Curriculum Units by Fellows of the National Initiative
2011 Volume VI: Great Ideas of Primary Mathematics

# A Fraction of What We Know

Curriculum Unit 11.06.08, published September 2011
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## Rationale

Fractions are a lot like fruitcake. A few people like them, but most greet their presence with something between a blasé and a bubonic reaction. Also, akin to a fruitcake, a fraction is made up of multiple parts, and those subparts can be seen as having yet more subparts (which those who do not like fractions find just as unappetizing), yet together they constitute a dense, singular entity. Ultimately the choice of what to do with a fraction, like a fruitcake, varies between consuming it as a whole, nibbling only as little as one must to be polite, or, as many students studying mathematics in our schools are prone to do, throwing it away and looking for something else to eat.

To get beyond metaphor, what I hope you might have a mental image of for a fraction, is some numerator and denominator pair with a line segment in between. If we say "fraction" in general these may be any values, but if we qualify it as a "rational fraction" we signify having integer values for each; possibly with variables standing for integer values, e.g. a/b, as we are often given as a model in algebra texts. [1] I am hoping you've pictured this because it is "rational fractions" that I want to discuss most in this paper, though I commonly say "fraction" in much of the text below.

Fractions are an important, common form of written rational numbers. When we state that a number is rational we intend to state that it represents a "ratio," between other numbers or quantities, not that it is "reasonable." [2] Indeed, for my students, they can seem quite unreasonable to work with them at times.

Over the past five years I have taught mathematics for students from 6 th through 12 th grade in a variety of secondary curricula. In every course I have taught, students have expressed frustration and confusion with fractions, and their success with mathematics study has been demonstratively hampered by poor skills in managing fractions. In the coming school year I will teach Algebra 1 and Algebra 2 courses at a magnate school for the creative and performing arts. Two of the more pronounced challenges to teaching secondary mathematics in this academic environment have been: 1) students falter with some primary skills to such a degree that it prevents their segue into secondary studies; and 2) many students I teach, who are artists and generally more inclined to humanities, have found secondary math courses to be unengaging and/or inaccessible.

I see fractions as an ideal vehicle to help my students attain more mastery of a ubiquitous topic from their primary skills, while at the same time using this mastery to find engagement with and access to algebra and advanced secondary studies. I will approach this by first considering some difficulties that my students have had with notation, language, and the way in which they have studied fractions in their primary school experience. Then I will state objectives and present strategies, with specific examples of assignments that I will use and other teachers may find helpful.

**Notation of Fractions**

The mathematical notation of a fraction can be very confusing to students, simply because it uses two value symbols (i.e. numerator and denominator) to represent a single value in a unified symbol (the rational fraction). As with any number and its symbolic representations, when we start to apply its value we discover that it can represent many types of complex relationships in many contexts. What started as a simple little alphanumeric bit of data can become very bewildering, very quickly.

According to an unofficial theorem of mathematician Roger Howe, the power of mathematics lies in its ability to compress large amounts of information into relatively very little notational space. [3] The symbolic form for any particular number has the ability to represent virtually any quantifiable thing. Equations and formulae have the capacity, in brief sets of alphanumeric sequences, to describe, organize, and predict results for vast, vast amounts of data that might otherwise require page upon page of verbal explanation.

The use of fractions allows helpful compressing of information in mathematical notation, but in the power that they bring to compressing, they create new challenges for the mathematics learner by way of the necessary, inverse acts of decompressing that information. Because the fractional notation can serve to represent many possible relationships, students are faced with a paramount task of interpretation and evaluation of multiple, potentially valid meanings when they "unpack" the compressed notation.

In the most basic sense, there are three key ways in which we might consider what the notation of fractions "looks like" to students. The first is illustrated in the following sequence. It has what I believe is an obvious clue to part of the problem students have with fractional notation:

$1 + 2 + 7 + 1/5 + 6 + 4 + 4/5 = ?$

As we read from left to right, we begin making easy progress until we meet the first fraction. Our "horizontal road" is suddenly obstructed by a "vertical wall." In the midst of what has become an intuitive act of adding numbers, even we who are teachers of mathematics must take a subtle pause, and decide what to do.

We can attempt to keep on flowing and allow the fraction to tag along on our current sum, forming a mixed number in transit. We can use the associative property of addition to regroup the integers in one summation and the fractions in another summation. We can also modify the given notation, either rewriting the integers as improper fractions with the denominator of 5 and the appropriate numerators for equivalent values or rewriting the fractions as the decimals 0.2 and 0.8 respectively.

Going through each of these processes successively is something that could serve as a reasoning exercise for students. Finding the sum of the problem is not meant to be difficult, but if we ignore the fact that it demands choices on the part of our students, even secondary students whom we expect to be competent with such primary tasks, then we may well miss the chance to identify their difficulties with fractions.

The second note that I want to make about the writing of a fraction is really a matter of typography or calligraphy. We might take for granted that the following three forms of fractional notation are interchangeable, but some students infer greater significance around the variations:

$$\frac{1}{2} \qquad ½ \qquad 1 / 2$$

The first, in the most vertical form, typical of the numerator "over" denominator conceptualization does not bear any functional difference from the second, in which the bar separating the "top and bottom" is diagonal and seems to condense the fraction (which could be seen as fixing it as a rational number). The third form can indeed be interpreted as more strictly representing "division of 1 by 2" than the first and second forms, but, for all that, it does not limit those other forms from representing the same division. It is also possible that the third form was necessitated by a word processing issue or the result of a style choice and had the intention of representing a rational number. Ignoring these variations in notation might be another missed chance at helping a student in accurately decompressing the meaning of a fraction.

The third issue that I call attention to in the notation of fractions is the misapplication of the decimal form. Whether it is because students use calculators so frequently and so early in their math classes, as I am inclined to believe, or because they see a unifying quality in the place value notation of decimals, as a continuation of that which they use for units, tens, hundreds, etc., students gravitate in large part toward decimal forms as a means of dealing with fractions.

