Curriculum Units by Fellows of the National Initiative
2010 Volume IV: The Mathematics of Wallpaper

Patterns, a Different Point of View

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Using music and art in combination with the application of math concepts entices me to create a unit that would engage all students in my math classes.

In this curriculum Unit, I will include ideas to combine music, dancing and math. The concepts of translations, reflections, rotations, and permutations are mentioned in conjunction with terms used in music. The concepts of relations and functions are as well mentioned, in order to be able to explain the relationship between inverse functions and their geometric representation using reflection.

Why am I writing this unit?

While I was in elementary school, I was a good student, but during Middle School and High School, my desire for learning almost disappeared. I understand that only minimal resources were available, but I don't understand why some of my teachers were not looking for training and striving to improve. At the same time I don't understand why the authorities in charge of the educational system were not doing anything to improve the situation. I don't want to sound negative about teachers in general because I had as well good educators who have given me the desire to be like them. However, there is a problem and students know that.

It is necessary to start with what has motivated me to become a teacher, and then the motivation to write this unit curriculum will follow automatically.

When I finished High School, I knew that I was unprepared for college, with low self–esteem and with a very bad math background. I had to enroll in a private institution to review what should have been taught in High School. The good result of my review work was that I really started learning. Finally, there were some concepts that started "clicking." Math started making sense when I started learning Physics, which became my favorite subject. Being able to apply the math I learned to the real world provided me the opportunity that my brain was looking for. After a year of preparation, I was accepted in the School of Engineering, where I learned more math and Physics concepts. To cover my personal expenses, and while attending college, I worked in the same private institution where I had studied. It was very interesting to be able to interact with students that were struggling with the same topics that had challenged me. At first, I was just proctoring, but after a year, I started subbing when teachers were late or absent. I found out that I really enjoyed sharing my knowledge. It became so natural to explain things in my own words. It was so nice to see the big smile on those students when they understood what I was teaching.

After completing my studies in Engineering I started looking for a job as an Engineer, but I was too young. Being only 22 years old was not looking good for the interviews. Without being able to find a job as an Engineer, I decided to stay in my current job and make a career out of teaching. After all, I was really already enjoying teaching Math and Physics.

Now, going from disliking math to teaching it, it has become the key to my style of teaching. I understand when students don't like math. I know how difficult is when the teachers in charge are not really paying attention to those little signs that students make, such as distractions and behavior problems, lack of participation, staring to the wall while lecture is taking place and writing anything in their papers thinking that their teacher will not read what they wrote. A lot of my students were surprised when I gave their papers back to correct their mistakes. I know how it is, because I have been there. That's why I attempt to reach all of those students who feel that math is not for them. Creating a curriculum unit that will address different points of view, I hope will empower all my students.

I will divide this unit curriculum in two parts that will become different lessons when the lesson plans are created. The first part involves concepts of inverse functions, and patterns. The second part involves the concepts of translations, reflections and rotations. Both parts are joined by the fact that the concept of inverse function, interpreted geometrically, involves a reflection.

Relations

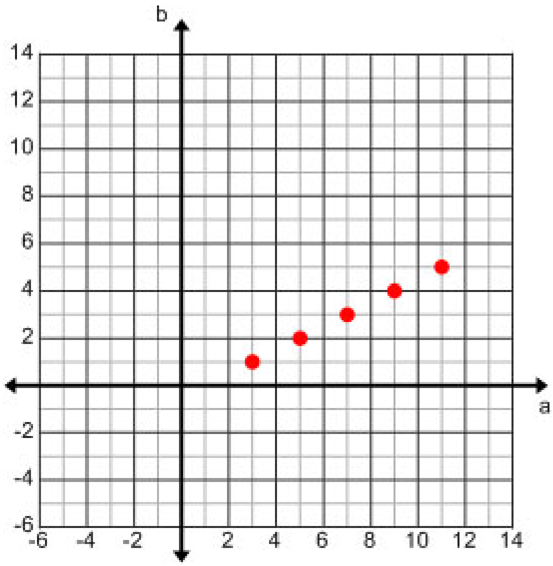
A relation is a rule that relates one kind of thing to another. The first kind of thing is often called an *input*, and the collection of all possible inputs is the *domain* of the relation. The second kind of thing is called an *output*, and the all the possible outputs belong to a set called the *range* of the relation. If the output *b* is related to the input *a* by the relation *R*, we write *aRb*. In principle, to describe a relation, we could just list each pair *(a, b)* such that *aRb*. The collection of all pairs *(a, b)* such that *aRb* is called the *graph* of *R*.

Most relations that we study in high school are numerical: the domain and the range are sets of numbers, and the relation is usually described by a formula. When the domain and range of R consist of real numbers, we can represent the graph of *R* as a collection of points in the *(x, y)*–plane. This gives us a geometrical picture of *R*. We also call this picture the graph of *R*.

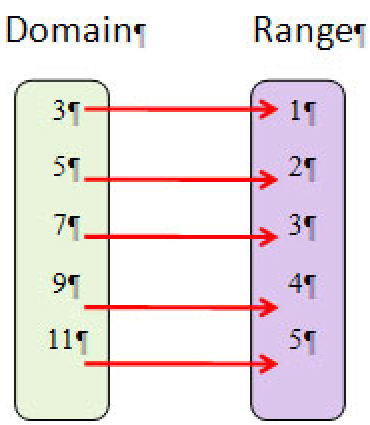
For example, suppose the domain A = {3, 5, 7, 9, 11}, the range B = {1, 2, 3, 4, 5} and the relation R is *aRb* if *b* = (a 1)/2. Then the graph G_R of R consists of the pairs

G_R= {(3, 1), (5, 2), (7, 3), (9, 4), (11, 5)}

Now if we plot these pairs on a coordinate grid, we get the following picture for G_R.



A relation can also be described using what is called a "Mapping Diagram." Here is a mapping diagram for the relation ***R***defined above.



Mapping diagrams are very useful when a student is trying to understand the concept of functions in comparison with relations. While in relations, input values could be repeated for the values of the outputs, in functions, this repetition is not admitted.

Functions

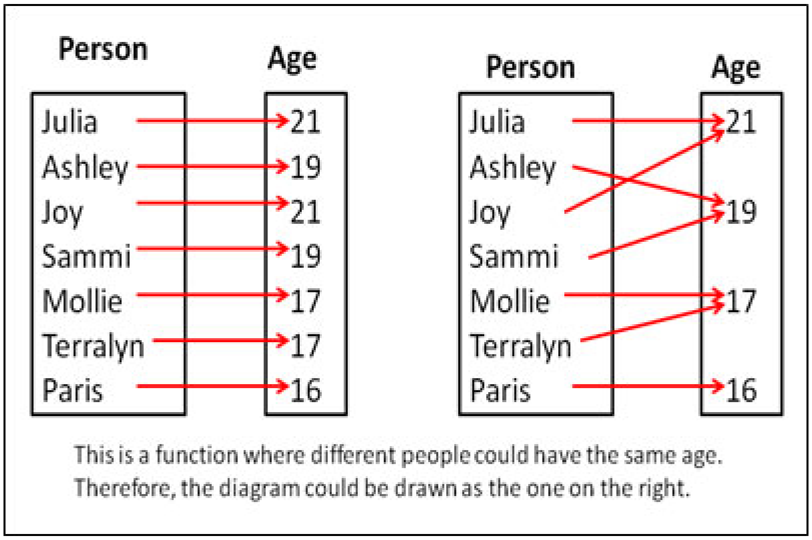
A function is a special kind of relation. A relation is called a function if it has only one output for any given input. The relation R given as an example above is a function. A relation that is not a function is the square root relation:

Q: aQb if $b^2 = a$.

