Curriculum Units by Fellows of the National Initiative
2011 Volume VI: Great Ideas of Primary Mathematics

Algebra Really Is Just Arithmetic

Curriculum Unit 11.06.04, published September 2011
by Michael Pillsbury

Introduction

I teach seventh grade mathematics in an urban school located in Charlotte, North Carolina. My school was classified as a full International Baccalaureate magnet school two years ago and this has changed the type of students enrolled in the school. All students must enter a lottery system and then be selected to attend our school. They must be on grade level as it relates to their end of grade scores in mathematics and reading. The school is diverse in its student body makeup with students from all over the world present in my classroom.

I have been teaching seventh grade mathematics for eight years and I still feel like I am a new teacher. I constantly seek out better strategies and a deeper level of understanding of my content area. I typically have few to any discipline problems in my classroom and the students work well in groups. I always keep in the back of my mind however that students are effective for only short periods of time in small groups. I have to refocus and direct them on a periodic basis.

Headlines often tell us how behind students in the U.S. are in mathematics. Should we believe them? What are the tests? Why is it that our colleges are full of students from these countries that are supposed to be ahead of us? Are the grade schools behind and the colleges ahead? Do other countries teach differently than we do? I can provide answers from my own personal experiences as a teacher. Most of the students that I have taught know mathematics only on a superficial level. They know how to plug and solve, they remember formulas and they remember rules told to them by their previous mathematics teachers. They do not understand mathematical concepts on a deep level. The length of time during which they remember these things varies and is often indicative of a successful versus a non-successful student.

My unit will address an area of mathematics that is often not fully understood by students, place values. My students are scheduled in honors level classes and they are expected to take high school Algebra the year after I teach them. It is important in my instruction that I prepare them for a high school level course by giving them a better understanding of mathematical content that they should already be familiar with. The title of my unit "Algebra Really Is Just Arithmetic" is my way of saying that if you understand basic arithmetic then you will have a better understanding of Algebra. Exploring place value and operations as they relate to place values should help clear up many issues that students have with Algebra.

The activities and strategies that I will use in my unit will take the basic concept of place value and using expanded form allow students to better understand operations and why they have to borrow or carry. Students will then see the direct relationship that exists between arithmetic and algebra. The method of performing operations with expanded form and exploding dots will seem like a lot of unnecessary work to some but it is the underlying knowledge of numbers and place value that is my objective. Lack of that, in my opinion, is what got us in the boat we are in now. Mathematics is a way of compressing things to make them easier to work with. My point here is there is no sense in compressing something if you do not understand it to begin with. If students do not understand what they are doing with numbers other than going through the motions of what teachers have told them for years then how will they know what to do when these numbers are now variables?

The unit is designed to be taught over a twenty day period. Students are expected to work in small groups of three or four and present their discoveries to the class on an as requested basis. While teaching this unit I have to be careful not to reveal too much information. I expect students to learn through investigation. The teacher should be a facilitator and guide not a lecturer or revealer. Think of teaching this unit as farming. A farmer plants seeds and then nourishes the seeds into a grown crop that can be harvested. I will plant the seeds and the students will provide the nourishment. The overarching concept here is that, the students must obtain a deep level of understanding of place value and arithmetic in order to be successful at algebra.

Rationale

I have students who seem mystified by fractions and the concept of decimals and percentages. When I introduce variables into problems this further confuses them. I have often had students not be able to correctly say the number 4,567,878 (four million five hundred and sixty seven thousand eight hundred and seventy eight. This may just be an issue with vocabulary or it could be an indication that they truly struggle with the concept of place value.

I have chosen to take my students back to what they may consider elementary school content. I want them to learn place value and operations with numbers so that they will be successful in Algebra. Teachers may find this somewhat difficult in execution. Most students will want to take the shortcuts that they know and love. This is not the unit for that. To make sure that students stay on task I have included problems that will require a good to great working knowledge of place value to solve.

It is my goal to have students that participate in this unit be more successful in Algebra than those who are not exposed to it. By understanding arithmetic they will understand Algebra.

Curriculum Objectives

In June of 2010 the state of North Carolina along with most states in the United States adopted the Common Core Standards. These standards were developed by the states themselves and should be adopted by most if not all states. This unit will address all mathematical practices of the Common Core Standards. The unit addresses seventh grade expressions and equations standards. Because of its nature the unit will address many different standards on many different grade levels. I encourage teachers that are trying to do this unit to look at their standards related to operations and algebra. The scope of the unit is so broad that teachers should have no difficulty tying into either an existing state standard or Common Core Standard.

Background

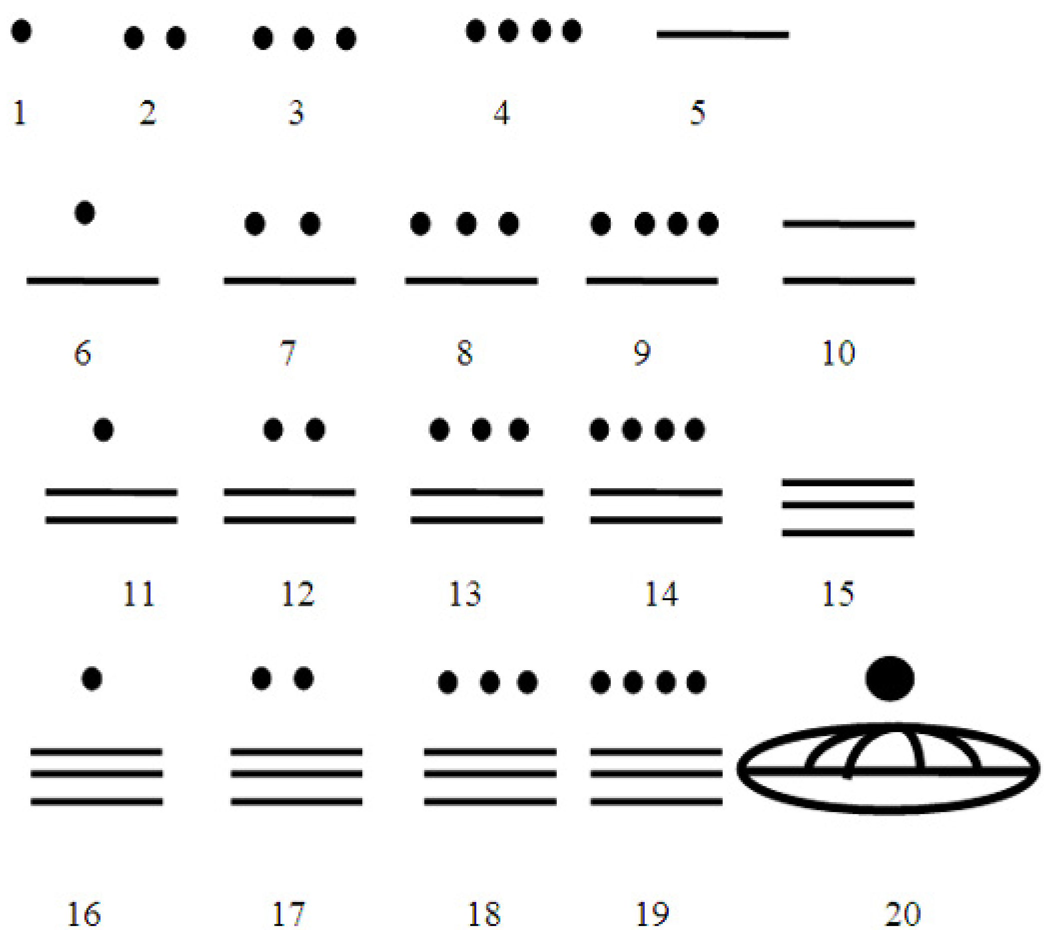
Students that are being taught this unit should have a working knowledge of the rules of exponents. They should also be familiar with the concepts of variables, operations involving integers and combining like terms involving variables. Students will be working with a significant amount of mathematical notation and they should know the order of operations as well as proper mathematical notation. This will help students to be able to complete some of the problems they will be presented.