In cases such as the example above, i.e. when an option was to convert to 0.2 and to 0.8, there were no units or context to refute this interpretation of the abstract and the conversions gave exact values. Confidence in this always being true is dangerous, e.g.:

1/3 of 1002 participants in a poll voted for candidate A. How many voted for A?

The tendency to convert 1/3 to 0.3 can lead to real problems of accuracy, since 0.3 converts far too neatly back to 3/10 and would tell us that 300.06 of the participants voted for candidate A. Converting to 0.33 might seem better, but it converts back to 33/100 and would tell us that 330.66 voted for A. Adding more places of significance, e.g. 333/1000, etc. would close these estimates in on the actual value of 334, which we get by properly applying a reasoning of 1/3 of 1002 participants, but this exhaustive approach would be an inefficient and ironically futile considering that students are trying to avoid such burdens.

If students have a strong enough conceptual understanding they will be able to directly divide 1002 by 3 and be successful in both the mitigation of a fraction and in processing the problem with respect for units and context. The moment when students so much as lay eyes upon the notation of a fraction they run into issues, before any mental processing of values even begins. Resolving this issue requires both visual and verbal understanding.

**Language of Fractions**

Students often do not feel ownership of the language that is used in mathematics and this is certainly true of their work with fractions. While many are able to phrase ideas in terms of "over" and "under," "out of," or "above" and "below," those students who struggle with proportional reasoning do not usually make effective or accurate use of formal terms descriptive of fractions, e.g. "ratio," "proportion," "quotient," "dividend" and "divisor," or of "reciprocal" and other important terms that arise in our consideration of fractions.

There is, fortunately, some language related to fractions that students are comfortable with and these terms may reveal clues about what we can do to help students find comfort with math jargon that surrounds fractions. Perhaps the most obvious and oft used examples are those of money. A "quarter" of a dollar is of the most accessible fraction for secondary students that we can expect. Another example is a "half" gallon, be it of milk, orange juice, or other product measured by volume. What makes these fractions easy while others are more difficult? The obvious answer might include the fact that these fractions are embodied in our everyday life by tangible objects, but there is more to it.

I suggest two aspects of these fractions that make them more easily understood than others. First, they are verbally managed numbers that can be used without necessitating a numeric representation through which to be interpreted. Second, at the root of this verbal, relative ease with quarter dollars and half-gallons is the fact that their names and their measures have become units unto themselves, such that we do not necessarily think of them as subdivisions or parts of greater whole units.

It would rob these student understandings of value if they were merely a convention of speech and students could not manage their use mathematically. However, there is ready understanding in the majority of students that there are 4 quarters in a dollar and 2 half-gallons in a gallon, so the automaticity that students demonstrate with these fractions is not a mere naming scheme, but also exemplifies that they are fluent enough to apply the appropriate multiplicative inverses. I believe we can use these verbal and 'stand-alone-unit' qualities to build student understanding of all fractions.

Language plays some other obvious, but critical roles in students' fraction management. Students struggle with fraction-word problems because they have not practiced them in significant contexts. [4] This is especially detrimental for secondary students developing proportional reasoning because although the application of fractions, usually in the forms of percentages, are presented as statistics in a variety of ways but without a real analysis of their values or what they mean in context.

As a last thought on language, it is important that students simply understand "what" they are studying. In his book Innumeracy, mathematician John Allen Paulos introduces what he considers some of the worst misconceptions about mathematics, the second of which is that mathematics is a completely hierarchical subject in which one topic or skill set follows another without commingling. [5] Students' general notion that they are learning "normal math" for years, and then suddenly are doing things like "algebra," works against their comprehension of the grand scheme in which they are working. Students should learn the terminology to appropriately consider their "math" studies.

According to Howe, "We will do well to regularly point out to students the methods we employ that are part of arithmetic and where and how they transfer to skills we would appropriately term algebra." [6] This could be very helpful to students in finding the nuances within the reiterated skills of managing fractions, because students might better understand a step forward is not an abandonment of previous skill sets.

**Primary Approaches to Managing Fractions**

Liping Ma, an international education researcher, claims that students use among three approaches to developing arithmetic skills: counting, memorization, and, what Ma calls, "extrapolation," by which she means a development of reasoning around properties of numbers (which are actually informal applications of both arithmetic and algebra), which is reliant upon student rigor, but neither upon the exhaustiveness of counting nor the inflexibility of memorized facts.

Students struggling with extrapolation in general are all the more prone to struggles with fractions, thus our efforts to address fractions in the algebra environment need to go beyond yet more attempts at rote learning of algorithms or drill practice for dealing with fractions. Of course, there does need to be forethought for the time that any version of an extrapolation approach requires. Ma points out that it relies not only on students having prior experience in taking this "open-ended" attitude toward learning new skills, but also additional time at each stage of study… implying efforts throughout algebra studies.

When we have difficulty justifying extrapolation, over algorithmic approaches, to administrators or skeptical colleagues, we do have some very supportive statements via the NCTM: "National Council of Teachers of Mathematics asserted in their Curriculum and Evaluation Standards (1989) that proportional reasoning is of such great importance that it merits whatever time and effort must be expanded to assure its careful development." [7] In other words, learning how to manage fractions is worth the time. I also hope to show in my strategies below that it can be coordinated in coursework so that it needn't "take time away from" learning algebra skills.

We should also expect lasting rewards for this investment of time. The perspective of solving the known from the unknown, i.e. using an open-ended problem solving approach based on what we know with confidence v. unqualified algorithmic exertions, initiates a lasting attitude toward learning and mathematics. [8] If this type of attitude toward the notation and communication of rational numbers can be established through work with fractions, it can only be a positive reinforcement of a similar attitude toward all algebra work that involves them.

