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The Power of the Number Line: Building the Bridge from Mathematics to Symmetry

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Introduction

> A *straight line* is a line which lies evenly with the points on itself.
>
> *From Euclid's Elements*

Someone once said that the need for a strong foundation in mathematics was akin to a tall building having a strong foundation beneath it. You don't necessarily see it, but if it isn't there, you would surely know it, as the building would likely topple over. It's like that for our students. If they are not taught well the foundational concepts of math, they are unlikely to be able to thrive in a world that requires a substantial knowledge of mathematics more than ever. Number operations, including conceptual understanding and fluency, are often considered the core of foundational mathematics. The main goal of this unit is to present two geometric interpretations of multiplication: 1) the area model (which is a refinement of the array model for products of whole number factors), and 2) the number line. In the number line model, multiplication can be thought of as uniform stretching of intervals on the line. This supplements the interpretation of addition (taking a number and adding a fixed number to it) as translation of the number line. Thus, the number line affords coordinated interpretations of both multiplication and addition, so it can show how they interact via the Distributive Rule. Also, it is a model that works equally well for all kinds of numbers: whole numbers, fractions, signed numbers and beyond. My first job will be to determine if my students understand the number line, then I will build instruction from there.

My school district has been the fastest growing district in the state for at least ten years, becoming more affluent as we have grown. Approximately nineteen percent of the students in my district come from low income households, but my elementary school, still called the "town school" by many, has over thirty-five percent of our children living in low income households. What that means in my classroom is that a child living in poverty could be sitting next to a child who lives in a half million dollar home. Our students are a blend of race, culture and ethnicity. Many families from foreign countries have settled in the area, making my school truly diverse.

The math curriculum recently adopted by my school district is TERC *Investigations in Number, Data and Space*. The units emphasize conceptual understanding of a topic, rather than learning rote algorithms or emphasizing number fluency. The program philosophy is such that if mastery is not achieved by all students, those who need more time will be exposed to the concept at a later time, but I find it important that my students have some depth of understanding of the material, so I never feel comfortable forging ahead unless most of my students have some acceptable level of mastery of the material. Some of the units help with this goal by making it possible to make connections between topics, rather than students perceiving each unit as a discrete topic, unrelated to any other.

State math standards for fourth graders in Delaware directly addressing multiplication are:

- Determine factor pairs that make up a given number;
- Show how multiplication and division facts up to 50 are related, using arrays, skip counting, and area models;
- Master multiplication facts and the related division facts up to the 10s tables;
- Model situations that involve the addition, subtraction, multiplication and division of whole numbers using objects, pictures, geometric model, and symbols;
- Represent the idea of a variable as an unknown quantity using a letter or symbol;
- Develop an understanding of the Commutative and Associative Properties of whole number multiplication as a tool to solve problems.

I teach in an inclusion setting where special needs students are educated alongside their regular education peers. That might mean I have some students who possess very little number sense and struggle significantly mathematically and some children who are able to work above grade level on math concepts. The struggling math students in my classroom aren't necessarily special education students, but often some of them are. There are two full time teachers who provide instruction to students and we share the planning and execution of lessons equally. We rely on co-teaching strategies so we are both always engaged with our students. As the special education teacher in the classroom, I do minimal pull out for instruction; instead we rely heavily on flex grouping based on the strengths and needs of *all* of our students.

Rationale

I decided to use the number line to teach multiplication and division because there are four units on multiplication and division at my grade level (4 th grade). These units provide a framework for teaching the concepts of multiplication and division and later multi digit multiplication and long division. Currently I use an area model to introduce multiplication. The benefits of teaching two conceptual models of multiplication are 1) It lets kids know that there are more ways of thinking about multiplication, and 2) The number line model extends readily to fractions and signed numbers. Additionally, the number line applies to measurement of time and distance, and geometrical transformations. I will develop interpretations of addition and subtraction using the number line. In this unit, I plan to do that using primarily two digit numbers. I know my students will have prior knowledge about addition and subtraction with regrouping using two and three digit numbers, but I teach a unit specifically about that later in the year, so I can revisit this concept later on.

