Curriculum Units by Fellows of the National Initiative
2010 Volume IV: The Mathematics of Wallpaper

Beyond the Wumps: Exploring Symmetry in Seventh Grade

Curriculum Unit 10.04.05, published September 2010
by Jane H. Fraser

Overview

Symmetry is everywhere. It is in nature, in buildings, in the tiniest building blocks of the human body. We are naturally drawn to symmetry. The 13-year-olds I teach are surrounded by symmetry, even if they are not aware of it: the McDonalds' arches, the logo on their t-shirt, the rhythm of their favorite song. And while they know more about symmetry than they realize, they really don't know much about the mathematics involved.

This unit will cover the basics of transformational symmetry and take the students beyond what they have learned up to this point about similarity and congruence . They will learn about the idea of *isometry*, a transformation that preserves distance. The ultimate goal is to have them make patterns by moving a single figure around on the coordinate plane, and finally creating a frieze pattern of their own design.

Rationale

The Pittsburgh Public Schools make up the second largest school district in Pennsylvania. Approximately 28,000 students attend grades K-12 in 66 schools. I teach 7 th grade math in a 6-8 middle school. We are considered a magnet school, which means we maintain an equal racial balance of our 330 students. All our students live in city neighborhoods, and the mean family income is about 64% of the Pennsylvania average. While we face many of the challenges of inner-city schools, we have not had to deal with excessive truancy or violence.

My students are what I would call "typical" 13-year-olds. Each year, I have about three or four gifted students in each of my classes. This past school year, I did not have any inclusion students, which is unusual since we are a full inclusion school. So I had classes of average-intelligence students. They are much more interested in socializing than in doing schoolwork, and definitely not very interested in math. Since we are a magnet school, a bit more parental involvement occurs than in some of the neighborhood schools, and consequently I think our students are slightly more likely to do their homework and a little more worried about their final grades.

The curriculum I teach is Connected Math 2 (CMP2). The goal of CMP2 is to develop student knowledge and understanding of mathematics that is rich in connections; connections between ideas and grade levels, between different subject areas, and connections within the community and the world outside of school. The CMP2 curriculum is organized around interesting, real-life problem settings. Deep understanding of mathematics is acquired through observations of patterns and relationships. The seventh grade curriculum moves into pre-algebra, with an emphasis on ratios and proportions.

CMP2 develops four mathematical strands: Number and Operation, Geometry and Measurement, Data Analysis and Probability, and Algebra. Most of the goals are revisited in later units and subsequent grade levels throughout the curriculum.

The seventh grade CMP2 curriculum I teach includes the following seven units:

Variables and Patterns (Algebra): describing patterns of change between two variables,

constructing tables and graphs, using algebraic symbols to write equations.

Stretching and Shrinking (Geometry): identifying similar figures by comparing corresponding parts, drawing shapes on coordinate grids and using coordinate rules to stretch and shrink those shapes.

Comparing and Scaling (Number): using ratios, fractions, differences, and percents to form comparison statements, scaling a ratio, rate, or fraction up or down to make a larger or smaller one, applying proportional reasoning to solve for the unknown part.

Accentuate the Negative (Number): comparing and ordering rational numbers, developing algorithms for adding, subtracting, multiplying, and dividing positive and negative numbers, using the Distributive Property.

Moving Straight Ahead (Algebra): constructing tables, graphs, and symbolic equations that express linear relationships, solving linear equations.

Filling and Wrapping (Geometry): understanding volume and surface area, exploring prisms, pyramids, cones, cylinders, and spheres, extending understanding of similarity and scale factor to 3-D figures.

What Do You Expect? (Probability): interpreting experimental and theoretical probabilities, determining the expected value of a probability situation.

As you can see, our 7th grade curriculum covers a variety of mathematical concepts, with the main emphasis on developing algebraic reasoning. Currently, students have ten periods of math a week; two periods per day (about 90 minutes total), usually (but not always) blocked together. While there is time to complete all seven units, time is always an issue and we follow a fairly strict pacing guide.