The beginnings of primary mathematics education, thus the beginnings of arithmetic, rely on "perceivable quantities," numbers that students have a working sense of (i.e. 1, 2, 3…). [9] As students enter algebra studies, we must recognize that for many of them fractions are not perceivable quantities. In order to make fractions perceivable, students must understand them via those integers that constitute their numerators and denominators, and from a proportional sense of the unified symbol.

Susan J. Lamon wrote the book on teaching fractions (well "a book," but an incredible one) in which she says, "Understanding fractions marks only the beginning of the journey toward rational number understanding, and by the end of the middle school years, as a result of maturing, experience, and fractional instruction it is assumed that students are capable of a formal thought process called proportional reasoning." [10] This assumption, though it would be lovely to be able to hold on to, is clearly invalid in many cases. Rather than taking a reactionary approach toward primary skills that we find lacking in our students, we need to be proactively incorporating them into our pedagogy.

## Objectives

I want my students to be able to deal with fractions effectively, whenever and wherever fractions are encountered in algebra or higher secondary studies.

It would be nice to leave it at that, but, of course, it would also be an impractically simplified statement of objectives. I propose two subdivisions to my overarching objective, as refinements that speak directly to improving effective student work with fractions in algebra studies.

First, students should have a more expansive understanding of fractions than as mere part-to-whole expressions. This idea of fractional notation, most prevalently taught in the elementary levels of math study in the US, becomes so fixated that students have difficulty moving from it to other valid interpretations of the form. So the overarching objective is extended to improve students' understandings of fractions in all their various purposes, as division, ratio, etc., via examples and practice in applying these interpretations. Fortunately algebra studies provide fractions as expressions of these relationships in many contexts, so the content is readymade for such adaptation.

Second, students should have a skill set for management of and communication about fractions that is built upon an extrapolation approach. Not only should students be able to recognize all the interpretations that are possible for a fraction, they should be able to accurately choose the appropriate interpretation to apply within the solution of any given problem, based on their "reasoning out" of the problem context. In other words, as students gain an expanded concept of what "a/b" might mean, they need to gain a complementary, diverse heuristic that allows them to choose the accurate meaning. Fortunately, again, algebra studies provide the opportunity to build this skill set..

## Strategies and Assignments

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The absolute key to a productive strategy for introducing remedial primary work into secondary curriculum is to keep "review" situated in "new learning." It is essential to be addressing the concepts and skills of managing fractions precisely where they naturally occur in the course of study that you are instructing (e.g. using rational fractions as coefficients and constants while solving for a variable in an equation, as opposed to stopping work in order to complete drill in performing operations with fractions). While some students will struggle with this rigor, it is an actively engaged means for overcoming their struggles with fractions and a far more meaningful approach. Basically, use fractions when and where they are most useful to the work you would have students do regardless, and accentuate work you would already have students do by using rational numbers.

In the sections below I consider how an instruction of fractions can take shape in algebra context. I will begin with ways to conceptualize fractions, then I'll consider both the notation and language that students use for fractions. Finally I will present examples of assignments that I plan to use in my own classroom. Each section is titled with a question, suggestive of what I hope students will answer in their work. The examples provided are meant to illustrate teaching approaches, but are far from comprehensive.

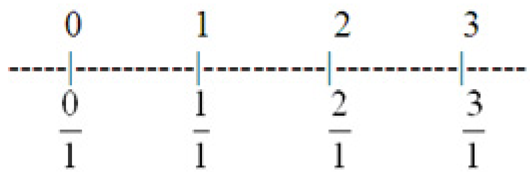
**What Does the Fraction Mean?**

As noted above, primary students in the US have a tendency to move on to middle and high school with a concrete ideation of a fraction as a part-to-whole identity, or a rational number with a fixed place on the number line. Actually, the mentality to consider the number line in this way is to be hoped for, but requires some care so that it can be used, first to expand on "part-to-whole," then to "see" the fraction as more than part-to-whole.

*The Number Line – Yardstick to Understanding Fractions*

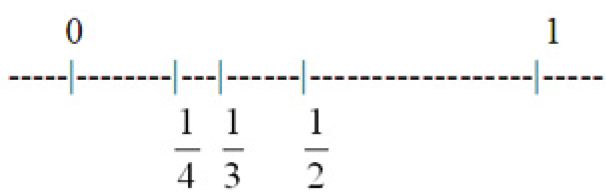
A major tool for building student concepts about rational fractions is the number line. There are many places where number lines naturally appear in an algebra course and they can be introduced as a graphic organizer at practically any point in our teaching, without hindering progress or altering content. Most importantly, number lines can readily model several concepts that we expect secondary students to have as a basis for instructing them in better management of fractions. These concepts are: the unit; the unit fraction; general fractions; equivalence of fractions; and fractions as division.

I've suggested that the first time students truly interact with fractions actually comes in the simple act of counting in whole number units up to 2, creating a relationship between 1 and 2 that is rational. The number line can be demonstrative of this defining of "units" the negligible denominator of "1" being added to make rational fractions of the positive integers (and/or all integers).



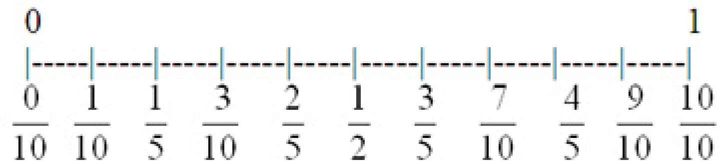

By understanding that integers can be expressed as rational fractions, students are building off of the most basic sense that they possess of perceivable quantities, recognizing ratio relationships of factors and multiples (albeit only in the most basic factor of 1), and implicitly using the associative property of multiplication to reconsider their most basic math knowledge. It is a brief step from understanding "units" to the "unit fractions" of one-half, one-third, one-fourth, etc., which are familiar to students.