Zero and Place Value

Our number system is called the decimal system and it is based on the number ten. The word decimal is derived from the Latin word decim. [1] There is no clear history of how we arrived at our current number system but the rough outline, and some specifics are known. A key feature in the overall scheme is the number zero and its value as a placeholder.

The number zero is important because of its use as a place value. Think about the difference of 1,001 dollars versus 11 dollars. The number zero is used to provide an unambiguous way of reading numbers. How else would we know that there are one thousand dollars in addition to the one dollar? We can write the number without the use of words thanks to zeros which hold the place value for us.

The ancient Babylonians had a base sixty number system. They had a separate representation for all the numbers up to sixty. [2] This is one of the main reasons our hour has sixty minutes and sixty seconds make up one minute. In our current system we have separate representations for all numbers up to ten. The ancient Mayans had a base twenty number system in which a dot represented the number one and a horizontal line represented the number five. The Spanish explorers discovered this system and the significance of this was that their concept of zero as a placeholder was very similar to ours as we write numbers today. The numbers one through twenty in Mayan would look like this:



Notice the one representation and then the special symbol which represented zero. The Mayan twenty would be equivalent to our number ten. The use of symbol representation used as a place holder was common among many cultures and the base number system that they used varied widely. [3] By the eighth century in India, our modern decimal system with zero as a place holder was developed. [4]

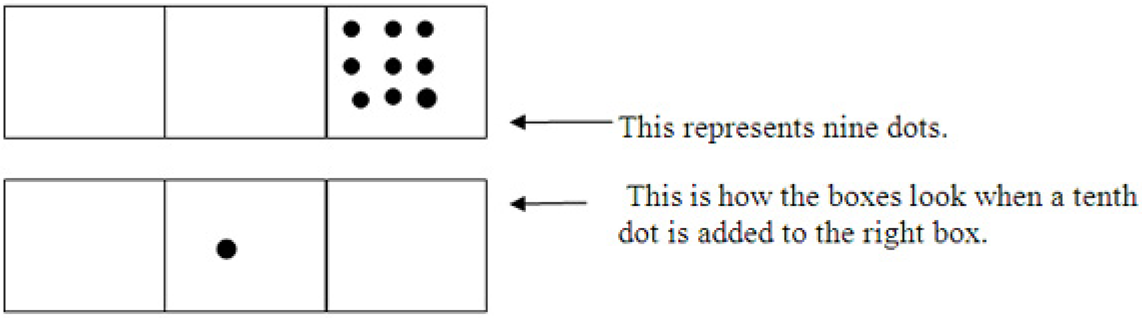
I will work to make sure that my students really understand the key role that zero plays as a place holder. The digits 0,1,2,3,4,5,6,7,8,9 are the basis for our entire number system. The number ten (10) is written with two digits with the 1 standing for 1 ten, and the zero indicating zero 1s. We know that the 1 represents a ten and not a one or a hundred, because of the single zero to its right. As simple as this idea is, it gives the decimal system tremendous power for representing even very large (relative to everyday experience) numbers compactly.

Strategy and Content

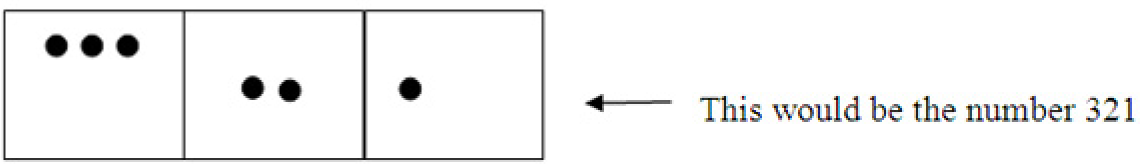
Exploding Dots

Exploding dots is a concept developed by James Tanton. [5] In its simplest form I think it is a great tool to introduce the concept of place value. Most students can tell you that in the number 345 there are three hundreds, four tens and five ones. By using exploding dots students will have an illustrated representation of the base ten number system or any base number system you choose.

Exploding dots consist of a series of boxes constructed together horizontally. I would recommend starting with no more than three boxes. Start with the box farthest to the right and draw in dots up to a maximum of nine. When the tenth dot is added the dots explode and a lone solitary dot will appear in the box to the left of the box. The boxes are called a machine. When this is carried out we will label it a 10 -> 1 machine.



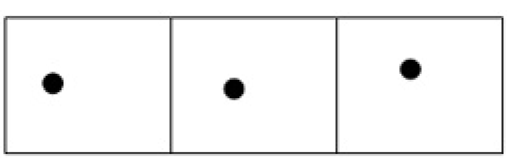
When trying to add the tenth dot to the far right box the teacher should make a loud sound simulating an explosion, wipe the dots away and then place the dot in the middle box. This will be true for any box. The maximum number of dots that can appear in any box is nine. The tenth causes the explosion and a dot to appear in the immediate box to the left. Question the students at this point and ask them how many dots are represented in total?



Write this box and the dots. First ask students how many dots does the first box represent? The answer should be 1. Ask them how many dots are represented in the second box? The answer should be 20. Then ask them how many dots are represented in the third box (going from right to left)? The answer should be 300. If the students don't get this immediately, the process can be explicated in more detail.