The second unit of the 7th grade CMP2 curriculum is "Similarity Transformations: When Shapes Shrink or Grow" (*Stretching and Shrinking*). In this unit, we graph a character named Mug, a member of the Wump family. Mug is basically a trapezoid with legs and a face. By applying different transformations to the original coordinate points, for example $(x, y) \rightarrow (2x, 2y)$ or $(x, y) \rightarrow (3x, 3y)$, the students transform the original Mug into larger members of the Wump family. They also move Mug around on the grid, or create "imposters" that are misshapen and therefore not similar.

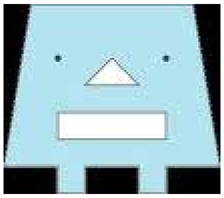

Fig.1: Mug Wump in his original shape

Stretching and Shrinking is a rich exploration of what it means for two or more figures to be mathematically similar, and by working within the context of a family of strange, gnome-like characters, students get to have a little fun while investigating all aspects of similarity.

In my curriculum unit, I will expand my students' exploration of the Wump family by introducing other types of transformations such as reflections and rotations. Symmetry is one area that is not covered in the 7 th grade CMP2 curriculum. There is one geometry book in 6 th grade, *Shapes and Designs*, in which two pages are devoted to symmetry. In the 8 th grade, there is an entire unit dedicated to symmetry (*Kaleidoscopes, Hubcaps, and Mirrors*), but that unit is scheduled near the end of the school year and is often not even taught because of time constraints.

According to the Pennsylvania Academic Standards for Mathematics, by the end of eighth grade, students should be able to:

- Use simple geometric figures (e.g., triangles and squares) to create, through rotation, transformational figures in three dimensions
- Analyze geometric patterns (e.g., tessellations and sequence of shapes) and develop descriptions of the patterns
- Analyze objects to determine if they illustrate tessellations, symmetry, congruence, similarity and scale

Overall, it seems that symmetry is not such a big deal in the scheme of middle school math, and certainly not even addressed in seventh grade.

So why teach symmetry to seventh graders?

Because symmetry is everywhere. Probably the most easily identified property of a figure, symmetry is about connections between different parts of the same object. Most students already have an intuitive understanding of symmetry; they can identify a symmetrical object and they can find repeating patterns. More sophisticated thinking is required to actually confirm symmetry and to construct figures with given symmetries.

Symmetry can be defined in terms of transformations of an object. While we are

transforming our Wumps by making them larger, smaller, or in some way misshapen, this is not considered what is termed isometry or congruent transformations. My goal for this unit is to work with our basic Wump figure, and later on with another simple geometric shape of the students' own design, and explore simple transformations of our figure through reflections, rotations, translations, and glide reflections.

Before we can go further in our discussion of symmetry, some basic key terms should be explained:

transformation: the transformation of a figure is achieved by applying a rule for moving the points of the original figure to obtain the new figure; the transformed figure is the collection of the image points of the original figure.

image: a copy of an original figure, made by a transformation

isometry: a congruent transformation of a figure. All distances are preserved in isometries; that is, for example, if point A and point B are 2 cm apart in the original figure, they will still be 2 cm apart after an isometric transformation. Equivalent terms for isometry are rigid motion, congruence transformation, and distance-preserving transformation.

line of symmetry: a line that divides a figure into halves that are mirror images

parallel: parallel lines are lines that never intersect; two non-vertical lines in the (x, y) - coordinate plane are parallel if and only if they have the same slope

perpendicular: two lines that intersect and form right (90°) angles

symmetry: when a transformation preserves a figure, it is called a symmetry of the figure. The figure is said to be symmetric under the transformation.

Let's now begin a (very simplified) explanation of what symmetry is all about. If you open a symmetry textbook, written for students or mathematicians, it can be very intimidating. And while I feel quite intelligent carrying books like this around with me, the information for the most part is way above my head. So I'll attempt to make this as accessible as possible to my readers.

There are four kinds of isometries that we will be exploring in class: reflection, rotation, translation, and glide reflection.