These "singular" part-to-whole identities of rational fractions are of course also well modeled by number line representations and using such representations might serve to root understandings we expect high school students to already have of unit fractions (i.e. 1/n, where there is 1 part of "n" equal divisions of a whole). The following is a very basic example, but one that may be useful as a reminder to students of the work that they have already done with fractions. It might also be considered a means for discussing inverse operations in algebra, since the relationship of each unit fraction to the whole unit can now be understood in either additive or subtractive direction… from part-to-whole or from whole-to-part.




Once students are confident with both units and unit fractions, "general fractions" (i.e. m/n, where there are "m" parts of "n" equal divisions) can begin to be understood simply as multiples of unit fractions. Students understand multiples of integers quite firmly, unless something has gone horribly wrong, by the time they enter algebra study. Thus they should be ready to use these sensibilities with rational numbers. In many ways, if students have clear understandings of the intertwined relationship of units, unit fractions, and general fractions some other issues they might have in algebra will be obviated. For instance, 3 is three 1s, and therefore 3/15 is three 1/15s. Such nuanced understanding as is actually representative of students understanding the distributive property.

Students' understandings of these three connected concepts can be extended into demonstrations by using a number line to show unit fractions and general fractions as part of a common set of related fractions. The following example might be useful to early algebra studies because it shows a pattern in the number sequence of the denominators. Analyzing such sequences is a particular focus in algebra courses, as a means of identifying functions and deriving values algebraically. This example also presents students with another way of imaging the base-ten system as a set of fractions.



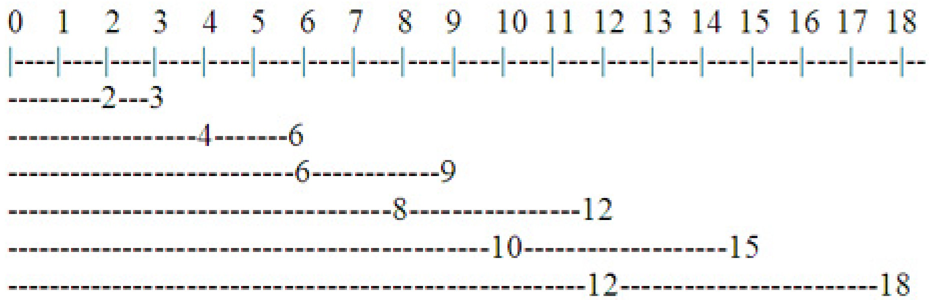

Another feature of the example just given is that it shows that certain conversions have been made in order to create the given sequence. These "simplifications" have given preference to rewriting the fractions in their most basic, irreducible forms of a ratio between the numerator and denominator. We frequently ask students to produce solutions or answers in simplest form, and this example can be used to demonstrate the utility of doing so. If one of the fractions with a denominator of 5 was not in simplest form, e.g. 1/5 was written as 2/10, the sequence of denominators would no longer be palindromic. Of course, if they were all written with a denominator of 10 there would be a uniform palindrome to their sequence. Thus the recognition of such patterns in algebra, e.g. palindromes, can be used as an indicator of "good form" in fractions.

Other consequences of these simplifications are that, since they have allowed two fractions to be interchangeable they imply substitution, a concept near and dear to the heart of algebra, and they lead directly into the concept of "equivalent fractions." The concept of equivalent fractions ranges beyond renaming a single place on the number line however. In order to fully conceptualize equivalence of fractions more examples are necessary to show the transitive property of equality that such fractions embody.

Number lines can be useful in showing that multiplication is simply a stretching of number values. [11] For fractions, number line representations can demonstrate why fractions are "equivalent" by showing that unit fractions can be "added up to reach the same place." Alternately, a number line can show how the rational relationship, that is represented by a rational fraction, can be interpreted as infinitely many pairs of numbers set apart at "equivalent" ratios of distance. The following two examples are both for the rational fraction 2/3:

As one distinct "place" with many equivalent "names" on an interval number line;


Or as an infinitely expressible ratio relationship of multiple pairs on the number line.

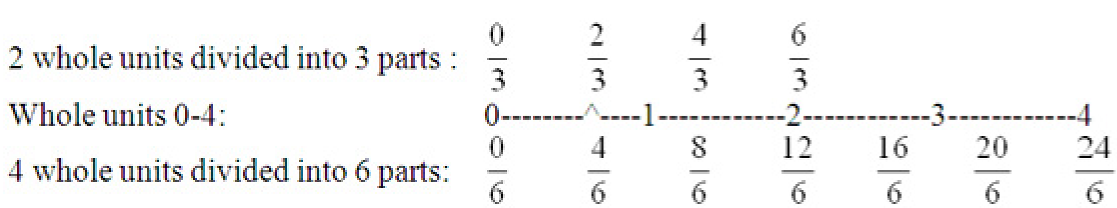



etc. Specifically, we mean that in each of the intervals indicated above, the subinterval cut off by the labeled point is 2/3 of the whole interval.

With these methods for demonstrating units, unit fractions, general fractions, and equivalent fractions to students, we will be better prepared to take on the concept of a fraction as division. It is a major conceptual chore for secondary students to break from the idea of a fraction as only a part-to-whole relationship and move toward seeing fractions as quotient values or division operations.

In much the same way that number lines can demonstrate fractions as rational numbers of a fixed location, number lines can demonstrate the location of a quotient value for the division of the numerator by the denominator. The number line can maintain this quotient value for equivalent fractions as well, providing another demonstration of why fractions are said to have "equivalence" while showing that the division of either numerator by its respective denominator results in the same quotient.

As a very simple example (using 2/3 again), it is possible to show why 2/3 is equivalent to 4/6, and how these two fractions may be viewed as equivalent division statements rather than merely as part-to-whole values. Using the established idea that whole "units" can also be expressed as fractions, on one number line we can show the equivalent results of dividing 2 whole units into 3 equal parts and of dividing 4 whole units 6 equal parts, with 2/3 and 4/6 aligned at the same place on the number line.