Students will then be asked to write down what numbers different boxes and dots represent. The concept is that they will quickly understand that a 10 -> 1 machine represents our base ten number system. The importance of zero will become more evident as you proceed with these exercises.

Different base number systems can and should be represented with exploding dots as well. A machine that is 2 -> 1 represents a binary system.



The number represented here is 111_2.

If we wanted to convert this number back to base ten then we simply write it in expanded form $1(2^2) + 1(2^1$

) + 1 (2^{0})= 4 +2+1= 7 in base 10. I think it is important that teachers show students at least one example of a number written in another base system. Exploding dots can help illustrate why 9 is the highest digit in base 10 and 1 is the highest digit in base 2.

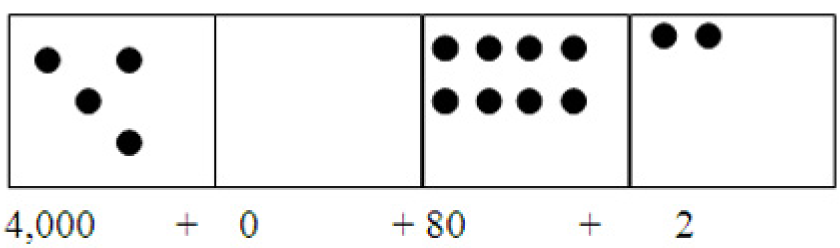
A widely held belief in education is that students who learn a foreign language will in turn learn the English language better. That is because they are forced to see something familiar to them, the English language, through the lens of learning a foreign language. I believe that if students are exposed to other base number systems and understand how they work then they will understand the base ten number system on a deeper level. It is beyond the scope of this unit to investigate this idea in detail, but some teachers might find it interesting.

Expanded Form

Being able to write numbers out in expanded form is important. It may be seem like the long way of doing problems by students but it does force them to see the significance of place values by writing numbers in expanded form and then being able to regroup them and perform various operations.

Expanded form can be shown in different ways. If we take the number 4,082 it can be written 4,082 = 4,000 + 80 + 2 .

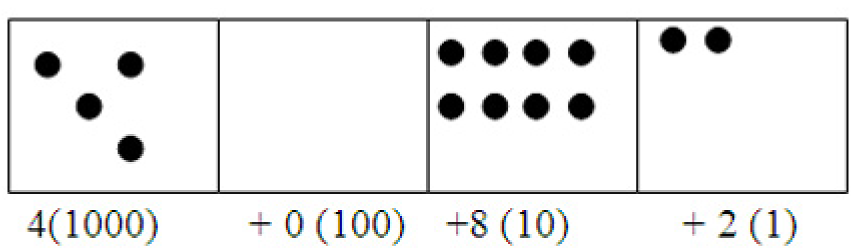
In exploding dots this number would be written as :



In this form it is not necessary to explicitly indicate that there are no separate 100s present, since each non-zero digit is represented in the correct place. The boxes themselves function as placeholders, and an empty box replaces a zero in the corresponding place. That is, we don't have to write

4,082 = 4(1000) + 0 (100) +8 (10) + 2 (1)

Using Exploding Dots:



In the above form a zero has been added to indicate that there is a zero amount of hundreds. This zero must appear in the compressed version of the number, since in the compressed form, the only way to indicate the unit that a digit represents is by its place in the number. We can also write our number as:

4,082 = 4(10 x 10 x10) + 0(10 x 10) + 8(10) +2(1)

In this more refined version of expanded form the powers of ten are being broken into the factors of 10 that they are made from, with the exception of the units digit. Finally, we can rewrite our number a fourth time:

4,082 = $4(10^3) + 0(10^2) + 8(10^1) + 2(10^0)$.

In this form, the various base ten units are now being written as powers of 10, using exponential notation. The representation using exploding dots would be the same for each expanded from version.

I think this is the best expanded form as it clearly shows a relationship to our base ten number system and allows for the importance of zero to be discussed as a place holder. This concept is referred to as the *order of magnitude*. For example the order of magnitude of 4,000 is three which is not only equivalent to the number of zeros but also the power of ten involved in the expression 4,000 = $4(10^3)$. The order of magnitude of 12, 456 is four as the highest power of ten is $1(10^4)$ which is the ten thousands place. The order of magnitude for 5 is zero. This is written as $5(10^0)$ and five is the highest number and it is multiplied by ten to the power of zero.

Adding with Expanded Form and Exploding Dots

Adding numbers in expanded form can be done in a variety of ways. Let's look at adding 4,082 + 988. In expanded form and moving numbers around because of the commutative law of addition I it would look like $4(10^3) + 0(10^2) + 8(10^1) + 2(10^0) + 9(10^2) + 8(10^1) + 8(10^0)$. When we add the numbers together and combine like terms we get $4(10^3) + 9(10^2) + 16(10^1) + 10(10^0)$. Since our number system is based on the number ten then we do not have a number higher than nine as any single digit. Starting with our last number in the 0 magnitude place we have to reduce the digit to a single digit below nine that is in the single digits spot of the number appearing. The digit is being reduced by removing the next highest magnitude. This might be a better demonstration to students of what is occurring during the addition process. This will allow students to see that the excess dots equal the number times a particular magnitude in addition with expanded form.

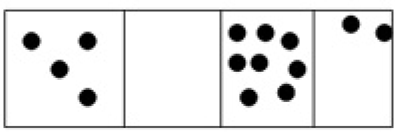
Addend A	4(1000)	+ 0(100)	+ 8(10)	+2(1)
Addend B		+9(100)	+ 8(10)	+8(1)
Subtotal A	**4(1000)**	**+ 9(100)**	**+ 16 (10)**	**+10(1)**
			+1 (10)	- 10 (1)
Subtotal B	**4(1000)**	**+ 9(100)**	**+17 (10)**	**+0(1)**
		+1(100)	-10 (10)	
Subtotal C	**4(1000)**	**+ 10 (100)**	**+7(10)**	**+0(1)**
	+1(1000)	-10 (100)		
Total	**5 (1000)**	**+0 (100)**	**+ 7(10)**	**+ 0 (1)**

This is the first number.

Here is the second number to be added.

When they are added the result cannot have more than 9 powers of ten. So we subtract ten powers of ten to the right and then add one power of ten to the left.

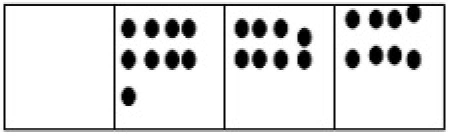
This process continues until we have 9 or less powers of ten in each place.



This represents our first number to be added.