Reflection symmetry is also called mirror symmetry, because if a line is drawn through the center of the object, one side is the mirror image of the other, across the line. If you think of the capital letters A, M, or Y, for example, you can see that they each have a reflection symmetry across an imaginary line running vertically through the center of the letters. The capital letters B, C, and D have reflection symmetry across a horizontal bisecting line. For all these letter examples, a point on one side of the line will exactly match the corresponding point on the other side. A point and its image under reflection will lie at the same distance from the line or axis of reflection, and the line connecting them will be perpendicular to the axis.

Reflections are the most fundamental isometries, because any other isometry is a combination (the technical term is composition) of them.

A reflection can be specified by giving the line of symmetry. If you look at Figure 2 below, you can see this. The big block letter F is reflected across line p. Point A and its image point A_1 lie on a line that is perpendicular to the line of symmetry and are equidistant from the line of symmetry. The same thing occurs with points B and B_1, and points C and C_1.

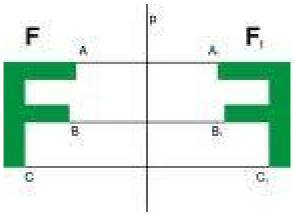

Fig. 2: Reflection symmetry

Rotation symmetry involves rotation (by other than a full turn) around a center point of the original figure. The rotation is a symmetry of the figure if it preserves the figure. All figures have what is called an identity rotation, which is what happens when you make one complete turn (that is, a 360° turn). Any rotation that is not an *identity rotation* is called *non-trivial*.

A rotation can be specified by giving the center of rotation and the angle of the turn. In the simple figure shown below, if each "leg" is rotated 120° around the center point, the figure is preserved. So rotation by 120° around the center is a symmetry of the figure.

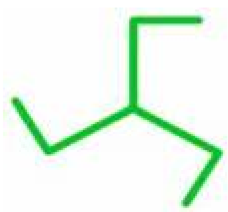

Fig. 3: Rotational symmetry

A translation is a slide on a plane along the path of a straight line. It is described by giving the direction and the length of the slide. This can be done by drawing an arrow with the appropriate length and direction. If you draw the segments connecting points to their images, the segments will be parallel and all the same length. The length is equal to the magnitude of the translation. A translation moves each point of the plane the same distance in the same direction.

Translational symmetry occurs when a figure is preserved by a translation. Each translation has a direction and a distance.

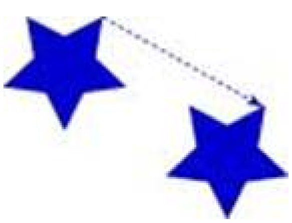

Fig. 4: Translation

There is one more type of symmetry transformation we'll be using in class, and that is a *glide reflection*. Glide reflection can be thought of as a combination of a reflection and a translation, as seen in the footprints below:

Fig. 5: Glide reflection

In each type of symmetry, characteristics such as side lengths, angles, size, shape, and distance are maintained. This is all a consequence of distances between points being preserved, because if you know the distances between points of a figure, you know the shape completely. In fact, if you know the distance from a point to any three non-collinear points, you know exactly which point it is. Reflections, rotations, and translations relate points to image points so that the distance between any two original points is equal to the distance between their images. This is the key property defining an isometry. A figure is symmetric under a particular transformation if it is left unchanged by the transformation.

By using combinations of these different transformations of an object, you can create *frieze patterns*. A frieze is a pattern which repeats in one direction. If you look closely, you will see frieze patterns all around you: around the tops of buildings, on wallpaper borders, pottery, needlework, and so on. If you go on the NCTM's *Illuminations* website, you can play around and create your own frieze patterns. Frieze groups provide a way to classify designs on two-dimensional surfaces which are repetitive in one direction, based on the symmetries in the pattern. There are seven different frieze groups.

Students already know about similarity and congruence of figures from *Stretching and Shrinking*. They know that with similar figures, basic shape and angles will remain the same, but side lengths (and therefore distances) may be greater or smaller. In this unit, they will study reflections, rotations, and translations by measuring key distances and angles. This unit will show them how they can make patterns by moving the same figure around on the plane. They will learn how to describe a particular transformation so that another student could reproduce it exactly.