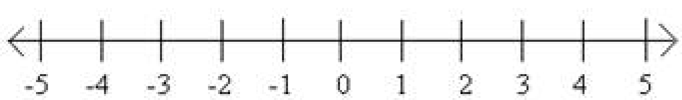
"I'm terrible at math." "I just don't have a math mind." "I wasn't taught how to do that." That is a familiar chorus many teachers would hear in their classroom on any given day. Now imagine a group of elementary school teachers singing this chorus. Guilty of this faulty thinking myself, I have perpetuated the notion that some people are just "born to do math" and some are not. It is time to let go of this kind of thinking! Most everyone is capable of learning how to think mathematically given the proper circumstances; that includes quality instruction from the earliest years. As Hung-Hsi Wu (2009) states so eloquently, "We want students to be exposed, as early as possible, to the idea that beyond the nuts and bolts of mathematics, there are unifying undercurrents that connect disparate pieces."

Research is beginning to show that one of the reasons for the discrepancy between Asian students and American student's performance in math is due to the lack of mathematical knowledge of *teachers*, particularly at the elementary school level. One discovery by researcher Liping Ma (1999) is that in China there is a much different approach to *understanding* mathematics and therefore, in the instruction of mathematics. Even though Chinese teachers are generally less formally educated than their teaching peers in the U.S., they demonstrate a much greater understanding of how math works and why. They also understand the content, the underlying concepts of math in a deep and substantive way.

I use some variations of the number line in my classroom, including line plots and likelihood lines, but after recognizing the importance of the number line in mathematical concepts, I wanted to take a concept I teach and make it better. Knowing that my students will be required to understand symmetry where mathematics is represented geometrically gives me substantial motivation for providing them with this conceptual understanding now. Most importantly, if I understand the "how and the why" of the concepts I teach, then my students will too.

Content Objectives

The Number Line



This is a number line. The number line provides a model for explaining the four basic operations for all rational numbers. It is by nature a spatial object: a set of points one can pass through as a journey along a path (Ryan, 2007). Use of the number line allows the student to "situate themselves bodily and spatially in the mathematics in a powerful way (Lakoff and Nunez in Ryan, 2007). It is actions+gestures+words (Ryan, 2007); a dependence on our visual-spatial skills that permits us to conceptualize mathematically the world around us. The numbers on the number line represent a number of units of length. Specifically, a given number on the number line tells the distance between the point that it labels and the origin. This means that the distance between any two points is given by the difference of the numbers labeling those points. 0 represents the point of origin and there are numbers on either side of it; numbers to the left of 0 are called negative numbers.

Thus, the sign of a number labeling a point tells you the direction to go from the origin. We call the combination of the direction and length the *oriented distance* of the point from the origin Number lines can also be combined so that one is horizontal and it is intersected at 0 by a vertical line, and then you have a grid which is called the Cartesian coordinate system. The number line is considered to go on forever on either direction. If it didn't, it would be called the number line segment!

When I began to plan a unit on symmetry, I would never have thought to focus on the number line and multiplication. I kept asking myself, "What does this have to do with symmetry??" But then I found out! The number line, this seemingly simple tool—although it is really quite sophisticated–is foundational for interpreting math geometrically; *moving one space*—a translation by one unit, which is addition of 1 on the number line is a key part of understanding the symmetry of patterns. We see the beauty of symmetry all around us: in the fabrics we wear, in art (M.C. Escher), architecture (Moorish tiling) and nature (flowers, a beehive). Mesmerized by the beauty of these things, one rarely stops to consider the mathematics that gives rise to it. Doing one thing and then another to a geometric shape is a mathematical operation. Hence, symmetry is infinitely mathematical, therefore to truly understand it; one must first know the mathematics that gives rise to it.

My first job is to determine if my students understand what a number line is and how it works. Even though this particular unit is about multiplication and division, it is worth taking the time to talk to students about how a number line potentially goes on forever, that 0 is called the origin—it has an important place on the line: that all distances are measured from here- and that the units can be broken down into even smaller units. The most important concept about the number line though, is that it shows ***distance***. For example, I will call two students to the front of the classroom and place them some distance apart. Then I will ask another student to come forward and measure off in steps how far apart the two students are. He might say something such as "Five steps." Then I will ask another student to measure off in steps how far apart the two stationary students are. Her answer might be "Three steps." Next we'll use a meter stick to measure the distance so we have a fixed unit. I can ask, "What will happen if so and so starts at this number three spot and takes five steps?" Most of my students are going to say that it is somewhere over to the right (in the manner in which we count left to right). But—I will have that student start at the "3" and actually walk five steps to the left (secretly, of course!) Oh my—this can't be right, they will think! That is when I have their rapt attention and can introduce negative numbers! So, now I must ask, "When we are working on the number line, what else do we need to know besides the distance?" The answer to that is "**direction!**" There is one math standard regarding negative numbers at my grade level, and that has been selected to be "compacted," which essentially means there isn't time to teach it, however knowing about negative numbers is critical in algebra so whenever I have an opportunity talk about negative numbers, I will do so.