The domain A of *Q* is the non–negative numbers, and the range B of *Q* is all real numbers. Since every non–negative number has two square roots, one positive and one negative, *Q* is not a function. For example, 1*Q*1 and 1*Q*(–1), so the input 1 has two outputs, 1 and –1.

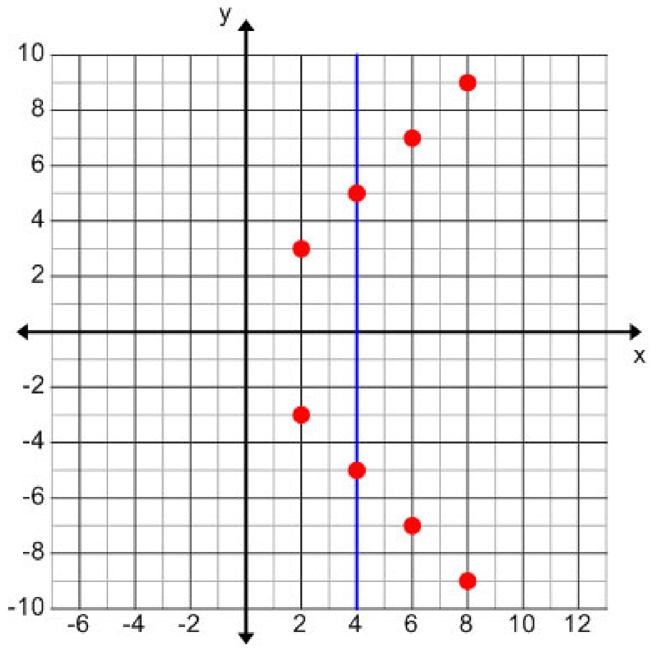
A simple idea of function is that a function is a set of ordered pairs where one quantity depends on another one to exist. This idea creates the function–notation ***f(x)***. For example, in ***f(x)*** = ***3x 1***, the expression **3x 1** depends on x to exist. As a consequence, the concept of independent variable for x and dependent variable for y is used. In this case, the function is written as **y = 3x 1**. In some cases, the independent variable **x** is also called **argument**, while the resultant y value is called **image**.

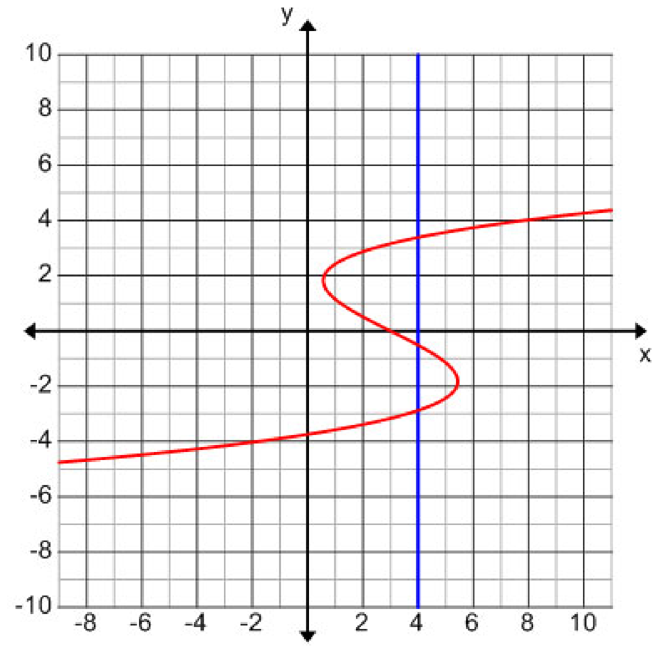
Another example of a function is the relation between people and their ages. At a given time, one person can only have one age; while several people could have the same age. This relation about ages, it is indeed a function. The following mapping diagrams illustrate this idea.



The Vertical Line Test

The vertical line test is a simple way of checking if a numerical relation is or is not a function by looking at its graph. If ***R*** is a numerical relation, then for a given number **x** in the domain of ***R***, the values y such that ***xRy*** are recorded in the graph of ***R*** as points (x,y). For the given ***x***, all the related values of y give points (x, y) that lie on the vertical line through ***(x, 0)***. If ***R*** is a function, there should be only one such y, and therefore the graph of ***R*** will intersect each vertical line in at most one point. In other words, on a graph of a relation, if a vertical line passes through more than one point, the relation is not a function.





The two relations whose graphs are shown here in red color do not represent functions because the blue vertical line is touching the graph in more than one point.

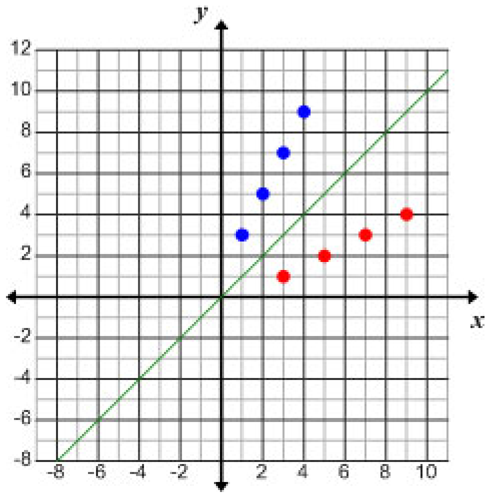
Inverse Relations and Inverse Functions

In general, if a relation ***R*** pairs elements ***a*** from a domain with elements ***b*** from a range, the **inverse relation**$\boldsymbol{R^{-1}}$ pairs elements ***b***with elements ***a***. That is ***b***$\boldsymbol{R^{-1}}$a exactly when ***aRb***. That is, if ***(a, b)*** is an ordered pair of the relation ***R***, then ***(b, a)*** is an ordered pair of $\boldsymbol{R^{-1}}$. This means that the graph of $\boldsymbol{R^{-1}}$is gotten from the graph of ***R*** simply by interchanging the x and y coordinates. The mapping

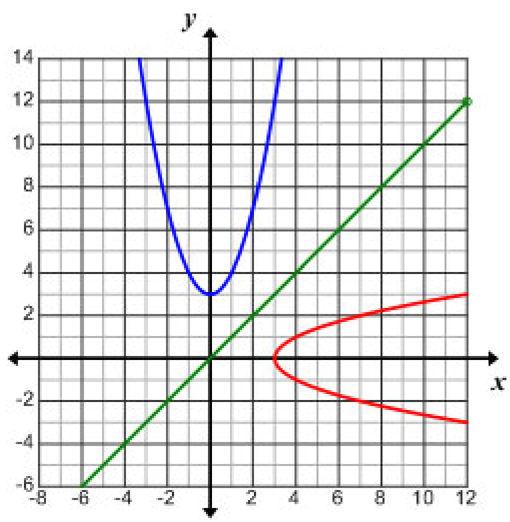
S: (x, y)→ (y, x)

of the plane converts the graph of any relation ***R***to the graph of the inverse relation $\boldsymbol{R^{-1}}$.