4(1000) + 0(100) + 8(10) + 2(1)

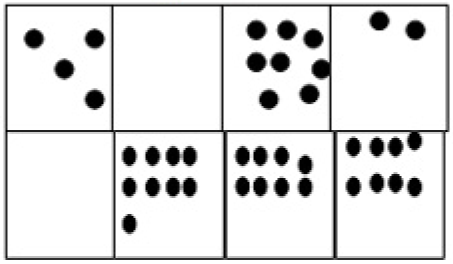
We would then add a box below reflecting the number 988.



This represents our second addend.

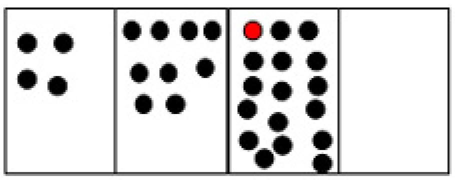
9(100) + 8(10) + 8 (1)

We would place one box over another and simply add them together. For purposes of demonstrating this, the boxes would not explode until we needed them to.



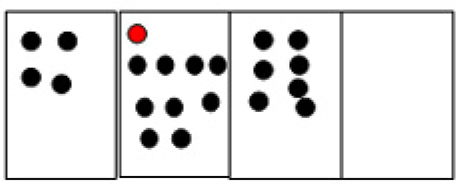
To the right the total number of dots are represented, in their unexploded state.

4(1000) + 9(100)+ 16(10)+ 10(1)



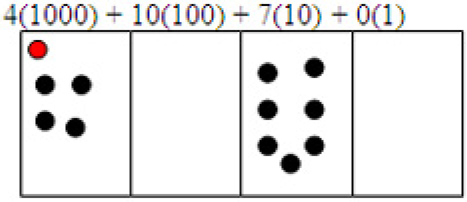
The dots in the ones box explode because there are ten. The red dot is the new dot created in the box to the left or the ten's place.

4(1000) + 9(100)+ 17(10)+ 0(1)



Here again we explode the box with more than ten leaving seven dots in the ten's place and now 10 dots in the 100's place. The red dot represents the new dot created.

4(1000) + 10(100) + 7(10) + 0(1)



There are ten in the hundred's place so we explode those to get our final answer.

5(1000) + 0(100) + 7(10) + 0(1)

Note: In representing a number in standard base ten form, we cannot have more than 9 copies of ten in any place.

Note how whenever when the addition process creates more than 9 copies of ten we have to convert 10 of them to the next larger unit, and then add this to the place value to the left. This is a big concept and should not be overlooked. For instance when there is $16(10^1)$ in the above problem the question is should I just subtract 7 to get to 9? How many do I subtract? Simply write the number out. $16(10^1) = 160$.Then break it into its base ten components: 160 = 100 + 60. The 100 belongs in the next larger place. So we know we have to convert 10 tens into 1 hundred or 1 (10^2). This of course is how the number is represented when added to the hundreds column. Remember the value never changes, only the location of the values. I always use exploding dots to guarantee a clear illustration.

Algebra Relationship (Addition)

In Algebra numbers are represented by variables. Students should be familiar with variables. While doing this unit you may need to familiarize students with the concepts of variables if they have not been exposed to them previously. I would use a variable to represent the powers of ten and the digit indicating how many copies of a given power of ten are present to be the coefficient of the corresponding power of the variable. The above problem would look like this algebraically:

The number 4,082 could be shown as $4x^3 + 0x^2 + 8x + 2$ or simplified it would be $4x^3 + 8x + 2$. The number 988 would be shown as $9x^2 + 8x + 8$ and it is simplified as much as possible. Here again we can use an array to add our numbers. The important concept here is that the x's could represent any number so we do not have to get our coefficients below nine, as we do when working in base ten.

Addend A	$4x^3$		$+8x$	$+2$
Addend B		$+9x^2$	$+8x$	$+8$
Total	$\mathbf{4x^3}$	$\mathbf{+9x^2}$	$\mathbf{+16x^1}$	**+10**

If we now substitute a ten back in for the variable x then we get the following

$4(10^3) + 9(10^2) + 16(10^1) + 10(10^0)$ is the same number as subtotal A in the addition table. This illustrates the fact that addition in base ten can be thought of as having two stages. In stage 1, digits are added as if they were coefficients of polynomials, and we were doing polynomial addition. In stage 2, we remember that x = 10, and do regrouping of coefficients that ended up being larger than 10. In the standard algorithm for addition, these two stages are mixed together.

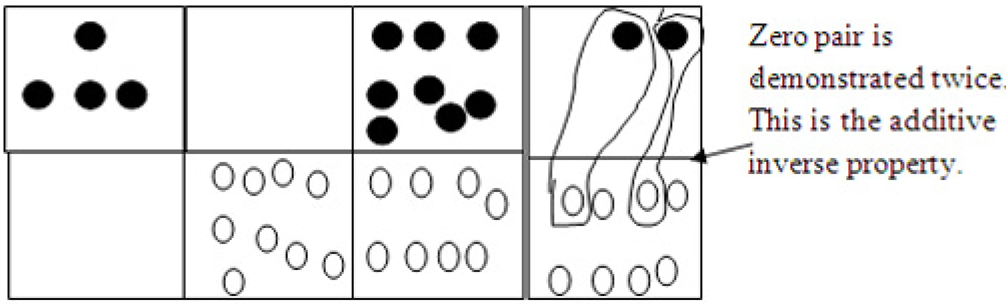
Subtracting in Expanded Form and Exploding Dots

Subtracting numbers in expanded form can also be done with the same concepts in mind. Here the idea of borrowing is revealed. We must start from the right to make sure that we have a high enough number to subtract so the top number is greater than the number being subtracted. The example below shows 4,082-988.

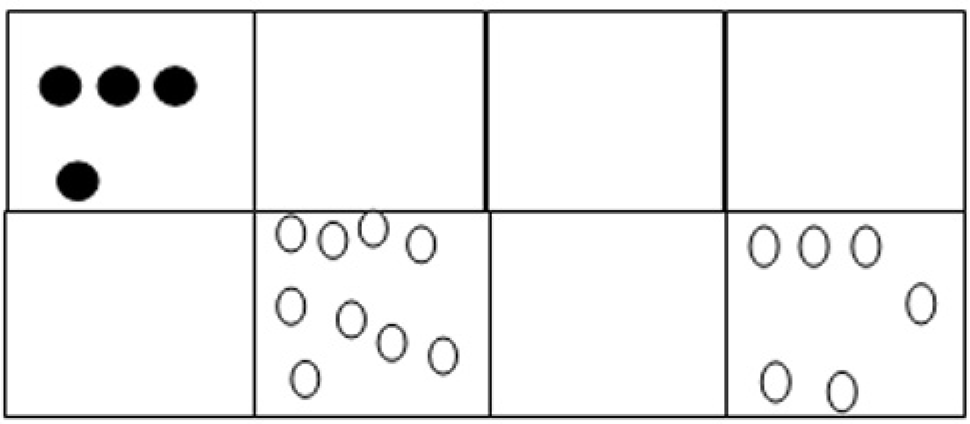
Minuend	$4(10^3)$	$+0(10^2)$	$+8(10^1)$	$+2(10^0)$
Subtrahend (-)		$-9(10^2)$	$-8(10^1)$	$-8(10^0)$
Total	$\mathbf{4(10^3)}$	$\mathbf{-9(10^2)}$	$\mathbf{+0(10^1)}$	$\mathbf{-6(10^0)}$

This is of course not an answer since standard base ten notation does not use negative multiples of the base ten units. This is the point where students should understand how the concept of "borrowing" is used. Showing this with exploding dots will help illustrate what is being done.