Addition and Subtraction

Prior to introducing the concept of multiplication on the number line, I want to take an opportunity to look at addition first. The reasons are twofold: Addition on the number line represents a translation; and, my students need to have a firm grasp of place value. With addition, there is a conceptualization hierarchy:

1. addition is combining heaps of things, which has a length measurement analog;
2. addition is also combining rods end to end which can be automated using the number line;
3. addition can be accomplished automatically using the number line by combining rods in a standardized way: put one rod on the number line with its left end at zero, and the second rod adjoining it on the right, then you can read off the sum from the location of the right hand end of the second rod.

4. if you add a fixed number to a variable number, the answer is always some units to the right of the other number, so that addition can be thought of as giving a translation $x \rightarrow x + a$ of the number line.

If you think about a simple problem such as Fallon had 43 pennies and her mother gave her 68 more, this can be represented in the four ways above.

1. In combining heaps, count 43 pennies and put them in one pile. Then count 68 pennies and put them in another pile. We can combine the two piles, but still need to figure out how many there are altogether. It makes sense to sort them into piles of ten. Students may want to do this with the two groups to begin with, but if not, let them see that it is much easier to work with piles of ten, rather than random groups of coins.
2. Another way I want to show this problem would be to use ten rods and cubes. On the floor, we can lay out four ten rods and three one cubes. Then we want to add the six ten rods and eight one cubes. We have to talk about why we would want to rearrange the rods and cubes in order for all the rods to be together end to end, and then the cubes all together. This kind of representation can be a key to understanding place value. Since we'll have eleven ones cubes we've got to figure out what to do with them. I expect some students will know that once you have ten, you want to put them together to form a tens rod, and at that point you can exchange one rod for the ten ones since they are easier to work with. I think it will be beneficial to take some time to show how this procedure with the ones and tens rods can be extended to three digit addition in order to emphasize the quantity aspect of the base ten decomposition of a number. Even doing four digits is possible, albeit a bit unwieldy! I have access to a large supply of manipulatives, and I can work in the hallway. I think it will be quite amazing for my students to actually see one hundred tens rods lined up in a row in order to show what a one thousand rod would look like! More importantly, this is a superb way to demonstrate the absolute efficiency of our place value system.
3. Using the number line and base ten rods, one begins at 0 and counts 43 units to the right (or four tens and 3 ones), then starting at the number 43, you count 68 (6 tens and 8 ones) units more until you end at the number 111. You want students to be able to see that the first line segment of 43 units is shorter in length than the second segment of 68 units and the second segment of 68 units is shorter than the two line segments put together, and the two line segments end to end are larger than each segment individually. The child can see she is combining and separating because a number line is a template with which to actually see and compare the length of the line segments or distance.
4. Finally, we can use the same numbers to show the problem on a number line. With 0 being the origin or beginning point on the line, we will count to 43 to show that Fallon had 43 pennies to begin. From 43, we will count on 68 more units to represent the pennies her mother gave her. Starting at 43, we will count 1, 2, 3, etc. and stop counting when we say "68." The last digit we land on will be 111. Counting by ones is not the most efficient way to add, but in the beginning I want my students to have complete understanding that each place on the line represents a unit of one. Therefore, we began with $x=43$, then showed that $43 + a$, or $43 + 68= 111$.

Different terms can be used to talk about the movement on the number line. I can call the moves on the line "jumps," or I can say "stepping out" by a certain number. I want to teach my students that there are different ways of describing the movement on the number line, however no matter what one calls them; the terms refer to the movement along the line.

I then want to take some time to solve subtraction problems using the same numbers in previous problems. I would ask, "If Fallon has 111 pennies now, and she started with 43, how many did she add? I need to make certain that my students understand that subtraction is the inverse of addition; it is an undoing of the addition problem. Students are often unfamiliar with the language of word problems and don't immediately recognize this as a subtraction problem. I want to spend enough time using the number line to explain addition and subtraction before moving on to multiplication, but I will teach a unit later specifically on regrouping with three and four digit numbers, so I know I will have an opportunity to revisit all that I have taught here.