It is easy to see that the mapping ***S*** leaves points (x, x) on the diagonal line fixed. In fact, as we will show later, geometrically, ***S*** is a reflection across the diagonal line. The pictures below illustrate the relation between the graph of a relation (in blue) and the graph of the inverse relation (in red). In both examples, the original relation is a function. In the first example, the inverse relation (in red color) is also a function, but in the second example, it is not, since the graph clearly does not pass the vertical line test.



The blue points above the line y = x represent a relation; while the red points below the line y = x represent the inverse of the given relation.



The blue graph above the line y =x represents the relation $y = x^2 + 3$; while the graph below the line y = x represents the inverse, $x = y^2 + 3$

Restricting the Domain to get Inverse Functions

As the second example above shows, the inverse relation of a function is not always a function. However, sometimes if we restrict the domain of a function to a smaller set, the inverse of the restricted function is again a function. For instance, in the second example above, if we restrict the function $x \rightarrow x^2 + 3$ to the domain x > 0, then the inverse relation will consist only of the top branch of the graph of the full inverse relation. This branch of the graph does satisfy the vertical line test, so it defines a function. This idea is used for defining important functions, including the inverse trigonometric functions.

In Algebra–II and in Trigonometry, we study the inverse of trigonometric functions. To be able to define them, the domain of the original trigonometric function has to be limited to a small part of the original domain. We can see on the attached graph that when the function ***y = sin x***is "inverted," the new graph becomes the reflection of ***sin x***over the line ***y = x***. This reflection occurs because all the elements ***(a, b)*** of the function are being switched to ***(b, a)***. Therefore when graphing the inverse function, the *x*–axis becomes the *y*–axis and the *y*–axis becomes the *x*–axis. Therefore, to define the inverse function **y = arc sin x**, we *limit the domain* of ***sin*** to the interval from **π**/2 to **π**/2. The inverse of this restricted function is again a function. Similarly, to define arccosine, we limit the domain of cosine to the interval from **0** to **π.**

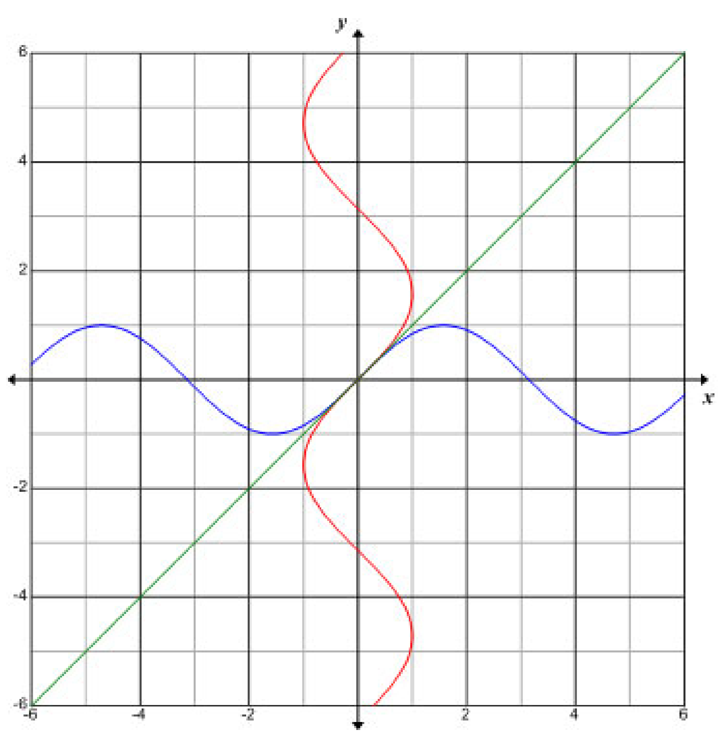
It is important to mention that students need to be taught the different ways that books, teachers and calculators use for symbols for the inverse of trigonometric functions.

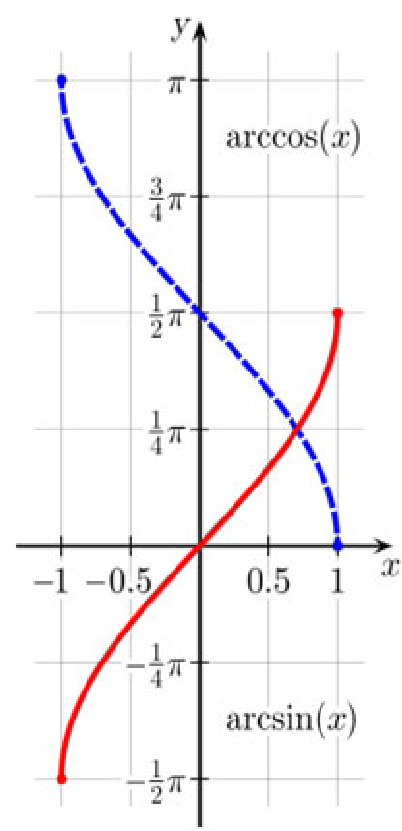
Inverse Notation	**Caution!**
arc sin *x* = sin-1 *x*	As stated at the left, the exponent of -1
arc cos *x* = cos-1 *x*	denotes "inverse".
arc tan *x* = tan-1 *x*	It does not mean 1/sin *x*, as in *x*-1 = 1/*x*.

When the trigonometric function **y = sin x** is inverted, interchanging x for y; the result is not a function.

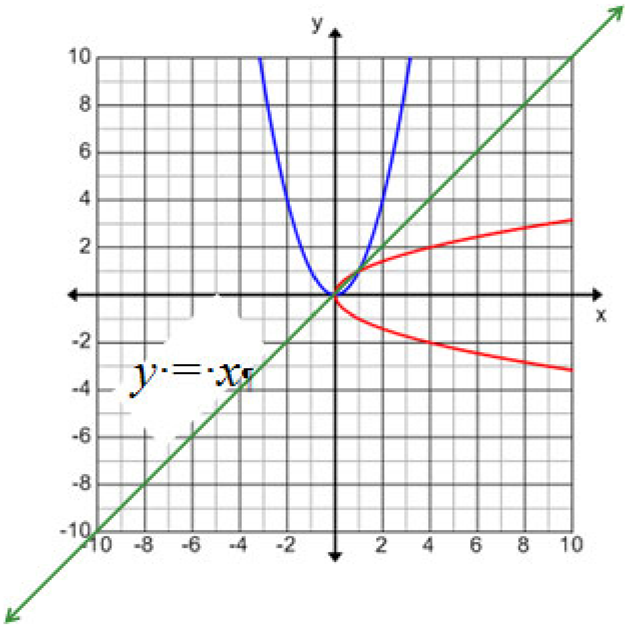
The first graph illustrates the reflection of the graph of the function **y = sin x** in blue color over the green line y = x. The red curve is the graph of the relation **y = arc sin x**; obviously, this red graph badly fails the vertical line test, so the result of this reflection does not produce a function. Therefore, to continue with the study of functions, the domain of **sin x** must be limited. We can see those limitations on the graph next to it.