Subtraction with exploding dots is a little different than addition. I introduce the concept of anti-dots. The great thing with subtraction and exploding dots is how clearly it illustrates the concept of borrowing. If we take the same two numbers we were using in expanded form 4,082- 988 we can illustrate what happens:



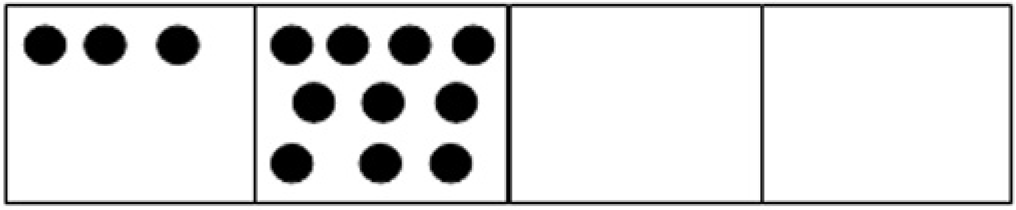
This can get real messy in such a confined space but as you can see I am matching up an anti-dot with a dot and creating a zero pair. This is using the additive inverse property (see Appendix B). When we are completed matching up all the possible zero pairs we end up with this:



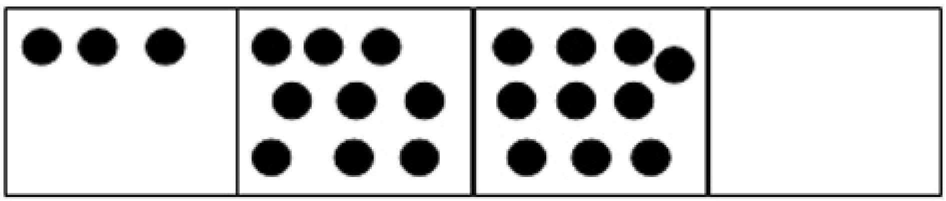
If we look at the exploding dots above we clearly have: 4(1000) -9(100) + 0(10) -6(1)

This is exactly what we had when we finished subtracting with expanded form. The problem is I cannot write this number out in condensed form as it has negatives in it. What I can do is spread the dots down from the thousands place and then I can eliminate the anti-dots by having enough dots to make zero pairs. This is where the concept of borrowing occurs.

The first thing I could do is implode a one thousand to create ten hundreds.



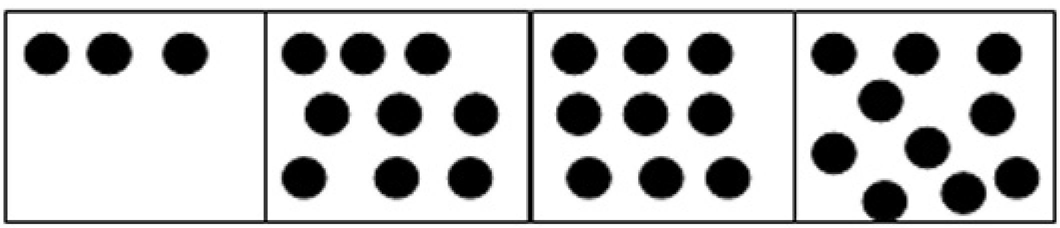
Then I could implode a dot from the hundreds place to create ten dots in the tens place.



This actually still represents 4,000 in an unexploded sense. Check it to see.

3(1000) + 9(100) + 10(10) = 4,000

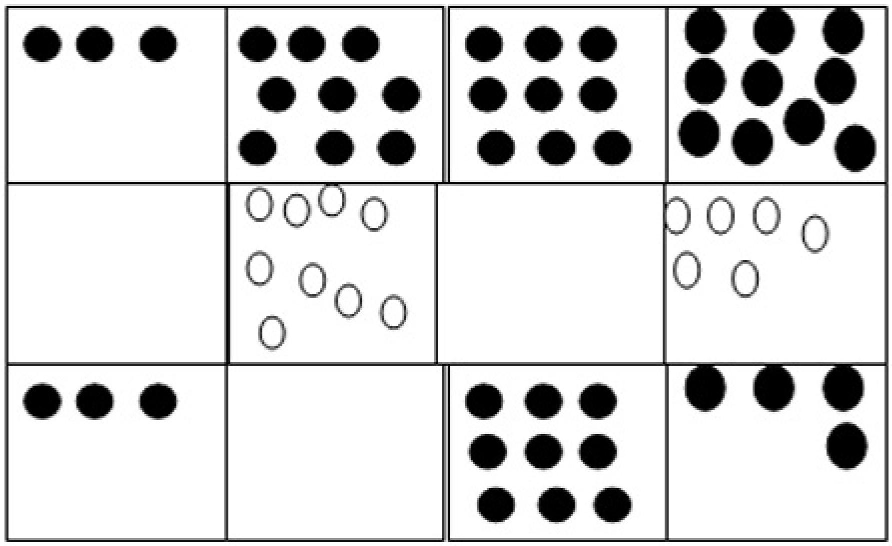
We have one more step and that is to implode a dot in the tens place and create ten ones.



We can again check our number in expanded form as it is represented in the unexploded dot form to make sure we still have 4,000.

3(1000) + 9(100) + 9(10) + 10(1) = 4,000

Now we have an equal or greater number of dots then anti-dots and we can eliminate all the anti-dots through the additive inverse property and get our results.



3(1000) + 0(100) + 9(10) + 4(1)

The clear dots are called anti-dots and represent the number being subtracted. Some students might want to call them negative numbers. Based on prior knowledge students should realize that adding a negative number is the same as subtracting a positive number.. When more anti-dots appear in a box than dots then obviously there is a problem. The key here is to implode a dot in the next box to the left. When a dot is unexploded then ten dots will appear in the box to the right. This illustrates the concept of borrowing that students should be familiar with but it focuses them on what is really happening when they borrow.