Multiplication and Division

In the multiplication unit I currently teach we use the array model, so students can visualize what multiplication represents: combinations of equal groups. The foundation for this unit is an area model for multiplication: equal rows and equal columns make a rectangle. I know it is common to teach multiplication as "repeated addition" but studies show that it actually confuses students later when they start multiplying fractions, so I don't want to teach them this terminology. In the context of stories, children are reintroduced to multiplication because they were first introduced to the operation in third grade. I place a certain number of colored square tiles for the overhead on the glass and ask how these tiles can be arranged into rectangles.

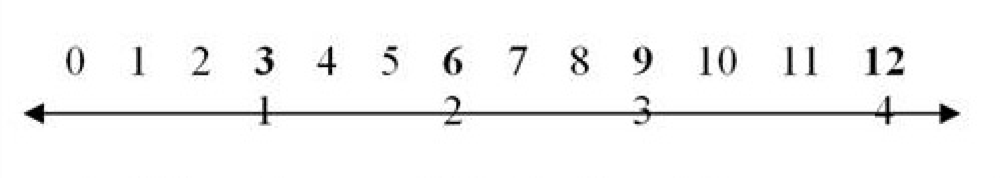
If I place twelve tiles on the overhead, one student may come forward and make three columns with four tiles in each column. Remember, rows are horizontal and columns are vertical. We will talk about what this means: If we pretend that the tiles represent cookies, and the student wanted to share his cookies with his friends, then his picture shows that he and two friends would get four cookies each. So the number sentence representing the pictures looks like this: 3 x 4 = 12: three rows, four tiles in each row, so each child gets four cookies. Some students will forget that they are part of the equation, so that is why he is sharing with two friends, he and the two friends make three. I would invite a different student to come forward and find another way to share her cookies. She may decide to take the twelve tiles and arrange them into two rows with six cookies each. That means she will share her cookies with one friend and the two of them will have six cookies each. I would probably point out that she likes cookies very much! The number sentence representing her picture is 2 x 6 = 12. The next student may decide to place the tiles in six rows of two, 6 x 2.

Once my students have exhausted all possibilities for possible rectangles with twelve tiles, I want to do some comparing of the rectangles side by side. I want to ask, "Is the two by six rectangle the same as the six by two rectangle? Is sharing twelve cookies between two people the same as sharing twelve cookies between six people?" The answer is "no." There is a fundamental difference mathematically between these two representations. I will analyze each pair of seemingly similar rectangles with my students. I want them to recognize that even though they appear to be the same, mathematically they are not. However I will teach them that 2 x 6 = 6 x 2 because we can convert a 2x6 rectangle to a 6x2 rectangle by rotating it 90°. We do end up with the same answer, but we don't arrive at that answer in the same manner. I will give my students many opportunities to explore this concept so that they understand it fully.

There are many problems we work out with different numbers of tiles so students understand that the rows and columns must always be uniform. The numerals that represent the rows and columns are called factors and the total number of tiles is the product. When skip counting with students we call those numbers multiples. A given number of copies of a given number are a multiple of that number.

Next, I will introduce multiplication by means of a measurement model using the number line. Since I will have already introduced the number line and how to use it to do addition and subtraction, I think my students will be ready to grasp the notion of using the number line for multiplication. A number on the number line represents the signed distance from the origin. Multiplication may be interpreted geometrically as dilation of the number line by a constant factor. If we have an equation: y=ax, let a equal the constant. Now replace "a" with a number, 3 for example, and let x represent the positive numbers 1-12. So, y=3x1, 3x2, 3x3, etc. A transformation occurs on the number line based on the scale factor. Again, this model offers another geometric model for multiplication as stretching of the number line. The length of each interval is stretched by the given constant factor.

This is a very different way of thinking about multiplication which is not often taught to elementary students. I am still focused on factors and products, but the process is quite different. We are not building a model, but rather using a model to move along at intervals to find a product, or moving to the left if we know the product and one of the factors. If we have a number line with the units marked in segments by ones, 4 x 3 would look like this:



I will introduce multiplication in a similar manner used in the array lesson: If one wants to jump to 12 on the number line, what are different ways one can get there? You want to have equal groups or spaces between jumps on the number line, and you want to land exactly on 12. Students will see that they can make twelve jumps of one or six jumps of two, three jumps of four and four jumps of three, two jumps of six, and finally one jump to twelve. We will write number sentences to describe these jumps so they will see the relationship between jumps on the line and factors and products. We will continue to explore this method using different numbers such as 15, 16, 18, 20 and 24. These are the same numbers students use to explore arrays in the unit, so we can make a connection to the array model now using paper strips of a number line. Still working with the number 12, we can cut the strip representing the product into equal pieces, and stack these pieces on top of each other to make the array/area model. Following are some of the key differences between the area model and the number line model:

1. The array model clearly shows the two factors, but it does not make as clear that the product is a number.
2. In the area/array model, the product is measured in different units than the factors, but on the number line, both factors and the product have equal status.
3. In the array model, the two factors are more alike than in the number line model. In the array model, one factor is the size of a row, the other is the size of a column; but these can be interchanged by reflecting or rotating the array.
4. In the number line model, one factor is the length of the original interval, and the other is the dilation factor, represented for whole numbers by the number of intervals laid out.