The second graph illustrates the limitation of the domain of **y = arc sin x** in red color and **y = arc cos x** in blue color. One can see that for **y = arc sin x**, the domain has been restricted to [–π/2 , π/2]





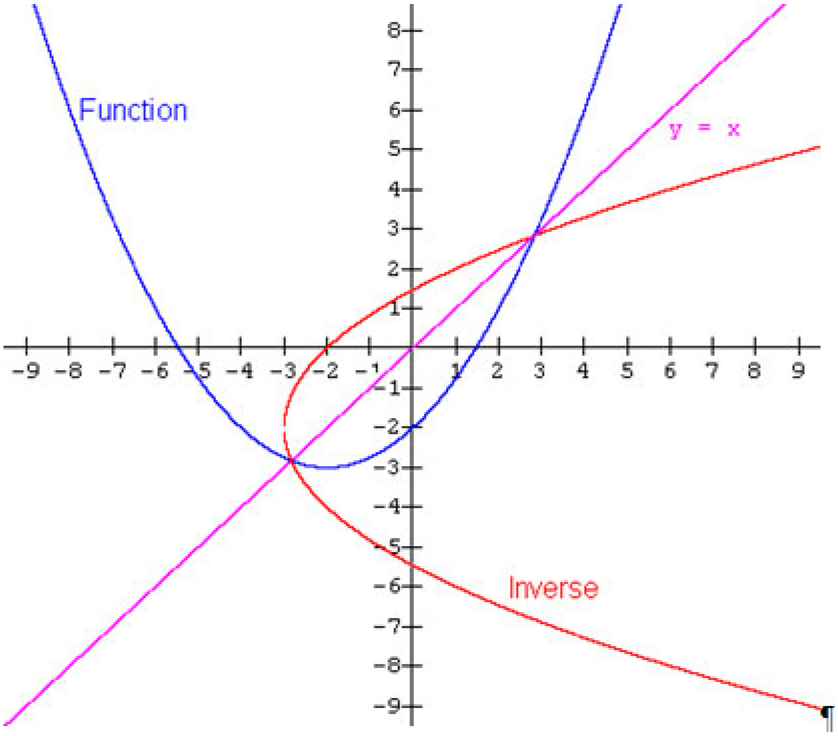
Similarly, when certain functions such as the ones who come from the parent function $\mathbf{y = x^2}$ are inverted; the result is not any longer a function. If one follows the algebraic instructions to inter–change "x" for "y" the new algebraic expression would be $\mathbf{x = y^2}$. Then, when solving for y, one would have $y = \pm\sqrt{x}$. However, only the positive or the negative portion of the expression represents a function. In the figure below, the function in blue color is $f(x) = x^2$ and the ***"inverted"*** function is shown in red color.



Another example is shown below. The function in blue color is $f(x) = 1/4(x + 2)^2$ 3 and in red color, the result of inter–changing the x for y is shown as the inverse of f(x). However, in both examples the resultant is not a function; therefore, the graphs have to be limited to only the top or bottom portions exclusively.

In the first example, to preserve the concept of function the domain of $\mathbf{y = x^2}$ can be limited to $\mathbf{x \geq 0}$; therefore, the inverse $\mathbf{y = \sqrt{x}}$ would be a function with domain $\mathbf{x \geq 0}$ and range $\mathbf{y \geq 0}$. In other words, it would be the upper side of the red curve. On the other hand, if the domain of $\mathbf{y = x^2}$ is $\mathbf{x \leq 0}$, the inverse would have the domain $\mathbf{x \geq 0}$ and the range would be $\mathbf{y \leq 0}$.

The graph of an inverse function can help to create patterns obtained by reflecting a given graph over the line y = x and therefore, being able to use the new graph in an artistic design. Designs are usually used to create wallpaper, shirts and other patterns applied to fabrics. This idea can be developed as well with the lessons involving patterns and sequences.



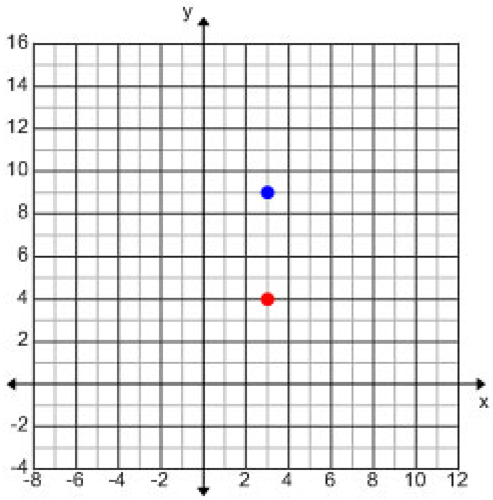
Applying Translations and Reflections

For a line l in the plane, the *reflection in* ***l*** is the transformation of the plane that takes a given point ***p*** to the point ***q*** opposite to p across ***l***. Both ***p*** and ***q*** should be the same distance from ***l***, and the line joining ***p*** and ***q*** is perpendicular to ***l***.

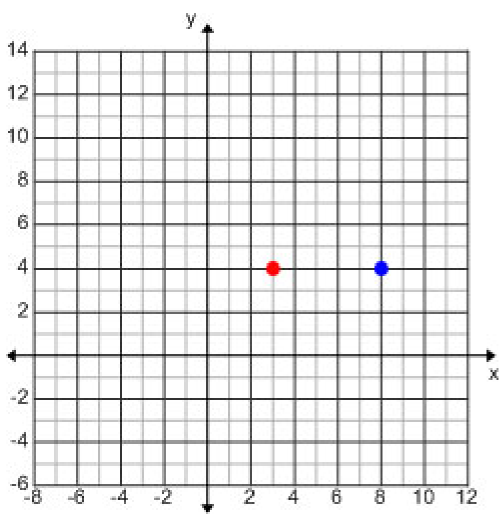
For example, the coordinate reversal (x, y) → (y, x) is a reflection in the line y = x. It is easy to check that the line segment from a point (x_o, y_o) to the reflected point (y_o, x_o) is parallel to the line y = x, which is perpendicular to the line y = x. Also, the midpoint of the line segment from (x_o, y_o) to (y_o, x_o) is the point $[(x_o + y_o)/2, (x_o + y_o)/2]$, which lies on the line y = x. Therefore, (x_o, y_o) and (y_o, x_o) are indeed the reflections of each other in the line y = x.

I will provide more examples of reflections using music and melody. I will also provide example of translations.

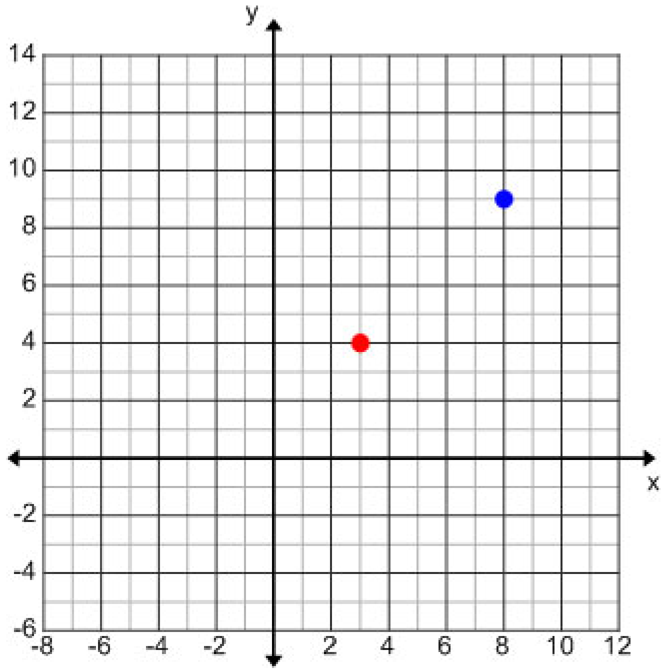
The concept of **translation** relates to a change in the position of a point or object. For this explanation I will refer simply to a point. The translation may be vertical, horizontal or a combination of both. For example, if point ***A*** is analytically located in the coordinate plane as ***A***(3, 4) in red color, and we want to translate this point 5 units up, the only coordinate affected will be the vertical coordinate **y = 4** increasing by 5 units. The translated point will be (3, 4 + 5) = (3, 9) in blue color. This is called vertical translation.