Objectives

After completing this curriculum unit on symmetry, students should be able to do several things. They should understand the important properties of symmetry, as I have explained them here. They should be able to recognize and describe symmetries of figures, and make their own figures with specified symmetries. Students will do symmetry transformations of figures, and they will be able to write coordinate rules for specifying the image of a general point (x, y) under particular transformations.

Strategies

CMP2 is a problem-centered curriculum, and I plan to teach this supplementary unit in the same way I teach all other units throughout the year. I follow the inquiry model of instruction, and it is through a series of investigations, discussions of solutions, embedded mathematics, and appropriate generalizations that the students become better, more confident mathematicians.

Or at least this is what we hope. It is a struggle. It has been a struggle for all the years I've taught CMP2, and it will continue to be a struggle, because it is very difficult to convince students that they are capable of solving problems on their own without my constant help. Many parents don't like the way we teach math, because it's not the way they learned, and many students absolutely hate it at first because I don't just give them the answers. It is enormously empowering to them, though, when they grasp a new concept that they have struggled with, and used their own reasoning skills to master.

The CMP2 classroom is generally run under the Launch, Explore, Summarize model, which I will briefly explain.

In the Launch phase, the teacher launches the problem with the entire class. During this phase, the teacher helps the students understand the problem setting, the mathematical context, and the challenge (what are they asking you to find out?). During this time, teachers can also review definitions, review previous concepts, and connect the problem to past experiences of the students. It is very important to make the problem very clear, but leave the solution open. There is often a very fine line between too little and too much information. If you tell students too much, you might make the problem too easy, or discourage students from finding their own strategies.

During the Explore phase, students work singly, in pairs or groups to solve the problem. I prefer pairs or groups of four. My role during this phase is to circulate around the room, making sure the groups stay on task, and asking assessing and advancing questions to keep them moving in the right direction. During this time, I look for any misconceptions that may arise, and look for different solutions so that I can call on the appropriate group when the time comes.

The Summarize phase is when the class comes back together, and students present and discuss their solutions. This phase does not necessarily occur when absolutely everybody is completely done; rather, it is a judgment call the teacher makes. Not all groups have to be done, but most students should be well on the way to a solution. During the summary, the teacher helps the students reach the mathematical goals of the problem and connect their new understanding to prior learning.

This will be a very hands-on type of unit, so there should be plenty of room for students to spread out while they are working with their group. Gather together as many old magazines as you can, because students will be cutting pictures out of them. Students will need scissors, glue, graph paper, pencils, and rulers. Colored construction paper, poster-sized graph paper and colored markers should also be available so that students can present their work to the rest of the class. We will also be exploring several websites that deal with symmetry, so a nearby computer lab would be most helpful.

Classroom Activities

This unit will be taught between *Stretching and Shrinking* and *Comparing and Scaling* (the second and third books), and it should last about 8 days. The activities described here are intended to be a brief, fairly simple investigation of symmetry transformations. As I mentioned previously, symmetry is not covered extensively in Pennsylvania's middle school curriculum. The activities are enrichment activities, a way for students to take their understanding of similarity a step further to create some fun and interesting math drawings.

Lesson 1: Learning about symmetry transformations.

This is the introductory part of the unit. It should take about two days to complete. Introduce students to the four different symmetry transformations: reflection, rotation, translation, and glide reflection. The examples in *Kaleidoscopes, Hubcaps, and Mirrors* are excellent and easy to understand. Most students have learned about types of symmetry in elementary school, so some of this information will be a review. Don't expect them to remember everything, though; it has been a while since they've seen this.

After this brief review, take your students into the computer lab so they can see more examples of symmetry transformations, and actually try out some transformations on their own. Before you take your students into the lab, however, you really must investigate on your own. A Google search for "interactive symmetry" or "symmetry websites for kids" will produce hundreds and hundreds of resources. Some sites are excellent, but do your homework and try out the games and interactive lessons first, because some sites do not explain transformations correctly, or they are confusing and just plain wrong. NCTM's *Illuminations* is an excellent site. You can fine-tune your search by grade level and subject, and the learning activities are professionally presented and mathematically accurate.