For simplicity and sufficient spacing, I have used 12 dashes to subdivide the space between whole integers. The point where I've put a carrot symbol, between the 8 th and 9 th dash, between the integers 0 and 1, is thus 8/12, another fraction equivalent to 2/3 and 4/6 that might be added to the diagram. In this case it would show 8 whole units divided into 12 parts. Any number of equivalent fractions can be added in this way to demonstrate that the fraction as division is an important understanding of what a fraction represents, and that fraction as division connects directly with the concept of equivalence of fractions.

*Linear Functions: Equations, Tables and Coordinate Graphs as Fractional Reasoning*

Use of a linear function to demonstrate the meaning of a fraction is highly practical in algebra course work. Because of the emphasis given to multiple representations of a linear function (i.e. the equation, table and coordinate graph), use of these functions as a model for fractions provides an immediate diversification of ways in which to see a fraction working. It is also in the management of linear functions that all the notions of a fraction, discussed above via number lines, might come most naturally into play for a central element of Algebra 1 course content. I'm referring, of course, to slope.

The way that we understand the slope of a line is to consider the ratio of its "rise over run," a fundamental fraction. Because the slope of a line in two-dimensional space inherently behaves as a fraction, even when the numeric value of the slope is given as an integer, it acts out, as a fraction, the entire series of representations I have given above with number lines. This should not be neglected in how the concept of slope might be introduced to students. The multiple representations of linear functions can help them visualize fractional reasoning; and, conversely, fractional reasoning can help them in their work with the representations of linear functions. If students can appreciate the utility of fractions in making their work with linear functions easier.

The following example, using a linear equation and a corresponding table of values, can illustrate how this might be approached. I use 2/3 again for the sake of comparison with the examples above. I also keep the example simple by nominally having a y-intercept of 0, which I recommend for initial work but which can quickly have rigor added by using more complex linear functions.

Equation (in slope-intercept form): $y = \frac{2}{3}x$

Table of Values ($0 \le x \le 6$):

| x | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| y | 0 | $\frac{2}{3}$ | $1\frac{1}{3}$ | 2 | $2\frac{2}{3}$ | $3\frac{1}{3}$ | 4 |

If this table were extended, the y-value 6 would occur with x-value 9, 8 with 12, and so forth, just as in the pairs of intervals illustrated above. This can be discussed with students, so that they understand that if they extend this table they will find, for every fraction a/b that is equivalent to 2/3, that there is a table entry with b in the top row and a in the bottom. Thus the line of the linear function essentially represents every fraction that is equivalent to 2/3.

Analysis and discussion can be carried on to evoke thinking about the fraction and how it "behaves." In looking at the table students can identify patterns (again, a hallmark of algebra studies) in the multiples of 2/3. This short sequence illustrates the notion of the fraction part-to-whole being added up (e.g. three 2/3s add up to 2), as well as the notion of equivalent fractions both simply, e.g. 2 is to 3 as 4 is to 6, and, more complexly, e.g. 2/3 is to 1 as 3 1/3 is to 5.

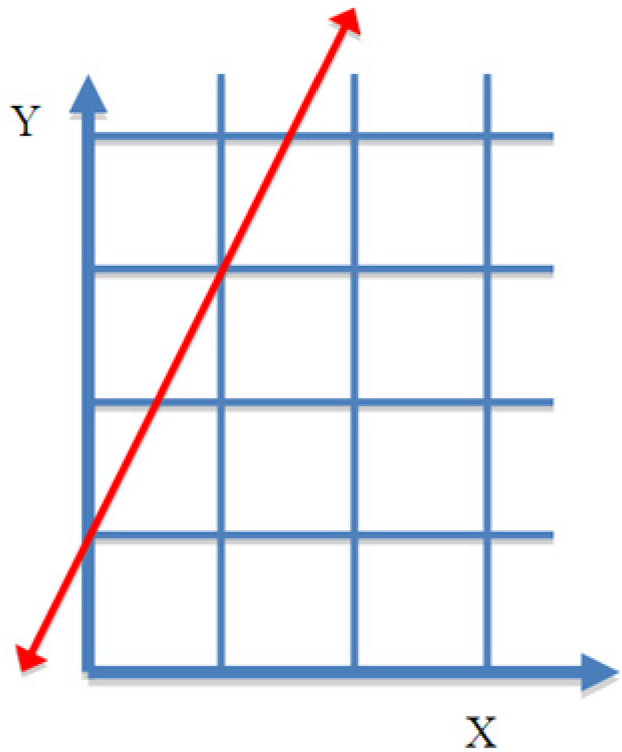
Again, there is a great deal of importance to student understanding of the fraction as division. This is well represented by the transition of values along a row of the table (e.g. paying attention to the "y" row as the "x" row values decrease from x = 3 to x = 1, it is clear that 2 ÷ 3 = 2/3). And it is also evident that the fraction acts as division in the form of a multiplicative inverse between the "x" row and "y" row (e.g. 2/3 of 6, or 6 times 2/3, or subsequently 6 times 2 and that product divided by 3, are all equal to 4)

In using this type of conceptualization with students, I see myself more like an interpretive guide on a nature walk than as a direct instructor. I will ask students what they notice and encourage them to find as many interpretations of the fraction as possible. This will serve not only to expand and strengthen their sensibilities about rational fractions for future use, but should have an immediate impact on their skills for interpreting information between equations and tables.

Fractional reasoning will go on to serve students well in their work with linear functions as we guide them in interpreting and creating coordinate graphs. Student understanding of unit fractions as being "divided into 'equal' parts," and general fractions as "multiples of these equal parts," can help them understand why intervals on the axes of a coordinate graph must be regular, i.e. with equal distance between units. Students who struggle with understanding that regular intervals are necessary for a coordinate graph to accurately represent a linear function can also be referred back to the ways in which they have already seen this represented on number lines. It is worthwhile to explicitly point out to students that the x and y axes are simply number lines set perpendicular at an origin point, and as such must have "even spacing" just like other number lines.