If the same point ***A***(3, 4) in red color is translated 5 units to the right, the movement is on the x axis; therefore, the affected coordinate will be **x = 3**, increasing 5 units. The translated point would be (3 + 5, 4) = (8, 4) in blue color. This is a horizontal translation.



Now, if the same point ***A***(3, 4) in red color is translated 5 units up and 5 units to the right, both coordinates will be affected, obtaining a new point (3 + 5, 4 + 5) = (8, 9) in blue color. This is a combination of horizontal and vertical translation, which can be called "oblique," although some textbooks printed after 2008 call them "diagonal."



Music and Mathematics

The reason for this second part is to embrace those students who love music but dislike math. Two thirds of the students on each of my classes are musicians or at least play an instrument. I am able to create music from scratch. I am a musician as well as a math teacher.

If I am able to create a lesson where my students can relate to music and learn the mathematical concepts, their engagement in math lessons will happen.

As background, I have to mention that I have a band and that I compose songs. Therefore, there are musicians available to help me out to complete the project. At the same time, a lot of my students like dancing which is a topic that I will address on the last part of this unit. They read and write music too. My plan is to compose an original song, or use one that I have already created, and write down the notes in a musical staff, also called a pentagram. The staff has 5 lines and 4 spaces between the lines. Sometimes is necessary to add lines at the top or at the bottom of the *pentagram* to include high or low sounds. The pentagram (staff) is the frame (as x–y axes) that allows the writing of symbols called notes that represent sounds. The set of individual notes create a melody. The melody might be played by several instruments and each sound for each instrument must be represented as a symbol in the pentagram.

After the notes are written, I will reflect them over the third line of the pentagram, obtaining a new set of notes. The third line is chosen because is in the center of the pentagram, similar to an "x–axis." The question that I will pose to my students is: Is the "new" melody pleasant to the ears or not? Is it the same melody? Is the new song a translation of sounds in a period of time, in other words, a delay? (This means that it does not start at the same time than the original song). We would need to repeat the experiment a few times to come up with a conclusion.

A reflection in a pentagram

The idea of writing a melody of an original song is just to make it a challenge. However, students could choose any contemporary song as long as they are able to write and read the melody and its reflection in a pentagram. For the purpose of this unit, a reflection in a pentagram will be defined as the reflection of the notes over the third line, as mentioned on the previous paragraph.

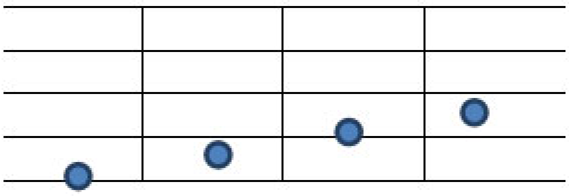
The concept of translation can be as well introduced at this level to combine the idea of translating functions and **"translating"** melodies.

The process of translating melodies, notes, and in general, songs, is called in music **"transposition."** This musical concept of transposition is geometrically seen as a ***vertical translation***. This means that if a song or melody is composed to be performed in the C note, but a new singer needs the same melody in the A note, the entire melody /song must be transposed to the respective note. However, the structure of the melody must be kept intact the same way as a function would be when it is being translated. Thanks to the frame (staff) used to write music, we can guarantee that this structure will be kept intact. For example if Frank Sinatra would sing a song and later on, Whitney Houston would want to sing the same song, it is obvious that the melody will be kept, but the keys (notes) will be adjusted to the proper voice. Key is the name given to the note or set of notes that would make up a melody or a song.

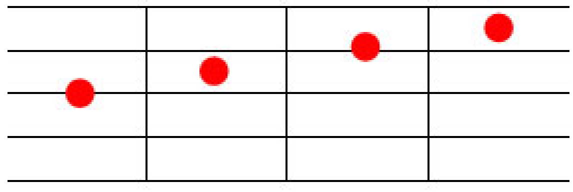
The translation of sounds in a period of time, also referred in music as **"time delay,"** is another type of translation. In this case, the notes written on a pentagram are displaced horizontally to the right or to the left. This time delay is a horizontal translation.

With time delay, the melody / song will start before or after the original song. For a certain style, the song with original notes and the new one with the time delay may continue, creating a special effect as a duet.

TRANSPOSITION

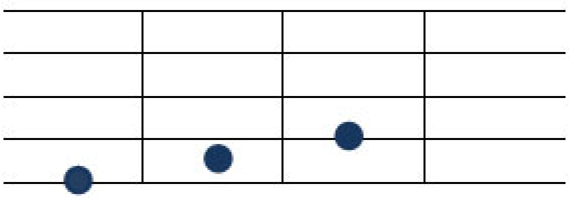


This represents the song with the original notes

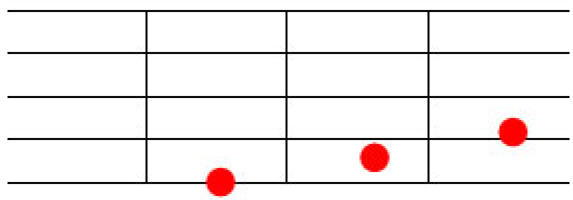


This is the ***"transposition"*** of the song, similar to a vertical translation.

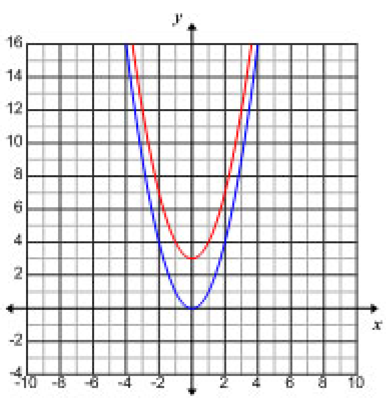
TIME DELAY



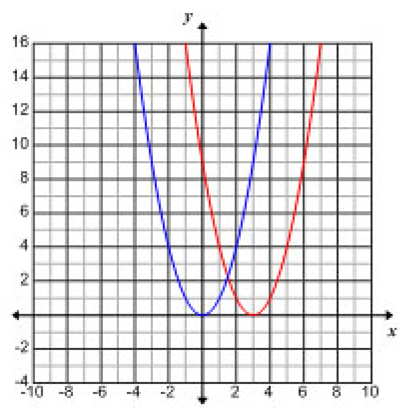
This represents the song with the original notes.



This is the ***"time delayed"*** of the song, similar to the horizontal translation.