Once my students complete these activities, they will work in teams to model multiples of the numbers 2-12 on the number line. For instance, if a team is working on dilation of the number line by two, they will show one jump from 0 to 2, and writ on an index card "x 1" to represent that movement, that is one jump of two. Index cards will be placed to the right of the number line. I want them to model factors from 0 to 12 using their given scale. They will use colored strips of paper to make their models of multiplication using the number line. What I envision they end up with is a sort of pyramid representing each number of jumps on the line. Using the example above, the team would draw a number line on one colored strip of paper. They would show one jump, from 0 to 2 with an arched arrow. They write the corresponding number sentence and tape that and the strip with the number line on the wall. Next, they use another piece of register tape to show two jumps, from 0 to 2, and 2 to 4. That shows two jumps of two, or "x 2." That is written on a card and together the number line and the index card are taped above the first number line. These should all be lined up over a numbered number line, so that any product can be read off readily. They will continue in this manner until they have modeled all the facts from 1x2 to 12x2. We will develop a strategy for marking off uniform intervals so this presents an opportunity for some problem solving. (For example, the children might decide to use the floor tiles as even intervals so they will know where to place the numerals on their number lines.) The largest product we will have to represent is 144, the product of 12 x 12. While that will require a very long piece of register tape, it certainly will provide a clear visual representation of multiplication on the number line! The whole activity should result in a colorful display that will make the dilation aspect very visible.

At some point during this unit, I will show my students how they can "chunk" a problem so that it is easier to solve. For example, if I give the problem 8x7, a student may not know the answer to that but he can solve 8x2=16 and he also knows that 8x5=40. He then uses those facts to figure the answer to 8x7: 16 + 40=56, the answer to 8x7. This is a nice introduction to the distributive property which looks like this:

8x7=8x (5+2)

=(8x5) + (8x2)

=40 + 16

=56

Since both models for multiplication are visual, it is possible to demonstrate this visually with square tiles and number lines. On the overhead, I will construct 8 rows of 5 tiles, and 8 rows of 2 tiles. We will find the total for each rectangle and add those two numbers. On the number line, we will have our strips with the same numbers: 8 jumps of 5, and 8 jumps of 2, and once more we will add the two together to reach a total of 56. Students later learn a strategy called the "break-apart method" for two digit multiplication, which is the essence of the distributive property, so including this piece of instruction here will provide a foundation for teaching and learning that will follow.

With both the array model and the number line model, I will show how division is the inverse of multiplication. Many of my special needs students struggle with this concept when presented with word problems so I know it will take a lot of practice and reinforcement for them to internalize this idea. Fortunately, the number line makes measurement division quite clear by its very nature. Once they know how to find a product by moving a certain number of jumps on the line, I will show how division is just the undoing of multiplication. A problem such as this: "We are having a barbecue. If we have 16 hot dog buns, and each person will eat two hot dogs, how many people will that feed?" can be solved by beginning at the number 16 on the number line, then skipping back by jumps of two to land at zero. By circling or highlighting our jumps, and then counting, students will clearly see that eight people will be able to eat two hot dogs each.

Strategies

One thing we know about math and how children learn is that is important to have dialogue about how and what they are thinking: dialogue between teacher(s) and student and student and student. The conversations we have with our children helps us analyze our students' understanding and determine if there are misconceptions that we can help to alter. Deborah Ball (2003) says there are important things we need to know about learning and teaching math. There is this notion of doing math in a social context, that is, as a part of other activities. She says to focus on the activity is "the work of learning and doing math." I see two ideas here: solving problems in the context of something else, in making connections to what kids know already so they don't view math as an isolated subject, and then there is the interaction between and among individuals, a sharing of thoughts, ideas and strategies. I will watch and listen to my students to see who does or does not understand the concept of equal groups. I can do this during independent activities and partner and small group activities, all of which will be found in the unit.