Algebra Relationship (Subtraction)

Subtraction is done in a very similar way as addition. I have constructed an array to show exactly what would occur. I have already commented on how to make this as parallel as possible to the straight numerical calculation. I suggest that you also now substitute x = 10 and show that the regrouping process leads to the same answer.

Minuend	$4x^3$		$+8x$	+2
Subtrahend (-)		$-9x^2$	$-8x$	-8
Total	$\mathbf{4x^3}$	$\mathbf{-9x^2}$		**-6**

If we go back and substitute 10 in for the variable x we get the same answer that we arrived at in expanded form.

$4(10^3) -9(10^2) +0(10^1) - 6(10^0) = 4(1000) -9(100) + 0(10) -6(1)$

Multiplying in Expanded Form

Multiplication can be done using the same principles. I think it is actually easier to see.

Simply carry out the multiplication using an extended version of the distributive property. This means that each term in one factor must be multiplied every term in the other factor. This could be called the Each With Each rule since the total product is the sum of the products of each component of the first number multiplied by each component of the second number. Demonstrating the Each With Each rule can be done using the area model. It is beyond the scope of this essay but it could be demonstrated to students.

I have set up a multiplication array to show this more clearly. What I am demonstrating in the array below is 4,082(988). I am however breaking the condensed numbers up and rewriting them in expanded form:

{$4(10^3) \times 9(10^2) + 4(10^3) \times 8(10^1) + 4(10^3) \times 8(10^0) + 8(10^1) \times 9(10^2) + 8(10^1) \times 8(10^1) + 8(10^1) \times 8(10^0) + 2(10^0) \times 9(10^2) + 2(10^0) \times 8(10^1) + 2(10^0) \times 8(10^0)$

Multiplication Array

	$9(10^2)$	$8(10^1)$	$8(10^0)$
$4(10^3)$	$4(10^3) \times 9(10^2) = 36(10^5)$	$8(10^1) \times 4(10^3) = 32(10^4)$	$8(10^0) \times 4(10^3) = 32(10^3)$
$0(10^2)$	$9(10^2)\ 0(10^2) = 0(10^4)$	$8(10^1) \times 0(10^2) = 0(10^3)$	$8(10^0) \times 0(10^2) = 0(10^2)$
$8(10^1)$	$9(10^2) \times 8(10^1) = 72(10^3)$	$8(10^1) \times 8(10^1) = 64(10^2)$	$8(10^0) \times 8(10^1) = 64(10^1)$
$2(10^0)$	$9(10^2) \times 2(10^0) = 18(10^2)$	$8(10^1) \times 2(10^0) = 16(10^1)$	$8(10^0) \times 2(10^0) = 16(10^0)$

Once the multiplication has been carried out then addition has to occur. I have written in arrows to show how we add the like terms which means we are adding together the multiples of the same power of ten In the above problem, for 10^3, this sum would be $32(10^3) + 0(10^3) + 72(10^3) = 104(10^3)$. I don't add different powers of ten together here. This relates directly to the algebra example later in the unit. The same problem arises as with addition before. We will have to subtract from the place value to the right and add to the place value to the left in order to get the correct numbers. The numbers must be placed in an array and lined up by their respective magnitudes. Notice how the powers arrange themselves diagonally. When we add up our like terms we get the following answer:

		$36(10^5)$	$+32(10^4)$	$+32(10^3)$	$+0(10^2)$	$+64(10^1)$	$+16(10^0)$
			$+ 0(10^4)$	$+ 0(10^3)$	$+64(10^2$	$+16(10^1)$	
				$+72(10^3)$	$+18(10^2)$		
Subtotal A		$\mathbf{36(10^5)}$	$\mathbf{+32(10^4)}$	$\mathbf{+104(10^3)}$	$\mathbf{+82(10^2)}$	$\mathbf{+80\ (10^1)}$	$\mathbf{+16(10^0)}$
Since we have ten too many ones we add one ten to the tens column						$+ 1(10^1)$	$-10(10^0)$
Subtotal B		$\mathbf{36(10^5)}$	$\mathbf{+32(10^4)}$	$\mathbf{+104(10^3)}$	$\mathbf{+82(10^2)}$	$\mathbf{+81(10^1)}$	$\mathbf{+6(10^0)}$
Since we have eighty too many tens we add eight hundred to the hundreds column					$+8(10^2)$	$- 80\ (10^1)$	
Subtotal C		$\mathbf{36(10^5)}$	$\mathbf{+32(10^4)}$	$\mathbf{+104(10^3)}$	$\mathbf{+90(10^2)}$	$\mathbf{+ 1(10^1)}$	$\mathbf{+6(10^0)}$
Since we have ninety too many hundreds we add nine thousand to the thousands column				$+ 9(10^3)$	$-90\ (10^2)$		
Subtotal D		$\mathbf{36(10^5)}$	$\mathbf{+32(10^4)}$	$\mathbf{+113(10^3)}$	$\mathbf{+0(10^2)}$	$\mathbf{+ 1(10^1)}$	$\mathbf{+6(10^0)}$
Since we			$+11(10^4)$	$-110(10^3)$			

have one hundred and ten too many thousands we add eleven ten thousand to the ten thousands column							
Subtotal D		***$36(10^5)$***	***$+43(10^4)$***	***$+3(10^3)$***	***$+0(10^2)$***	***$+ 1(10^1)$***	***$+6(10^0)$***
Since we have forty too many ten thousands we add four hundred thousands to the hundred thousands column		*$+4(10^5)$*	*$-40(10^4)$*				
Subtotal E		***$40(10^5)$***	***$+3(10^4)$***	***$+3(10^3)$***	***$+0(10^2)$***	***$+ 1(10^1)$***	***$+6(10^0)$***
Since we have forty too many hundred thousands we add four millions to the millions column	*$+4(10^6)$*	*$-40(10^5)$*					
Total	***$4(10^6)$***	***$+0(10^5)$***	***$+3(10^4)$***	***$+3(10^3)$***	***$+0(10^2)$***	***$+ 1(10^1)$***	***$+6(10^0)$***

Algebra Relationship (Multiplication)

The use of the distributive property is evident when representing this situation.

Multiplication is carried out by each term from one number with each term from the other number. In this problem a polynomial is being multiplied by another polynomial. Students should quickly learn how to multiply polynomials by using the array method. This will help avoiding having to teach the FOIL method with no reasoning. The reason that FOIL works is through the each with each concept. Here I again I am substituting the variable x for the powers of ten.