In blue color, the parent function $y = x^2$.
In red color, the vertically translated function $y = x^2 + 3$



In blue color, the parent function $y = x^2$.
In red color, the horizontally translated function $y = (x - 3)^2$

The graph of $\mathbf{y = x^2 + 3}$ is a vertical translation of $\mathbf{y = x^2}$, three units up. The same way as a song / melody is being **transposed** from D to A. This process is continuously used in the music industry where composers create a song in a certain key, which later has to be adjusted to the voice of different singers. As a matter of fact, the same song may be recorded by different singers using the transpositions of the original song. Sometimes the transposition would go up, other times it would go down. The translation of the melody as well, may be used to create harmonic sounds applied continuously in duets, chorus and other performances. A harmonic sound is a combination of two sounds in different notes, one being a transposition of the other one.

The graph of $\mathbf{y = (x - 3)^2}$ is a horizontal translation of $\mathbf{y = x^2}$ three units to the right. It is important to mention the tendency to see $\mathbf{y = (x - 3)^2}$ and think of a translation to the left. However, solving $\mathbf{x - 3 = 0}$ will provide where the vertex is translating (moving) to.

There is a parallel concept between geometric translation in math and transposition in music. In both, there is a move from an original stage to a new one. While in math, the original and the new stages can be expressed as a set of points; in music the presence of a pentagram (staff) is needed. The pentagram plays the role of the Cartesian coordinate system. Depending on the action desired, the **transposition** is done moving all the notes up or down, while the **time delay** is done moving all the notes right or left.

In math, the coordinate system with its x–and y–axes is the structure to graph the functions (or relations). The pentagram (staff) provides the frame to write the music, which is translated into a geometric configuration. After having a geometric configuration, there can be geometric transformations. The question that arrives is if the transformations have any musical meaning. If so, what is that meaning?

The other part to explore in this curriculum unit at this point in relationship with music is the idea of reflecting the entire song with respect of the line that would be exactly in the middle of the entire song. This would imply to have the complete song written continuously on a single pentagram; look for the middle point and draw a vertical line to the pentagram, then, reflect the entire song over this vertical line. In reality this would mean to play the song in backwards.

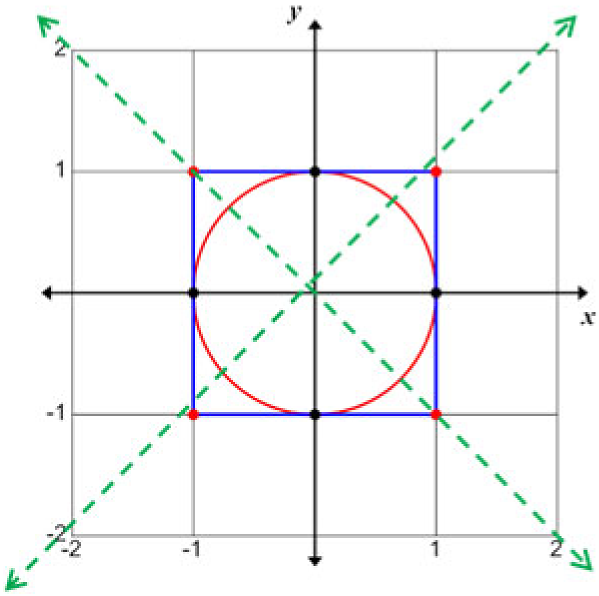
There are different ways to perceive the geometric translations, reflections and rotations of the notes from the pentagram and therefore, get students interested in music and math.

The Unit Circle, the Symmetries of a Square and Dancing

It is necessary for the understanding of this unit to provide the ideas of what a unit circle is and what the Symmetries of a square are.

The **unit circle** is the name given to a circle with radius equal to 1. The unit circle is usually placed with its center at the origin of the coordinate system. In other words, analytically its center is ***O***(0, 0). The unit circle is used in trigonometry to easily graph all the trigonometric functions, but in this unit we will refer only to sine and cosine.

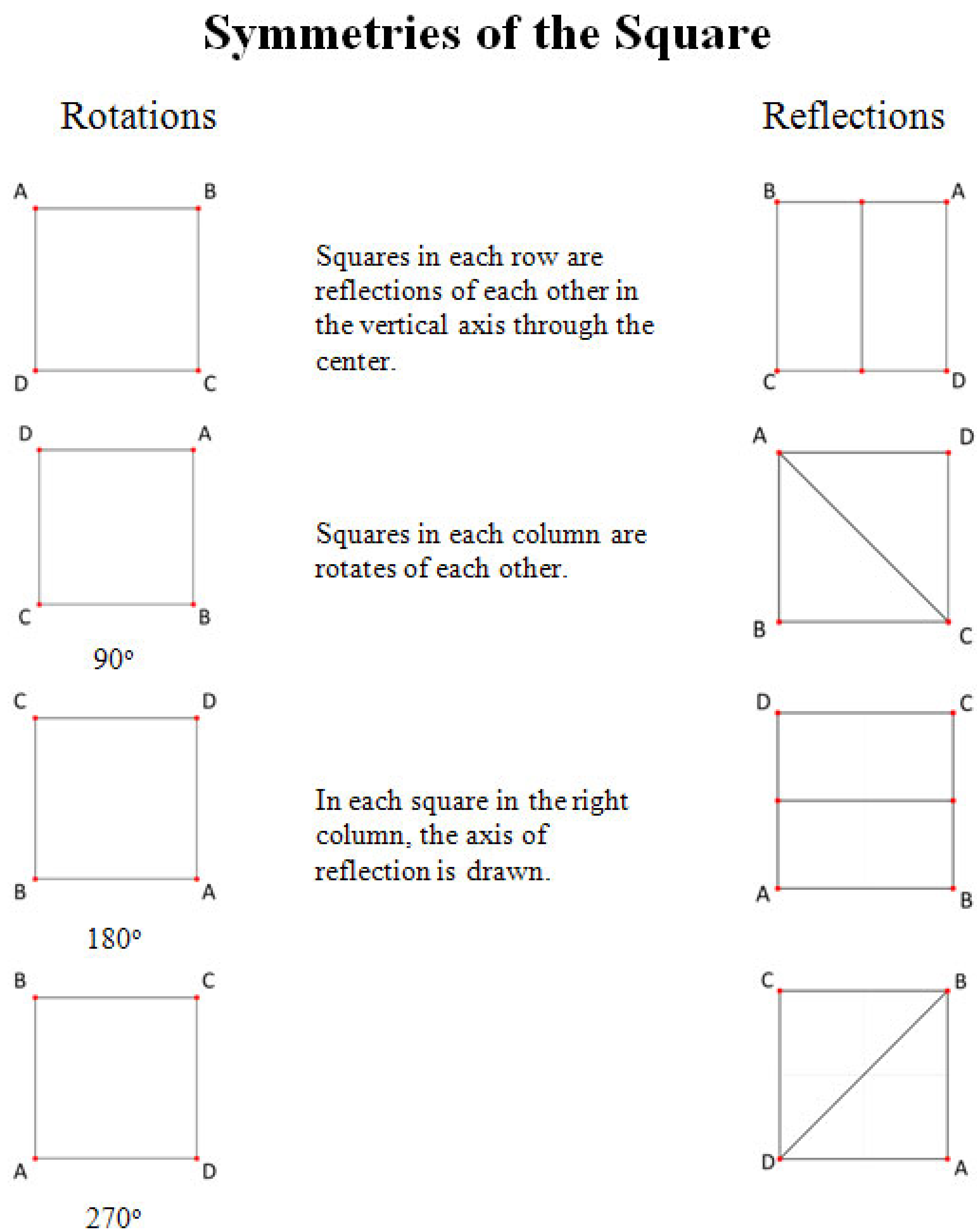
We will also think of the square as sitting in the coordinate plane, centered at the origin and with sides parallel to the coordinate axes. We take the side lengths of the square to be 2, equal to the diameter of the unit circle, so that the unit circle just fits inside the square, and is tangent to it where both intersect the coordinate axes. See the figure below.