Since this unit developed from a seminar on symmetry, I want to spend time examining the multiples for interesting patterns. I will give each student a grid and have a large one on the overhead or Smart Board that we can look at as well. One thing I may do with a special needs student is to have them color each set of facts on the grid sheet, using a different color each time, so it will be easier to find the patterns within the numbers. I would ask, "Does anyone notice any patterns or anything interesting as you look at the arrangement of numbers on the grid?" Some of the responses I expect to hear are "All of the multiples of two are even numbers," or "All of the numbers under ten end with a zero," or perhaps "The odd numbers will go even, odd, even, odd as they go down the row." Some students will be able to recognize that for the multiples of three, when you add the digits in each box it goes 3, 6, 9, 3, 6, 9. Something similar happens with the multiples of nine. When you add the digits in each box, they always add up to nine! Kids love making these kinds of discoveries! They feel like they are unlocking the mysterious secrets of numbers. I love doing this kind of analyzing because students are beginning to realize that there are patterns in numbers, there is order in them and math doesn't have to be awful or scary or confusing.

I want to use our array models to look further. I assign student pairs certain numbers to work with (I would use 5, 7, 8, 9, 12, 15, 18, 20, 24, 36, 48, etc.) They use the square tiles to form arrays for their assigned number(s) and then they copy the arrays onto one inch grid paper. They have a sheet of large construction paper on which to glue all the arrays they can find for their number(s). Next will be the important conversation about what they have made. I will ask if anyone has a poster with only one array, only one way to make that group. Of course, whoever has 5 or 7 or 17 or any other prime numbers will say that they do, so now we can label those as *prime numbers*. I also want to know if anyone has an array that is a perfect square—it has the same number of squares or tiles going across and going up or down. Again, some students will recognize that they have arrays that fit the description (4, 9, 16, 25, etc.), and that is when we put a label on them: the square numbers. I want them also to recognize the number sentences they have written on their arrays: 2 x 2, 3 x 3, 4 x 4, etc. I said earlier that symmetry has math, but I guess you could say that math has symmetry too!

I consider multiple learning styles in my classroom, so I will incorporate other materials that will allow students entry points into the nature of the subject. The number lines and square tiles already provide us with items to manipulate for the visual and tactile-kinesthetic learners (which most ten year olds are at this stage). I have access to the website "Brainpop" and our librarian recently purchased some new instructional DVD's explaining multiplication. Because I'm a big literacy person, I use children's literature whenever I can.

Anything to facilitate understanding! One of the really neat books about multiplication and division is called *Each Orange Had Eight Slices*. In this book, it begins with "On my way to Grandma's I saw two fat cows. Each cow had two calves. Each calf had four skinny legs…" and so on and then the author asks, "How many cows? How many calves? How many cows' legs in all?" Another book that emphasizes strategies for figuring out the times table is *The Best of Times* by Greg Tang. He shows children that multiples of four and eight can be figured out using multiples of two and that you can figure out the nines by rounding up to ten and subtracting. One more appropriate children's book is *One Hundred Hungry Ants*. The author writes a tale about one hundred ants and how they split into different groups on their journey, such as two groups of fifty, four groups of twenty-five, etc. All of these books reinforce equal groups, sets of numbers, breaking apart larger numbers into equal groups, and thinking mathematically and the authors do it in an engaging way.

One of the most important things I do instructionally in my classroom is to get them up and moving as much as possible. So any time I can think of a way to present information so my students can use their whole bodies, I do it. I want to revisit what I said earlier about the social nature of mathematical discourse. I frequently ask my students to work in partner pairs or small groups to engage in problem solving. One way they have to communicate their reasoning or discoveries is to make a little summary poster with words and pictures illustrating their understanding of what they've learned.

Activities

The reader will see that there are core activities presented here, but keep in mind that fourth graders need multiple opportunities to explore and practice a concept in order to know, to own it, so to speak. Therefore, I haven't included a lesson for every day of the unit, but just know you should provide many learning opportunities for your students as I will do.

Activity One: Introducing….the Number Line!