Another useful site I found was *onlinemathlearning.com*. This is a directory of free geometry math games available online. The games are all categorized and reviewed, so it will help you pick and choose what you want your students to try. Go to "transformation games" for dozens of activities on symmetry.

The Symmetry Web quest (at *adrianbruce.com*) is an interactive website that gives hundreds of examples of different types of symmetry. It includes pictures of kaleidoscopes, African masks, and the Taj Mahal. It is a good way for your students to see examples of all four isometries they have learned about, and it includes a couple mini-quizzes so they can test themselves.

Despite the wealth of resources at your disposal, remember that you are dealing with middle school kids, so don't expect their enthusiasm to last longer than a class period. This computer lab experience is meant to be just a little taste of what transformational symmetry is all about, and the rest of the unit will be completed in the classroom, using scissors, glue, paper, and pencils.

The final part of this first lesson tests the students' ability to identify the four different isometries. Give each group (three or four students per group, no more) a stack of magazines, scissors, glue, and a large piece of colored construction paper. Their assignment is to look for examples of reflection, rotation, translation, and glide reflection symmetries in their magazines and make a poster showing all four. Reflection symmetry will probably be the easiest to find, but your students may struggle a bit more with the other three. Let them struggle. Try to include some art, architecture, and nature magazines in the mix, and they will eventually be successful.

Lesson 2: Transformations of figures on coordinate grids.

Many computer-based geometry programs use a coordinate grid as the drawing window. You create figures by specifying the endpoints of line segments. Students should be well-acquainted with graphing on a coordinate grid after completing *Variables and Patterns* and *Stretching and Shrinking*.

In this lesson, students will use the original figure of Mug Wump from *Stretching and Shrinking* (see Figure 1) and use transformation rules to reflect him across the x-axis and the y-axis.

Begin the lesson by looking once again at Mug. By now, students may be sick to death of looking at him, but tell them that they will have a chance to move him around like a computer animation and have a little fun with him now. This lesson is meant to be a very simple introduction to how mathematics is used in the creation of animated games and movies. A very thorough exploration of transformations on the coordinate plane is presented in *Kaleidoscopes, Hubcaps, and Mirrors*. Rotations, translations, and glide reflections are also explored in this book, but for our purposed here we want to keep it relatively simple and just work with reflections. Once students are comfortable with reflection transformations, they will be able to expand their understanding further and create transformation rules for the other three types of transformations. Please feel free to take this lesson further, but I have planned to complete to this activity in two days.

The students should work on this problem in groups of two to four students. Begin by having students graph the original Mug on the coordinate grid. For this exercise, tell your students to forget about the eyes, mouth, and nose and just concentrate on the body. Remind your students that Mug was created by connecting coordinate points; designs are created by specifying the endpoints of line segments. This is a key concept, because isometries are transformations that preserve distance. After Mug has been drawn in his original position, ask your students to draw what Mug's shape would look like reflected across the x-axis. Tell students to record the coordinate points of the new figure.

After they have reflected Mug across the x-axis, ask them to do the same thing across the y-axis (they should always go back to the original figure). Again, they should record the coordinates of this reflected image.

A completed chart of the coordinate pairs for Mug and his reflections is shown here:

Mug (*x*, *y*)	Reflection across *x*-axis	Reflection across *y*-axis
A (0, 1)	(0, -1)	(0, 1)
B (2, 1)	(2, -1)	(-2, 1)
C (2, 0)	(2, 0)	(-2, 0)
D (3, 0)	(3, 0)	(-3, 0)
E (3, 1)	(3, -1	(-3, 1)
F (5, 1)	(5, -1)	(-5, 1)
G (5, 0)	(5, 0)	(-5, 0)
H (6, 0)	(6, 0)	(-6, 0)
I (6, 1)	(6, -1)	(-6, 1)
J (8, 1)	(8, -1)	(-8, 1)
K (6, 7)	(6, -7)	(-6, 7)
L (2, 7)	(2, -7)	(-2, 7)
Connect L to A		

Take some time to discuss the results with the class. Ask students to describe any patterns they see in what happened to the coordinate pairs when the original points were reflected across the axes. Make sure they understand that all distances have been preserved, and this is why reflection is a transformational isometry. After the students have recorded and analyzed the coordinate points for all three figures, they should have no trouble coming up with the rules for creating the reflected images. For the reflection across the x-axis, the rule is (x, -y) and the rule for reflection across the y-axis is (-x, y). You may also want to have the students compose these two reflections, and see what happens.