It should be made clear to students that the line drawn for a linear function can serve as a third number line representation, consisting of points whose values reflect the rational relationship between the corresponding x and y values along the axes. As a general approach for discussing fractions throughout student work with linear functions in algebra, I suggest using proportional reasoning as the avenue for students to first recognize what a slope is (versus presenting a definition).

Before throwing the word "slope" at students, give them the graph of a simple linear function (e.g. the function: y = 2x + 1):




From an accurately drawn coordinate graph of the given function, most any student will be able to see that there are 2 whole units of change along the y axis for each 1 unit of change along the x axis. So long as students understand that the measure of one unit of the x axis is equal to the measure of one unit of the y axis, they can incorporate their most primary counting skills to "see the slope." Even without labeled intervals as shown here, it is possible to have students "excavate" the ratio of 2:1 for the differences between "y" values and "x" values.

This also reenacts the number line representation of units given above, in which units are written as fractions with the common denominator of 1, while going a step further and farther to introduce this orientation in a way that insinuates that fractions are the most natural, intuitive numbers that can be worked with in two-dimensions (although perhaps that is somewhat overly optimistic). Students can, in this way, write their own definitions of what a slope is and have the concept and term unified by knowledge about rational fractions.

Students will then be better equipped with understandings of units and the way in which fractions are composed of ratios of units, in order to resolve examples such as, "Find the slope between the points (4, 9) and (1, 3):"

$$\frac{y_1 - y_2}{x_1 - x_2} \Rightarrow \frac{9-3}{4-1} \Rightarrow \frac{6}{3} \Rightarrow \frac{2}{1}$$

Strong proportional reasoning is necessary for students to take their understanding of the example above and apply it to obscure coordinate points, such as (19, 39) and (-3, -5). When students can correctly translate the coordinate notation to fractional notation, they will be able to resolve 39 -(-5)/19-(-3) to 44/22, and thus 2/1, almost as easily as the simple example illustrated above. I should also note that in these examples I use a convention to make the larger absolute values into minuends and smaller absolute values into subtrahends for the subtraction operations, since the order was arbitrary in the act of finding the absolute differences for y and x values respectively. This is not necessary, but it can simplify student work by reducing how many negative integers that they might otherwise have to manage.

A last point, before moving on to considering fractions in mathematical notation and language, is to help students anticipate dealing with rational numbers. Early on in an algebra study, students should be forewarned to expect rational number answers when a teacher knows that they will appear as the correct solutions. Students often believe that they have made a mistake when they arrive at, e.g. x = 4/5, because it is unsatisfactory to their conditioned sensibilities not to have a nice round integer.

Asking students to look for fractions where they might not be obvious (i.e. no coefficients or constants deviate from integers in the original problem) is a way not only to prevent confusion, but to spur proactive critical thinking about when and why rational numbers (versus integers) make sense either as values in intermediate steps, or as final problem solutions. I often find the forensic evidence of correct, rational number answers scratched out or erased on students' papers, while they sit and stare at a problem trying to discern where they went wrong. Simply preparing students to get away from neat, "easy" integer values is a critical step toward their negotiation of fractions in algebra.

**What Is the Notation Trying to Tell Us?**

I consider the examples demonstrating fractions in linear functions above to be highly suitable for algebra courses. However, the place that I will most directly address my students' difficulties with rational fractions is where it appears, nonchalant and as presumptuous as a party-crasher, in the midst of problems that I know students would otherwise have efficacy and success with. Students in the early stages of an algebra course typically are not overtaxed, for example, by solving for the value of "x" when presented with: 2x = 10. They know that the solution is x = 5.

However, when presented with: 2 = 10/x, many students take a pause, either brief or indefinite, the length of which is in direct variation with their difficulty in processing the solution x = 5.

Verbally, we can help students sort this out by asking, "What should you divide 10 by to get 2?" Early work in Algebra 1 courses often involves students interpreting back and forth between written statements and mathematical expressions, and such questions can encourage intuitive access for students.

Also early in most Algebra 1 courses, there is formal reintroduction of the algebraic principles of equality, etc. that students have hopefully used via extrapolation in arithmetic. The formal work with these principles can help students renegotiate their understandings of fractions and deal with them effectively in examples like this. Because in many ways these are reiterations of skills that students have already learned under other guises in their primary/pre-algebra courses prior knowledge should be an aid in this.

A typical approach that is taken in algebra is to perform inverse operations. In this case such an approach is reaffirming of the notion that a fraction is division and the inverse is therefore multiplication, which can be applied as the multiplication property of equality and leads students to notation they are already more comfortable with:

$$x * 2 = x * \frac{10}{x} \Rightarrow 2x = 10 \Rightarrow \frac{2x}{2} = \frac{10}{2} \Rightarrow x = 5$$

This method is effective, but there is some irony in that we began with a division problem, performed multiplication to both sides of the equation, and are then in need of a different division (i.e. 10 / 2) in order to complete the solution for x. This is a great time to ask students, "What other options do we have?"

In the 6th grade text series that my district uses (which contains the last primary, and therefore the most recent, study of fractions in our continuum), the term "fact families" is employed for this type of conversion.[12] It might be more efficient to encourage students to rely on this prior knowledge in order to convert the problem into another "fact" from the "family," in a form that leads more directly to the solution for the variable:

$$2 = \frac{10}{x} \Rightarrow x = \frac{10}{2} \Rightarrow x = 5$$

This can serve as a foot in the door, but it could be detrimental to leave this as the only option shown to students – thereby limiting their perspective of other valid and important solution paths. Building students' skills at decompressing the notation of fractions is critical at this point and exploring other options now will serve them well later in their dealings with future, novel problem contexts and formats.