I'll begin this lesson by showing a number line on the interactive whiteboard. I will ask, "What is this object? What can you do with this thing? Where would you use it? Does it give you any information? Does it always look the same?" After our discussion, I will take my students outside to form a human number line. I will point out that each one of them represents the dot or the tick marks on the number line, and the space between each student represents the space between units on the number line. I'll have to do my best to space them evenly apart, but this is an opportunity to remind them that on a number line, the spaces between the markers are exactly the same length. Once we have our number line assembled, we'll practice some different kinds of problems. We'll begin with simply counting out loud one by one and jumping up as we say our number. On this first day, I'll keep it simple—easy addition and subtraction problems such as 4+8=12 and 24-12=12. I can have one student be the "problem solver." When I present a problem, he/she has to touch and count each person on the line. The person being counted raises their arms over their heads so we know they are part of the numbers being counted. I will also have a "recorder" who records the problem numerically on a student white board. I want to use problems that will give everyone a chance to participate, and the jobs of "problem solver" and "recorder" will be rotated. Once we've done several problems, we'll go back inside to complete the remainder of the interactive whiteboard presentation. My students will learn that the dots or ticks can represent counting up by ones, or they can represent multiples of two or ten or any number really. Other features in the presentation will include how the line theoretically goes on forever in either direction, there are negative numbers to the left of 0, and an overview of the kinds of problems one can solve using a number line (number operations, fractions, calculating intervals of time, etc.) Then on the next day we'll review our presentation and practice addition and subtraction problems on our human number line.

Activity Two: Practice with the Number Line

On this day I want my students to get used to using a number line on paper, so we will solve addition and subtraction problems with different scaling. Just to make it a little more fun, I'll let them write the number sentences that describe the movements of their number lines on their student white boards. So, they'll show the moves on a paper number line, but write the problems with white boards and markers. First, we'll use number lines marked in units of one and show how to move back and forth to solve easy problems. I will give them a variety of problems requiring the inverse operation, changing the order of the numbers. I want my fourth graders to see that you can solve problems like this: 4+4= ? and 4+ ? =8. Once they've worked on some single digit problems, I will ask if we can change the numbers (called scaling) on our number line so that we can solve two digit problems, and perhaps three digit as well. Hopefully someone will volunteer that we can mark our points on the lines by tens. Then we can work out the problems from there. It would be appropriate for my students to make up and solve their own problems, either individually, in pairs or small groups. This is a time in my classroom where I will see who is really struggling and pull them into a small group to work in a supported environment with fewer problems and/or variables. Again, this type of work can be done on successive days (but not for the entire math block) or I might send homework for reinforcement.

Activity Three: Introduction to Multiplication

I will use my recommended curriculum unit guide to introduce multiplication. This is where we will learn about arrays, also known as the area model of multiplication. Once I've taught several lessons in that unit and my students have a good understanding of the concept of multiplication, I will introduce multiplication on the number line. We'll form our human number line again, only this time we'll be skip counting to demonstrate dilation of the number line by a given numeral. For the easier facts, each student can represent one place on the line, but once we get to facts with higher values, I will give them construction paper with large numerals printed on them so we're not just doing the 1's, 2's and 3's of the multiplication table, we can model the 8's, 9's or 10's. Once we've played this for a bit, I will show my students on the interactive white board how we can use a number line to skip count and solve multiplication problems. As I said earlier, this linear model is quite different than what they are used to, so we want to have a conversation about how and why this is so.

The next part of this lesson is to give student teams strips of colored paper so they can make models of multiplication on the number line showing how each multiple results in a successively longer line. Prior to their working with their teammates, I will model with my own number lines made from colored strips of paper. These need to be prepared prior to this lesson. Using the number two in a dilation of the line, I will model how to show a jump of one, equaling 1 x 2, two jumps of 2, equaling 2 x 2, etc. I will assign one set of multiples to each team. The students may decide to use the tiles on the floor as landmarks for making even marks on their own number lines. We will post these number line sets in the classroom so they can compare them to their array models. On successive days, I'll ask my students to make up story problems or algebraic expressions (actually, a mix of types would be good) based on their number line and then students will solve each others' problems. By algebra, I'm simply referring to using a letter to represent the unknown. This could be a differentiation strategy used with students who are quite adept at math, although my state has a standard addressing the use of letters to represent unknowns, to which all of my fourth graders should be exposed.

Another activity I want to try is to make a big number line on a blacktop area outside using colored chalk. We will hop from spot to spot counting by the number called out. I know many of my students will not know their facts yet, as we will still be learning them in our number fluency program, but this is still a fun way to practice and use the number line at the same time. I could make large number cards with the multiples and hold them up as a mode of visual reinforcement. A variation on this would be to have my fourth graders hop on one foot from spot to spot, spin and jump, etc.