Once your students have come up with these two rules for reflection across the x- and y-axis, ask them to repeat the process all over again, only this time using a figure that they have created on their own. This new figure should be fairly simple, perhaps one of their initials in block letters or a small tree (try to discourage simple rectangles or triangles; too easy!).

The final step in this activity involves having the students trade their figure's original coordinate points with a classmate, and then seeing if the classmate can recreate the figure and also do the reflections across the axes.

Lesson 3: Creating Frieze Patterns

This final section of the unit should take three or four days, depending on how deeply you want to explore frieze patterns. I will give my students a basic introduction to the seven frieze patterns, and then give each group one of the patterns to work with. Each group will be responsible for producing a section about five feet long. All groups will use the same figure in their frieze pattern, so that the differences between the patterns will be easier to see.

A *frieze* is a pattern which repeats in one direction. There are seven different frieze patterns possible, each described by transformations that produce it. A *frieze group* is a mathematical concept used to classify frieze patterns according to the structure of their symmetries. All frieze patterns have translational symmetry. They can also have other symmetries and are classified according to those other symmetries. Here is a very simplified and very basic description of the seven patterns and the transformations involved:

Pattern 1: Referred to as the *hop*; this involves translational symmetry only.

Pattern 2: *Jump*: Translation and reflection across the horizontal axis (glide reflection)

Pattern 3: *Sidle*: Reflections across two vertical lines of symmetry. An alternative set of generators is a translation and reflection across a line perpendicular to the direction of translation. Or, a horizontal translation, and reflection across a vertical line. This pattern has evenly spaced vertical axes of symmetry. The spacing between adjacent axes is half of the smallest translation distance.

Pattern 4: *Spinning hop*: Two half turns. An alternative set of generators is translation, and one half turn (around a point on the central axis of the pattern). This pattern has evenly spaced centers of 180 rotations. The spacing between adjacent centers is half of the smallest translation distance.

Pattern 5: *Spinning jump*: Three reflections, one across a horizontal line of symmetry and two across parallel vertical lines of symmetry. Alternative set of generators: horizontal translation, and reflection in the horizontal axis of symmetry, and reflection in a vertical line. This pattern has both vertical lines of reflection, and centers of rotation. The centers of rotation are the points of intersection of the vertical lines with the horizontal axis of symmetry.

Pattern 6: *Step*: This is generated by a single glide reflection. This pattern looks like footsteps. The glide distance of the glide translation is one half of the smallest translation distance.

Pattern 7: *Spinning sidle*: Reflection across a vertical line of symmetry and a half turn. Alternative set of generators: glide reflection and one half turn (around a point on the central axis of the pattern). This is the trickiest pattern. Again there are vertical axes of reflection, spaced at half the smallest translation distance, and also centers of reflection. However, here the centers of reflection do not lie on the reflection axes, but halfway in between.

This simplified illustration below shows the seven frieze patterns. Think of the triangles as footprints, and you can see how the different patterns got their names.