	$9x^2$	$8x$	8
$4x^3$	$4(x^3)\,9(x^2) = 36(x^5)$	$8(x^1)\,4(x^3) = 32(x^4)$	$8(x^0)4(x^3) = 32(x^3)$
$0x^2$	$9(x^2)\,0(x^2) = 0(x^4)$	$8(x^1)0(x^2) = 0(x^3)$	$8(x^0)0(x^2) = 0(x^2)$
$8x$	$9(x^2)\,8(x^1) = 72(x^3)$	$8(x^1)\,8(x^1) = 64(x^2)$	$8(x^0)8(x^1) = 64(x^1)$
2	$9(x^2)\,2(x^0) = 18(x^2)$	$8(x^1)\,2(x^0) = 16(x^1)$	$8(x^0)\,2(x^0) = 16(x^0)$

Compare the above array to the array completed with the powers of ten. Once the multiplication is completed then simply adding by using like terms is done and the problem is complete.

	$36x^5$	$+32x^4$	$+32x^3$	$+0x^2$	$+64x$	$+16$
		$+ 0x^4$	$+ 0x^3$	$+64x^2$	$+16x$	
			$+72x^3$	$+18x^2$		
Total	$36x^5$	$+32x^4$	$+104x^3$	$+82x^2$	$+80x$	$+16$

If I substitute a ten back in for the variable x then we will see the relationship to the expanded work previously demonstrated. The answer below is the exact same answer we obtained in subtotal A from our array example.

$36(10^5) + 32(10^4) + 104(10^3) + 82(10^2) + 80(10^1) + 16(10^0)$

Use of Integers and Base Ten

Most students in an honors level class should be able to solve basic integer problems. In this method students simply ignore carrying and borrowing when adding or subtracting. They rely on their knowledge of base ten and integers. In the below problem students are subtracting 988 from 4,082. Students have been taught to borrow to carry out this subtraction. I am showing that they do not need to borrow in order to carry out the operation. They can treat the subtrahend as a negative number and add the two numbers together. The only difference is they will be writing the problem and answers out in expanded form and then arrive at a solution.

Minuend	4(1000)	0(100)	8(10)	2(1)
Subtrahend(-)		-9(100)	-8(10)	-8(1)
Total **(Showing Base Ten value)**	**4(1000)**	**-9(100)**	**00**	**-6(1)**

Take the resulting numbers and add them together. 4000+ (-900) + (00) + (-6) = 3,100+ (-6) which if difficult can be expanded to 3,000 +90+10+ (-6) which is 3,000 + 90 +4 or 3,094. I would have the class work with various problems of this nature using addition, subtraction, and multiplication. The students should be familiar with expanded form and comfortable with demonstrating different ways of arriving at solutions.

Algebra Connection

Replacing the 10 with the variable x can help here also.

Minuend	$4x^3$	$0x^2$	8x	2
Subtrahend(-)		$-9x^2$	-8x	-8
Total	$4x^3$	$-9x^2$	0x	-6
Substitute a 10 back in for x	$4(10^3)$	$-9(10^2)$	0(10)	$-6(10^0)$

This answer is the same as our total in the array showing subtraction in expanded form.

This answer is the same as our total in the array showing subtraction in expanded form.

Divisibility Rules

Place value is also a powerful tool to use when determining divisibility rules. Rote memorization is a tool that teachers often rely on to teach students. They want students to learn by memorizing a fact or formula. I often comment in my class that this memorization method is like giving my students an orange rather than giving them some seeds and having them plant, tend and harvest their own oranges and in turn planting more.

I myself cannot remember all the divisibility rules but I can use place value to help determine them. What about four? Let's take the number 334 and see if it is divisible by 4.First write it in expanded form.

$300 + 30 + 4 = 3(25 \times 4) + 3(2 \times 4 + 2) + 4 = 4(3 \times 25 + 3 \times 2) + 3 \times 2 + 4$

The final expression shows that divisibility of 334 by 4 comes down to the divisibility of 3x2+ 4 by 4. Since 3x2+4 = 10, which is not divisible by 4, neither is 334.

If students do several more examples, some of them will probably see that 4 will always go into any hundred value as 100 is divisible by 4. Likewise, 1000, and 10,000, and all higher powers of 10 are divisible by 4. We can therefore focus on the tens and ones place. If we divide 30 by four we can rewrite it as 34 = 30 + 4 = 3x10+4 = 3x(2x4+2) + 4 = (3x2)x4 + 3x2 + 4 and we know that 6 is not divisible by 4 so 334 is not divisible by 4. What we did learn is that twice the 10s digit plus the 1s digit should be divisible by 4. Exploring how 8 would work might not be so hard if we understand how 2 and 4 are determined. The place value that becomes significant with 8 is the hundreds place. For 8: 4 times the 100s digit plus 2 times the 10s digit plus the ones digit should be divisible by 8. The number one thousand, ten thousand, one hundred thousand…. are all divisible by 8. If this is the case then the focus needs to be on the hundreds, tens and ones place. If a number represented in the hundreds place and lower is divisible by 8 then the whole number is.

When looking at whether a number is divisible by three the same approach can be taken. Let's take an Algebraic approach to this. If we use a three digit number then it could be written as 100a + 10b+ c where a, b, and c are the digits of the number. Here arithmetic really is being dealt with as algebra! Obviously 100 and ten are not divisible by 3 but 99 and 9 are so we could rewrite this number to be 99a+ a+ 9b+b+c and we have the same number. If we regroup the number we get 99a+9b +a +b +c. We know that 99a +9b is divisible by 3 so the focus would then be on whether a + b+ c is divisible by 3. This makes the rule that all we have to do is take the sum of digits of a number and if they are divisible by 3 then the number is divisible by 3. Here again determining the rule for 9 should be fairly obvious.

In order to investigate this take a number that we know is divisible by 3 such as 300. If we assume that because it is divisible by three it is divisible by 9 we might be mistaken.

3(100) equals 3(99+1)

Since 99 is clearly divisible by 9 a three is left over and is not divisible by 9. How does that help with developing a rule?

Go back to the aforementioned divisibility by 3 rule. With a three digit number 100a + 10b+ c is the same as 99a+1a+9b+1b+c and we know that 99a+9b is divisible by 9 so a+b+c must be divisible by 9. In simple words the digits have to add up to a multiple of 9.

Classroom Activities

Mathematical content and background knowledge has been provided previously in this unit. Teachers should feel free to make their own activities based on the ability of their class. The activities are broad in scope to allow for differentiation. Information provided relating to the decimal system and the concept of zero might be good introductory material.

Exploding Dots

In this activity students will be introduced to the concept of exploding dots. Credit should always be given to James Stanton the creator of the concept exploding dots.

Students should be shown a box and then dots added until they explode! The first machine that should be demonstrated is a 10 -> 1 machine. The teacher should not explain what is happening but should instead just place dots until they explode.

