it is one less than the number of digits used to write it. Thus, the order of magnitude of 123 is two. An important idea in estimation is that if you can determine the order of magnitude of a number, you know something important about it.

In fact, when one is dealing with numbers that arise from measurement, that are used to describe the real world, it usually will not make sense to specify more than the leading four, or often three or two digits, and sometimes only the leading digit really makes sense. The radius of the earth can serve as an example. It is somewhere around 4,000 miles, but it does not make sense to specify it to the nearest whole mile. The bulge around the equator caused by Earth's rotation; the lumps and dimples like the Himalayas and the deep ocean trenches; and other irregularities, mean that it only makes sense to talk about the radius of the earth to about ± 10 miles, which means the last decimal place in a figure like 3958 should be ignored. Use 3950 or 3960.

The seminar explored these ideas and their implications at some length. We learned the usefulness of one-digit arithmetic (and even, in some situations, of zero digit arithmetic!). The units produced by the seminar Fellows vary in level from first grade to high school, but they all incorporate these ideas in a meaningful way.

The first grade unit of **Carol Boynton** seeks to incorporate the idea of place value into the beginnings of arithmetic. It emphasizes the idea that a two-digit number is a sum of some tens and some ones, and already uses this in teaching the addition facts, thus preparing a smoother entry into general addition and subtraction of two digit numbers. The main point explicitly related to estimation is simply that tens are much larger than ones, and are what one concentrates on when comparing numbers. The second grade unit of **Sarah Kiesler** deals with similar ideas, but combines them with measurement ideas, and coordinating number with length. Both of these units incorporate ideas from the Singapore mathematics curriculum. **Vivienne Bartman's** third grade unit studies place value through a focus on the key idea that makes place value work: the zero. Her students will learn about Zero the Hero.

The upper elementary and middle school units develop real world projects that incorporate ideas and principles of estimation. Place value, relative place value and order of magnitude will help **Sharyn Gray's** class get a handle on issues of waste, and resource management. **Eunice Rebullida's** unit emphasizes order of magnitude, and develops a pictorial estimation technique based on array sampling. **Elaine Tam's** unit has the theme of rice, its consumption, production and storage. It includes cooking zongs, or "rice tamales." **Kathryn Kinsman** uses baseball as a hook to introduce the key ideas, then progresses to less recreational topics. **Brian Bell's** unit highlights the horseshoe crab, an important part of Delaware's shoreline ecosystem. The statistical methods used to estimate horseshoe crab population on Delaware's beaches also use a form of grid sampling, and Brian's unit incorporates projects to help his students grasp the issues involved in these methods. Finally, **Patricia Marasco's** project for high school students concentrates on developing a comfort in working with large numbers and gaining a practical understanding of approximate arithmetic. Teachers wishing to give their students a richer experience in estimation, and better appreciation for its potential can find both specific ideas and inspiration in these units, whatever grade they teach.

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