Curriculum Units by Fellows of the National Initiative
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Introduction

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The idea of symmetry, which has long been associated with art and notions of beauty, took on a new life in the 19th century, when it was found to be the key to understanding deep questions left over from classical mathematics. In the 20th century, these same ideas were found also to be fundamental for understanding the behavior of matter. In particular, the hydrogen atom, the simplest particle of normal matter, was shown to have an intricate structure entirely determined by its symmetry. This seminar approached the idea of symmetry through its appearance in decorative patterns, especially wallpaper.

The mathematical conception of symmetry differs from the usual description of symmetry in art, where it usually is described in terms of "balance" and "harmony," words that connote stasis. For a mathematician, symmetry is an active idea – it involves transformations. An object (or a design or a figure, or any sort of structure) is called symmetric with respect to a given transformation, if that transformation leaves the key features of the object unchanged. The transformation is then called a symmetry of the object. For example, take a square. It looks the same if you reflect it in either of its diagonals, or in either line that connects the midpoints of two opposite sides, or if you rotate it though 90° in either direction, or by 180° . It is also unchanged if you do nothing. (The transformation that doesn't do anything – that leaves every point, or part, just as it was – is often not thought of as a transformation, but it is, and it plays a very important role, like 0 for addition, or 1 for multiplication.) This gives four reflections and four rotations, eight transformations in all, and these transformations, taken as a collection, constitute the symmetries of the square.

The importance of thinking of symmetry actively, in terms of transformations, is that transformations can be combined, by performing one and then performing another. The technical term for this is composition. For example, if you reflect a square in a diagonal, and then reflect it in a line through the midpoints of two opposite sides, the combined transformation, or composition, is a rotation by 90° . Thus, the rotations of the square are already inherent in the family of reflections of the square.

The possibility of composing transformations endows the collection of symmetries of a system with an algebraic structure, analogous to addition or multiplication, but with much richer possibilities. The technical term for a collection of transformations that contains the result of composing any pair of its members is group. The algebraic structure then has implications for the physical structure. For example, as described above, the square has 4 reflections, and 4 rotations (counting the identity – rotation through 0°). The fact that the number of rotations is equal to the number of refections can be predicted by knowing the structure of the group of symmetries.

The seminar studied these ideas in the context of understanding the symmetries of planar figures or designs, in particular, regular polygons, frieze or strip patterns, and wallpaper patterns. We learned the four main types of isometry of the plane (reflections, rotations, translations, and glide reflections). We saw how these transformations could be combined into systems of symmetries of figures. Finally, we got some insight into how knowing the properties of transformations and how they combine allows one to classify the possible types of symmetry of patterns. For example, there are 7 essentially different types of symmetry that a frieze pattern can have, and 17 types for wallpaper. We also studied particularly elegant ways of constructing patterns with certain types of symmetry.

Each Fellow has selected from the broad array of topics touched on in the seminar to fashion a unit. Carol Boynton and Jane Fraser want to introduce their students to the geometric features of each of the main types of isometry. Shamsu Absul-Aziz and Katherine Radcliff will teach that the standard terminology for special types of quadrilaterals is in effect a symmetry classification of these figures. Several Fellows, including Elwanda Butler, Stephanie Colombo, and Rose Schmitt have designed projects that feature the seven types of frieze patterns. Finally, Holly Grandfield is emphasizing the geometric interpretation of the operations of arithmetic, using the number line. Each Fellow has selected and described an interesting feature of the elephant of an idea called symmetry.

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