Frequently the teacher should ask students how many dots are actually represented in the box. Once the students seem capable of determining accurately the numbers represented the teacher can move on to writing a list of numbers and having students draw boxes to represent the numbers.

Discussion is important at this point as it should be obvious the numbers represented are the numbers we use every day in our number system. The concept of digits and what they are as well as the importance of zero. I would then explore other number machines such as 2 -> 1. Binary as this is called is critical to computer programmers as is base eight and base sixteen. Exploring numbers written in these different bases could be beneficial to students in understanding base ten.

The final activity I would do with exploding dots would be to do operations with the dots. Stick with just addition and subtraction using exploding dots as it will model operations with numbers written in expanded form later on. Use the problems in Appendix A labeled Practice A as support for instruction.

Expanded Form

Write the number 1,234,649 on the board and ask students to write out this number in expanded form. Students may inquire as to what expanded form is. I would tell them to write out what they say when saying the number, for example one million, two hundred and thirty four thousand, six hundred and forty nine. Now tell them to write the number out with only one digit then all zeros added to the other numbers. This would be 1,000,000 + 200,000 + 30,000 + 4,000 + 600+ 40 +9. We could then move on to break down the numbers further by writing them as powers of ten times the one digit. This would give us 1(1,000,000) + 2(100,000) + 3(10,000) + 4(1,000) + 6(100) + 4 (10) + 9. Finally we could show the powers of ten in their exponential form which would be $1(10^6) + 2(10^5) + 3(10^4) + 4(10^3) + 6(10^2) + 4(10^1) + 9(10^0)$.

It is important that students see the order to numbers in base ten and the importance of zero as a place holder. Students should practice writing out numbers in expanded form.

Operations with Expanded Forms

The teacher should start with addition and move on through subtraction and then multiplication. Teachers can make problems progressively more difficult or differentiate the problems based on the academic level of the students. I would start off with some simple problems and addition. The teacher must be careful to add numbers that will require carrying or the purpose would be defeated. Having students discover what to do in order to be successful might be made easier if the teacher used exploding dots and modeled actually what was going on. Students would then try to do model this when they performed operations with the numbers written in expanded form. Please note some sample problems in practice B.

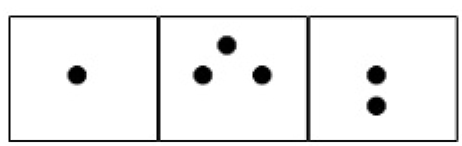
Integers and Place Value

The final concept of the unit is performing operations without carrying and borrowing or using place markers. It will reflect the students' ability to understand place value by performing operations using their own expanded form version of the numbers. Students should use problems from Practice C as a tool to help them understand.

Appendix A

Practice A Exploding Dots

If this is a 6 -> 1 machine what number is represented?



What number is this in base 10? ______________

2)

Using the boxes add 345+ 79. Add them together and show the unexploded view in box C meaning there could be more than ten dots in a box. In the total box show the total after you allow the dots to explode!

3)

Using the boxes subtract 345-79 by writing 345 in box A and 79 in box B. Write 79 as anti-dots .

Practice B

1) Using the numbers 2,345 and 345 write them in expanded form using the powers of ten and then

a. Add them

b. Subtract them

c. Multiply them

Be sure to show all of your work!

2) Using the numbers 43,699 and 7,366 write them in expanded form using the powers of ten and then

a. Add them

b. Subtract them

c. Multiply them

Be sure to show all of your work!

Practice C

Do the following problems without "borrowing" or "carrying." Write out your answers in expanded form.

1) 345
-98

2) 567
+78

Practice D

1) Express the following as a base ten number in expanded form and then determine two numbers that when added together total this number in expanded form. You will have to substitute the value of ten in for x.

$36x^5 + 19x^3 + 10x^2 + 9x$

1) Using the polynomial $-4x^4 + 6x^3 - 6x^2 - 5x^1$ determine two numbers, written in base ten expanded form, that when subtracted would give us this number written in base ten expanded form. You will have to substitute the value of ten in for x.

Practice E

Listed below is a problem that was presented to me in a Graduate level Discrete Mathematics course by Dr. Harold Reiter at the University of North Carolina at Charlotte. The problem is certainly a high level problem but it can be solved using the knowledge of place value and Algebra.

In the problem below abcde represent certain digits and their respective place values. The 4 also is a digit and its value is determined by its location or place. Solve for abcde.

4(abcde4) = 4abcde

Appendix B

Properties of Addition and Multiplication

When teaching this unit both the teacher and the students should be familiar and comfortable with the properties of addition and multiplication.

Rules for Addition:

Commutative Property a + b = b + a

Associative Property (a + b) + c = a + (b +c)

Identity Property a + 0 = a

Additive Inverse Property a + (-a) = -a + a = 0

Rules for Multiplication:

Commutative Property a × b = b × a

Associative Property a × (b × c) = (a × b) × c

Identity Property a × 1 = a

Multiplicative Inverse Property a x 1/a = 1

Distributive Rule a(b + c) = ab +bc

Works Cited

Bellos, Alex. *Here's looking at Euclid: a surprising excursion through the astonishing world of math*. New York: Free Press, 2010.

This book provided some historical context to place value as it relates to Babylonians.

"Decimal | Define Decimal at Dictionary.com." Dictionary.com | Find the Meanings and Definitions of Words at Dictionary.com. http://dictionary.reference.com/browse/decimal (accessed July 15, 2011).

Used this website to determine the origin of the word decimal.

Ma, Liping. *Knowing and teaching elementary mathematics teachers' understanding of fundamental mathematics in China and the United States*. Mahwah, N.J.: Lawrence Erlbaum Associates, 1999.

This book was helpful to compare teaching strategies of American teachers to eastern teachers.

Smith, David Eugene. *History of Mathematics* . New York: Dover Publications, 1958195153. Print. This book was helpful in providing historical information to zero and the development of numbers.

" Thinking Mathematics!." Thinking Mathematics!. http://www.jamestanton.com/ (accessed July 11, 2011).

Notes

1. Decimal | Define Decimal at Dictionary.com." Dictionary.com | Find the Meanings and Definitions of Words at Dictionary.com. http://dictionary.reference.com/browse/decimal (accessed July 15, 2011).
2. Bellos, Alex. *Here's looking at Euclid: a surprising excursion through the astonishing world of math*, 49.
3. Smith, David Eugene. *History of Mathematics*, 259.

4. Smith, David Eugene. *History of Mathematics* ,267.
5. "Thinking Mathematics!." Thinking Mathematics!. http://www.jamestanton.com/ (accessed July 11, 2011).