This leads us, somewhat naturally, to use of the associative property of multiplication. Because we will have considered the students' comprehension of unit fractions and general fractions carefully, we can count on this "prior knowledge" to the degree that we would like to be able to do so. And, if we can trust students to understand that a general fraction is just "so many" copies of a unit fraction added together, then we can trust them to use the associative property as an option for resolving our current problem. This is ideal because the associative property provides the means of simplifying the student approach without making it devoid of meaning.

Relying, again, on the understanding of what a unit fraction and a general fraction are, students can factor out the original problem in order to reveal the skeletal structure of the fraction, so to speak:

$$2 = \frac{10}{x} \Rightarrow 2 = 10 * \frac{1}{x}$$

I freely admit, this conversion via the associative property does not make the problem immediately easier to solve via the mathematical notation, it isn't necessarily meant to. There are very serious, demonstrative advantages to having students carry out this solution, because the problem has maintained its fractional format, and thus retained its identity of "fraction as division," without relying on either purely computational approaches (multiplication) or any "trick" of prior knowledge (facts).

As to the possible dilemma from this step to the final solution – that the notation is not very well arranged for performing a mathematical operation that will produce the solution – I argue that this is a very good dilemma for students to face. Facing such challenges can guide them to mastery via an extrapolation approach. What we essentially have done in this conversion is "dismantle the engine and look at the parts to see how it works."

Students must think critically about the significance of the unit fraction in this scenario, having done so they must then consider that the balance of the equation is reliant upon a division of 10 into "x" equal parts, and subsequent to this they must recognize that the value of one "x-th" of 10 is equal to 2. There may not be one quick piece of notation that most students will be able to write as a segue from the notation above to "x = 5," however, students who work through the thought process just described will be able to produce the correct solution with little or no mystery as to how they have been able to reason it out.

I suggest using a discussion around the resultant equation, 2 = 10(1/x), in order to make students construct a reasoned out approach to solving for x, before beginning to apply properties of equality or making further conversions with the associative property. If students just begin using inverse operations without prior consideration, they might find the solution quickly but not understand how, or they might end up with an equally difficult equation to solve. Requiring students to first think-out multiple solution paths and then to complete formal steps in written notation for these paths is a great exercise to solidify comprehension. I suggest having students write a two-to-three sentence explanation of whatever solution path(s) they might take or reasoning that they might use to guide their formal notation.

For example, a student might begin with 2 = 10(1/x) and divide both sides of the equation by 2, resulting in 1 = 5(1/x), in which case they might recognize that 5 multiplied by its reciprocal 1/5 is equal to 1. Thus x = 5. Another possibility is that a student might begin, again, with 2 = 10(1/x), but divide both sides of the equation by 10, resulting in 1/5 = 1/x, in which case they might recognize that 5 and x must be equal.

There are obviously many, many examples of varying complexity that can be considered as an illustration of problems that students would find easy "if only it wasn't for those darned fractions!" Other examples might be worked out via the methods of inverse operations, prior knowledge of facts, and the rearrangement by associative property that I have just shown, or in other sequences of steps based on other algebraic properties. Having students do so will strengthen their abilities to apply these properties at the same that that they improve their mastery of fractions.

As with solving for a variable in an equation that is composed with just one fraction (on one side of an equation), the application of the associative property based on understandings of unit and general fractions, should be most helpful to students working with cross-multiplication scenarios (which present fractions on both sides of an equation). The reason for this delineation is, again, to pursue the objective of having an extrapolation approach to add to students' repertoire and/or can simply be thought of as a way of showing students "why" the "trick" of cross-multiplication works. It should be given to students as a parallel when teaching cross-multiplication. I give a simple example, using 2/3 again for easy comparison to earlier examples, and this time with a variable value as a dividend/numerator rather than as a divisor/denominator:

$$\text{Traditionally cross-multiplied: } \frac{x}{6} = \frac{2}{3} \;\Rightarrow\; 3\text{x} = 12 \;\Rightarrow\; \frac{3x}{3} = \frac{12}{3} \;\Rightarrow\; \text{x} = 4$$

$$\text{Modified via associative: } \frac{x}{6} = \frac{2}{3} \;\Rightarrow\; \text{x} * \frac{1}{6} = 2 * \frac{1}{3} \;\Rightarrow\; \text{x} = 12\left(\frac{1}{3}\right) \;\Rightarrow\; \text{x} = 4$$

The desired effect is that students will consider the obvious contrast between the unit fractions on either side of the equation and find the proper balancing contrast between the coefficient variable value "x" and coefficient known value "2" (i.e. since x is being divided by 6, twice as much as the 3 that 2 is being divided by, x must be twice as large as 2 in order for the equation to be balanced). This is an example that I would most definitely require students to write at least a brief, two-to-three sentence explanation of.

Granted, I have used such a simple example here that many students would be able to quickly identify x = 4 at a glance. If they do so by rote fact there is reason to be concerned that they will not translate this recognition to other, more complex, or unfamiliar examples. If, however students quickly see through this simple example by already recognizing that the denominators (6 and 3) simplify to the ratio 2:1, and then use this to multiply 2 by 2 in order to find x = 4, then I would argue that they are effectively using proportional reasoning. If so, all is well, so long as students can articulate their understanding. The breakdown to unit fractions and coefficients is meant to allow for more "breathing room" in the notation, with which students might air out their thoughts.

As students are thinking through various methods, it is the right time to prod them toward making critical comparisons between these methods. "In the US, the dominant practice is to solve simpler problems with arithmetic methods and later problems with algebra. the two are rarely juxtaposed." [13] Students should explicitly compare arithmetic and algebraic approaches, the similarities and contrasts in these solutions will reveal to students how the methods they have studied in the past are related to the new methods they are learning in algebra. Specific to managing fractions, this should give students the opportunity to evaluate algorithms they may have already internalized so that they will know when to use such steps and when they will need to incorporate new steps to deal with fractions effectively. This is supported significantly by Susan Lamon's comments on proportional reasoning of secondary students:

> Students with less than mastery of proportional reasoning ability can compensate using rules of algebra, geometry, and trigonometry, but are eventually susceptible to confusion and difficulty in dealing with real applications that do not require these higher math techniques, but do rely upon a robust sense of rational numbers and the proportions they are used to represent. [14]

Two special notes I wish to make about notation, not "of" but "related to" fractions are for the symbols "%" and "÷." Percentage seems relatively well within the grasp of students entering algebra course, yet the notion that it literally means "out of 100" is often lost on students nonetheless. I reiterate this whenever using percentage in algebra coursework as a means of fortifying proportional reasoning and point students toward taking a careful look at the symbol itself in order to recognize that it is in the shape of a fraction. The division sign provides the same chance for simple symbolic comparison.