The square has reflection symmetries and rotational symmetries. To see if a square has symmetry with respect to a line, we apply a reflection over the given line. If the properties and characteristics of the square are preserved, then we say that the square has symmetry with respect to that line. With our standard square as described above, we can see that it has four lines of symmetry. It is symmetric across the coordinate axes, **x = 0** and **y = 0**, and also across the diagonal lines **y = x** and **y = x**.

> "In geometry and linear algebra, a **rotation** is a transformation in a plane or in space that describes the motion of a rigid body around a fixed point. A rotation is different from a translation, which has no fixed points, and from a reflection, which "flips" the bodies it is transforming. A rotation and the above–mentioned transformations are isometries; they leave the distance between any two points unchanged after the transformation."
>
> From Wikipedia, the free encyclopedia

The square is also preserved by certain rotations around the origin. The angle of rotation can be 90°, 180°, 270° or 360°. This last value will place the square back to where it was at the beginning. It is the same as rotating by 0°; that is, not moving it at all. We call this the *identity transformation*.



While researching how to implement the concepts of symmetries of the square that we learned in the seminar of "Math of Wall Paper," I came across applications in dancing. It is interesting to see that the re–positioning of the vertices of a square after each reflection produces similar movement as those done in a dance. If a dancer gets situated on each vertex of a square and then we apply all the reflections using the symmetries of a square, we will have the different positions that a group of dancers would have.

These ideas are most relevant to the type of dancing called square dancing. In square dancing, four couples start by standing at the corners of a square. As the music plays, they execute various movements, following the instructions of the person leading the dance, known as the *caller*. Each movement is a *call*. Many calls involve rather complicated motions, but some of the simpler ones are related to symmetries of the square, and could provide a way to connect dancing with learning about symmetries of the square.

Some of the simplest ***square dance calls*** involve each couple doing something in its own corner. Examples are ***dosado***, with each couple walking around each other, facing the same way all the time, so that they are back–to–back when passing behind each other. There is also the ***swing***, in which the couple holds each other as the pivot around the point halfway between them. ***Allemand***is when they grasp hands or forearms while turning around each other.

In more complex ***calls***, couple change places, or sometimes just the ladies or gentlemen change places. A ***call*** involving **rotation** is a ***Promenade***, where each couple joins hands and walks side by side around the circle. In a ***Right and Left Grand***, women face one way, and men the other and they walk forward clasping hands as they pass. In term of symmetries of the square, the men are rotating in one direction, and the women in the other. Usually, everyone walks all the way around the circle and returns to their original position, illustrating that rotation of 360° gives the **identity transformation**. In a ***Ladies Chain***, the women will all join hands with the ladies from the couple on either side, and walk sideways around the circle.

Right and Left Through is a ***call***in which couples pass each other to assume the place of the other. If two pairs of couples on two opposite sides of the square do it, the square is reflected in the perpendicular bisector of the sides. If this is done with one pair of opposite sides, then another, the result would be rotation by 180°. If a pair of couples on opposite corners of the square does it, the square is reflected in the diagonal. Doing this with one pair of opposite couples, and then the other will again produce a rotation by 180°.

By doing these and other calls, students could produce any symmetry of the square. They could be challenged to predict what symmetry would be produced by a specific sequence of calls. If they got to be experts at this, they could be challenged with calls that produce more complicated permutations. This could be a great combination of dancing and math!

Being able to combine math, music and art in a lesson or a set of lessons that would start with patterns, sequences and series is keeping alive the dream that any teacher has, which is to be able to reach all students in a classroom, not only those who already like math, but those who struggle. I have always thought that a student does not need to be a genius to learn math. This is as well my personal dream, since I had the intention to become a musician since I was a child, but I was denied that opportunity. However, I have succeeded in keeping my dream alive and I want to be able to transmit that hope to my students combining my two passions, teaching math and music.

Now, if a student is able to understand the process involved in expressing steps of dancing, the next step would be to use that knowledge to understand the concept of motion using trigonometry, and therefore, expose students to the option to take more advanced math classes. In this case, the next level of math would be AP Calculus.

From words to action

I have planned several ways of interacting between mathematics, music, art and dancing. Now, these ideas will be implemented following lesson plan formats and fitting them in 55–minutes periods. I really would love to implement the ideas in all my classes, but realistically, I better concentrate on one or two classes. These classes will be Trigonometry and Algebra–II.

Planning for Algebra–II

Algebra–II students commonly have all the basics from Algebra–1 and from Geometry, so I can easily refer to certain concepts needed to teach this curriculum unit. These are concepts already presented in Geometry and Algebra–1

As an anticipatory set, students already have the vocabulary needed to succeed in this portion of the curriculum unit. We have reviewed relations and functions, inverse of a function, translations and a coordinate system; we have studied about shapes analytically described in a coordinate plane as well as the ideas of domain and range and how important is to define them to preserve the concept of function.

To start the curriculum unit, a set of exercises of simple translations using shapes already known will be created. Using only the coordinate system and the description or definition of the shape to be translated will make the process smooth enough to implement these new ideas. After students feel comfortable with the transformations, the concept of inverse of a function will be included.

Students will start with a warm–up to practice translations and locating figures in a coordinate system. At the same time, students will get familiar with the language used for the activity. Warm–up will take up to 15 minutes including the solution of the exercises from the warm–up. The next 10 minutes, instructions for the activity of the day will be given. Students will receive the materials and utensils for the hands–on activity. On the following 15 minutes, students will engage in the hands–on activity cutting out the shape to be translated and placing it on a graph paper. A digital camera will be available to take pictures of the "before" and "after" products. Students will be assigned in groups what exercises should they concentrate on

Using the digital pictures, students will be able to record and later on, to see and write the coordinates of important points of the shape before and after being translated. When I refer to important points, I mean the vertex (if any), the y–intercept and the x–intercepts.

The last 15 minutes of the class will be devoted to student's presentations on their assigned exercises. Power points with the inclusion of digital pictures will be shown with the LCD projector while students are presenting. In addition to the geometric transformations, students will start their groups for their culminating project. Project assigned by the end of the report period will be a power point presentation per group about relations, functions, and inverse of a function. A written component describing these concepts and what they have learned will be as well required. The last portion of the project will have some graphs in a coordinate plane, some designs of clothing to include fashion and the bibliography, including Internet sites citation. The ending is however open to students' creativity.

For student's presentations and report project, a rubric will be created. Rubric will be given to students prior to the actual presentations. It is much better to provide students with the information of the rubric, so they will know what teachers expect from them. At the same time, having the rubric helps students to concentrate their efforts on certain parts of the project in order to obtain a much better grade.

The second lesson of the curriculum unit

Previously, students practiced with hands–on transformations, usage of a digital camera and presentations. The next step is to engage them in the different ways of using translations and / or rotations, including dancing, and music.