Activity Four: The Rules of Math Nobody Teaches

You might think I'm crazy for wanting to teach these rules and my students might think I have three heads, but I'm going to try this. It's not that I don't teach these rules now, but I haven't done a great job explaining and teaching them to mastery. I plan to directly teach the commutative property, the inverse rule, the identity rule and the distributive property. The commutative property works for addition and multiplication. It simply says that the order of the numbers in the equation doesn't matter; you will achieve the same result. So, 4+6=10 and 6+4=10, just as 4x6=24 and 6x4=24. One way to demonstrate how to remember the term commutative is to ask, "Who knows what a commuter is? Have you ever heard someone say, "I have a long commute to work?" Associate the word "commute" with movement, going somewhere. Then, have a couple of volunteers come to the front to demonstrate with number cards how they can move, or commute, and it doesn't change the answer. Then play the card game below.

As for the inverse rule, one operation is sort of the opposite of the other. Subtraction is the inverse of addition: instead of putting numbers together, you are taking them apart. For example, 7+6=13, therefore 13-6=7 and 13-7=6. It's the same for division, as division is the inverse of multiplication: 4x8=32 so, 32÷8=4 and 32÷4=8.

The way I plan to teach this is to have the students work in groups of five. Each group will have a set of cards with which to work. The cards will have numerals and the symbols for addition, subtraction, multiplication and division and they will have yarn attached so the kids can wear the numbers. Each child gets one card, so they either get a number or a symbol. One of the symbol cards will have the addition sign on one side and the subtraction card on the other side, and the other operation card will have the multiplication sign on one side and the division sign on the other. They will form number sentences (correctly) and then they will change the order in which they are standing and do a quick flip of the symbol card to make a number sentence that is the inverse of the original one. Once they've completed their set of cards, they trade with the group to the right, so every group of students gets multiple opportunities to practice. This can also be done on successive days for practice.

The identity rule states that for addition there is only one number you can add to a given number and still have the original number, and this is 0. For multiplication, you can multiply any number by 1 and still retain the original number, so for multiplication the identity is 1. Children know the identity rule, but perhaps implicitly—they haven't ever had to think about why this is so nor are they asked to explain it. For an activity, I will give one third of my students a large copy of the letter **i**. Another third will have large cards with the numeral 1 on one side and 0 on the other. The remaining third students will have strips with one number sentence on each side, one showing a partial addition sentence and on the other side, a multiplication sentence (like this: 4 x ? = 4, or 6 + ? = 6). They will then have to form groups of three to demonstrate the identity rule for either addition or multiplication. So a group of students should correctly form:

i = 6 + 0 = 6

Finally, I want to work on solving problems by what I like to call "chunking"; it is what I referred to earlier as the "break apart method," and what students *need* to know is the Distributive Rule of arithmetic. This works well for the multiplication tables with higher values, and the ones with which they struggle more on memorizing. Use problems such as 7 x 9. Following is one way to demonstrate the distributive property:

7 x 9 = 7 x (4 + 5)

= (7 x 4) + (7 x 5)

= 28 + 35

= 50 + 13

= 63

As a follow up to teaching each rule, each student will make a mini-book with the terms and the explanation and a number sentence(s) illustrating the meaning of the rule.

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Appendix: Implementing District Standards -

Determine factor pairs that make up a given number

Students will learn that two factors multiplied compose a number with a higher value.

Explore negative numbers by extending the number line using familiar applications (elevator, temperature, sea level, debt)

The focus of this unit is not on negative numbers, however there are opportunities to present and discuss negative numbers on the number line.

Add and subtract larger numbers (e.g., three digits + two digits) and explain how the operation works

Students will learn several ways to interpret addition.

Demonstrate mastery of mental math strategies for multiplying numbers (e.g., 25 x 8)

This unit will teach students how to use the distributive property on single digit multiplication problems.

Show how multiplication and division facts up to 50 are related, using arrays, skip counting, and area models

This is the heart of the unit, but will also include the number line as a geometric interpretation of multiplication.

Master multiplication facts and the related division facts up to the 10s tables

Exposure to multiplication and division facts with both array and number line activities will help facilitate retention of facts.

Model situations that involve the addition, subtraction, multiplication and division of whole numbers using objects, pictures, geometric model, and symbols

Again, this standard goes to the heart of this unit.

Represent the idea of a variable as an unknown quantity using a letter or symbol

In this unit, students will learn how to use letters to represent unknown quantities in multiplication and division problems.

Develop an understanding of the Commutative and Associative Properties of whole number multiplication as a tool to solve problems

Several rules of arithmetic are directly taught in this unit including commutative, distributive, identity and inverse properties and/or rules.