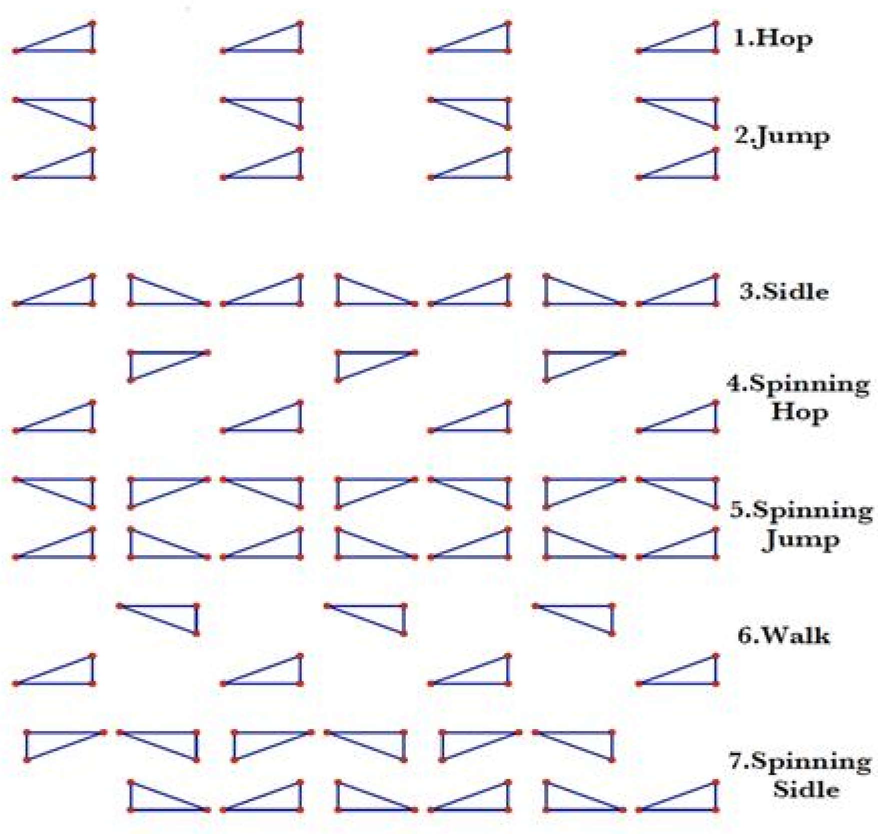

Fig. 6: Frieze Patterns

Again, the *Illuminations* website is very helpful. There is an activity on the site where students can experiment with the seven classes of frieze patterns. By playing around with the different frieze patterns and discovering the transformations that produce each pattern, students can deepen their understanding before sitting down to create their own.

Annotated Bibliography

Barker, William, and Roger Howe. *Continuous Symmetry: From Euclid to Klein*. Providence, RI: American Mathematical Society, 2007. The authors describe this text as intended for a one-semester course on geometry. Definitely beyond my level of understanding, but the information on frieze patterns was helpful.

du Sautoy, Marcus. *Symmetry: A Journey into the Patterns of Nature*. New York: Harper Collins Publishers, 2008. Written in narrative form, this is a very accessible and entertaining journey into the life of a mathematician.

Lappan, Glenda, and James T. Frey, William M. Fitzgerald, Susan N. Friel, and Elizabeth Difanis Phillips. *Kaleidoscopes, Hubcaps, and Mirrors: Symmetry and Transformations*. Boston, MA: Pearson Prentice Hall, 2009. This is the teachers' edition for the CMP2 eighth grade unit. It is an excellent resource for definitions, illustrations, and lesson plans.

Libeskind, Shlomo. *Euclidean and Transformational Geometry*: A Deductive Inquiry. Sudbury, MA: Jones & Bartlett Publishers, Inc., 2008. Written in textbook form with lots of exercises. Easier to read than some other textbooks; Chapter 5 on isometries and size transformations was helpful.

Seymour, Dale, and Jill Britton. *Introduction to Tessellations*. Parsippany, NJ:

Dale Seymour Publications, 1989. Basic explanations of symmetry transformations and excellent illustrations.

Websites

www.adrianbruce.com. Interactive website that gives hundreds of examples of different types of symmetry.

www.illuminations.nctm.org. Excellent resource for teachers and students, with many interactive activities involving transformation symmetries.

www.onlinelearning.com. Directory of free online geometry games.

Math Standards

M7.C.3.1 Locate, plot and/or describe points on a coordinate plane.

M7.D.1.1 Recognize, reproduce, extend and/or describe patterns.

M7.E.4.1 Draw conclusions and/or make predictions based on data displays.