For this part I will divide the students in three groups. One group will be those who will pursue careers in engineering, medicine, and such, where the level of math required to succeed in college is very high, and therefore, they will be taking Pre–Calculus/Trig next year and AP Calculus the following year. In other words, this is the group of students who like math or at least they feel very confident with their math skills.

The second group will be composed of those students who like music. They must read and write music at least at a basic level to be able to understand the instructions for this part. Basic vocabulary on music such as a staff (pentagram), notes, chromatic scale, transposition and time delay is also important. If not, these concepts must be addressed prior to the lesson. Another way to do it is coordinating with the music teacher.

The third group is composed by students who like dancing and / or students who are currently in the cheer–leaders group, dancing squad or simply want to be part of this group. A basic knowledge of vocabulary for a square, such as vertices, vertex, diagonal, sides and rotation is necessary. As an anticipatory set, students in the group who will work with music will have to bring a song written in a pentagram. It can be any song that they like, as long as they can read it and play it with their respective instrument.

The activity for this lesson will have three parts, one for each of the groups. There will be 12 questions about applied rotations and translations and a review from the previous lesson. There will be 4 questions per each group. The first 10 minutes will be for to the warm–up. Those who know how to solve their corresponding questions will come up to the board to show their solutions. Showing solutions will take approximately 5 to 10 extra minutes, including the explanation. There are 35 minutes left from the 55 minutes allocated for the lesson.

This time, we will learn the concept of reflecting an entire song (or a phrase of the song) using the pentagram and the song they have brought to class. There are two options for the axis of rotation. One option would be to have the axis of rotation as the third line of the pentagram. In this case, the "song" would be reflected through this third line; therefore, the notes written at the bottom will be at the top o the pentagram and the notes from the top will be at the bottom.

The second option would be to reflect the entire song having the axis of reflection as a line perpendicular to the pentagram and right in the middle of the entire song (or phrase). For this particular case, the ending of the song would be the beginning and the beginning would be the end.

Inviting the music teacher to coordinate this part would be a great example of working as a team. I will contact the parents of my students as well to let them know what their children are doing. It is not out of experimenting but reaching all the students.

We will learn the application of two types of translations in music, under the name of ***transposition***, comparing it to the idea of ***vertical translation***, as well as ***time delay*** to compare it with ***horizontal translation***. It is not my intent to teach music instead of math concepts. It is a matter of relating the math concepts with things that students are familiar with.

Planning for Trigonometry

Lesson probably will be introduced in the second half of the second semester when the concept of the Unit Circle has been already taught. Combining a coordinate system and analytic geometry from Algebra–2, plus the new ideas from trigonometry such as sine and cosine functions in the unit circle, will help to have an anticipatory set.

To start de actual lesson, students will do the warm–up activity being reminded of the concepts already learned in previous math classes. At his point we have covered already concepts from the first semester. The warm–up will consist of exercises reviewing the positioning of sine and cosine of 45° using the unit circle, reflection of functions over the lines y = x and y = x, equivalence between graphing a point in the unit circle and the corresponding coordinates from a coordinate system. Vocabulary will be introduced to remember lessons from Algebra–II and geometry as well from previous years. The activity should take approximately 20 minutes including teacher's explanation or students presenting their solutions on the board.

The next 15 minutes will be used to explain the relationship between trigonometric identities and the unit circle. Given and angle with initial side at the x axis and terminal side anywhere between 0° and 90°, we can represent sine of the given angle as the vertical segment from the circle towards the x axis; and cosine of the same angle as the horizontal segment from the circle towards the y axis.

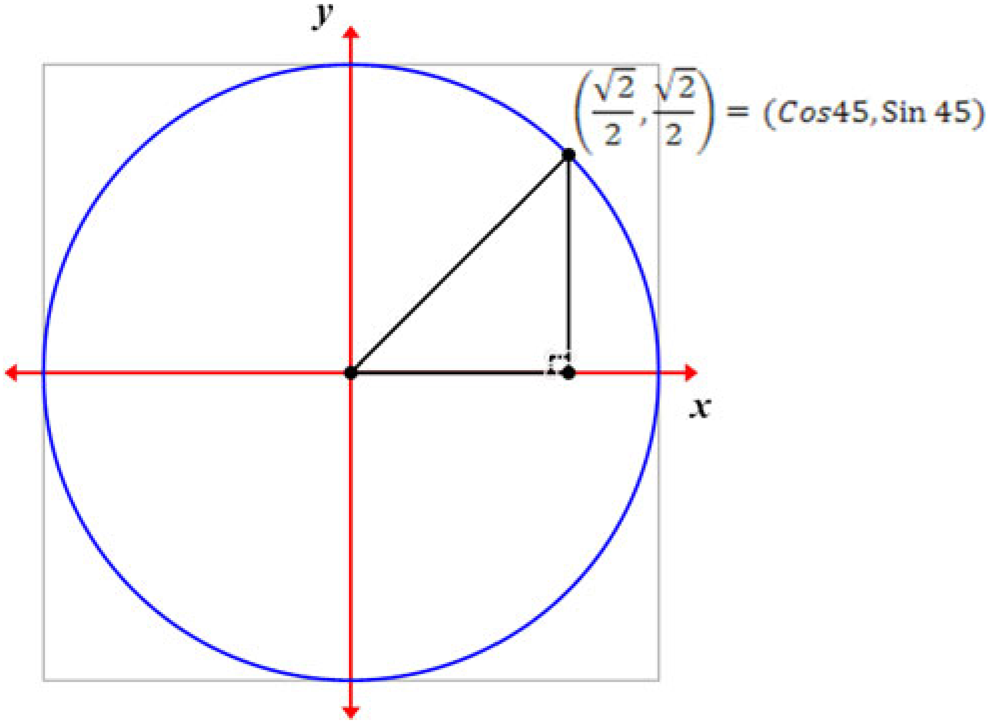
On the next 15 minutes, the demonstration of the identity $\sin^2\theta + \cos^2\theta = 1$ will be done. The first part is to apply concepts from Geometry such as trigonometric ratios and algebraic expressions to be able to demonstrate the identity. The second part is to apply the unit circle and the graphic approach of sine and cosine to demonstrate the identity.

To prove the identity $\sin^2\theta + \cos^2\theta = 1$ using the unit circle, we can include the theorem of Pythagoras, and conclude that the addition of the squares of the two segments, vertical and horizontal described on the previous paragraph will be equal to the square of the hypotenuse, which in this case is exactly the radius of the unit circle. By default, because the radius of the unit circle is 1, the addition $\sin^2\theta + \cos^2\theta$ will be l.

As well as the above identity, there are others that could be verified using the unit circle. One would just need to relate the identity with the Pythagorean Theorem.

The geometric representation of points in the unit circle using trigonometric terms is similar to the representation using analytic coordinates. For one side, in Algebra–II, I will be able to present the curriculum unit just using Algebra, for Trigonometry, I will be able to use sine and cosine. For example, for square dancing, the vertices of a square to represent the presence of 4 people, one on each vertex can be as well represented using sine and cosine. Using sine and cosine instead of algebraic analytic coordinates provides more flexibility for the "dancers" to move. If the position where the dancers are located varies on the circle, the angle will change. It is important to understand that there will be a fixed distance between partners while dancing.

On the following graph, the same representation is given using coordinate values and trigonometric expressions. This graph is just an example of what can be done. Although x could be any value, I have chosen the value of x = 45 ° as an example.



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