Before moving to some quick thoughts on how language symbolizes fractions for students, I want to make an argument for a strategy, or perhaps an anti-strategy, that has been implicit to all that I have discussed thus far. We must demand that students avoid shortcuts, at very least, until they demonstrate proficiency.

Liping Ma articulates how students acquire skill sets is the following way, "When a more efficient strategy replaces a less efficient strategy, the previous strategy becomes useless. Thus, students do not accumulate these strategies." Ma is referring, with some specificity, to counting strategies in this statement, however it can clearly be seen as an expansive observation for how students negotiate prior knowledge and new skill acquisition. Unfortunately, when the "more efficient strategy" is to use a "computational crutch," they are beguiling themselves as to the purpose of studying mathematics. So long as we provide the guidance students need to learn extrapolation approaches, work employing algebraic properties in place of calculators will be the greatest strategy for students to improve their fractional/proportional reasoning.

**What Is the Language Trying to Show Us?**

I began this paper with metaphor and have used it throughout. In my teaching career I have found the method of making analogy to be incredibly helpful, particularly for students who express disdain for math and/or a preference for humanities. Where fractional notation is troublesome for students, we need to have more effective language to communicate concepts that are being lost, and more accessible language in the literary guise may be more effective than additional technical jargon. This is no less, and in some ways all the more, true for the topic of fractions and proportional reasoning at large.

I've referred to the "shifty nature" of fractions above. I have also discovered Susan Lamon's use of the term "slippery character" in relation to a constant of proportionality. Adopting language that suggests to students that there is indeed some strangeness about fractions is a validation of their concerns and creates an opening for them to speak, even as they have been muted by their sense of uncertainty. This makes for very real discussions that open up what misconceptions have been interfering with student work with fractions, and gives students confidence to question aloud. This coordinates well with a strategy from Liping Ma. "Create a dialogue environment in which children are relaxed, happy, and willing to actively communicate with you, in their own words, about their everyday life, and make up word problems with quantities..." [15] In order to be effective mathematics teachers we need to find ways to encourage students to participate in discussion at a level that they feel comfortable to enter it at, and then to guide them in advancing their level of communication about mathematics.

Most importantly, problems with fractions should be given in word problem contexts as frequently as is possible and sensible to the curriculum at hand. Assignments should be based in the act of student writing, prompting them to articulate what fractions represent as often as possible. As we avoid using purely algorithmic or formulaic processing of notation, we should also avoid algorithmic or formulaic verbal instruction (e.g. when multiplying one fraction by another fraction, phrasing such as "top times top, bottom times bottom."). I am not suggesting that such statements are untrue axiomatically, or even that they should never be used, but such verbal tricks cannot be relied on in isolation if we hope to improve students' proportional/fractional reasoning.

**When Do Assignments Using Fractions Fit Best Into Algebra Studies?**

I hope that I have spoken to this question significantly in the content above. Answering this, and answering how it is possible to work on fractions in algebra class without interrupting the algebra course content, have been my central tasks. In my school district there are prescribed, standard curricula for core mathematics courses, including Algebra 1, Geometry, and Algebra 2. The level to which teachers can exert freedom to actually write curriculum varies from school to school and district to district, but on the whole I expect that the issues of teaching remedial/primary topics in a secondary course are similar in most classrooms and I hope to have provided methodology and strategic guidelines for doing so.

Of the types of assignments we can provide to students in the classroom during a regular school day, I think most of "beginning, middle, and end," which translate to: pre-activity problems (what I have always referred to as "warm ups"); core activity problems; and quizzes/assessments of knowledge and skill. I will give three examples, one of each of these types of assignment, with instructional notes and discussion of solution methods for each. The unifying theme that I have in mind for all three is that they might be easily linked to existing course content that a given teacher (such as myself) is mandated to use.

For a warm up problem I intend to have students exercise their thought process on how a fraction can behave as both division and a rational number (i.e. part-to-whole) simultaneously. I will pose students the following prompt and give them several minutes to work individually on it, before opening it to a brief, whole class discussion: "If I start with a certain amount of money and take half of it, and then take the remaining money and add half of its new value, I have 75 cents. How much money did I start with?"

This problem is seemingly simple, but it takes a step toward the sophistication that we want from students because it relies upon using the same fraction, ½, in two conceptualizations via two, possibly three, operations. To start with a value and "take half" can be perceived as either division or subtraction, both of which are valid. If we consider it division, then we are treating the fraction itself as division. If we consider it subtraction of half of the starting value, then we are treating the fraction as part-to-whole.

Discussing this nuance with students can be an invaluable way to help them see the distinction. This distinction between division and rational number identities can also be emphasized by the fact that subtracting ½ of a number from itself leaves you with the same result as dividing by 2, whereas adding ½ of a number to itself does not give the same result as multiplying by 2. The fraction is more "powerfully inclined" toward reduction.

In working out the problem I expect a fair number of students to successfully determine that the "starting amount" was $1.00. However, I also anticipate the very issues that have motivated this paper. For students who struggle, and for students who have found only one solution path, I will encourage them to attempt two solution paths: 1)work from end-to-start in order to trace back from the known to the unknown; and 2)employ a variable "x" for the starting amount and write some expression(s) for the steps described in the prompt. This provides a juxtaposition of arithmetic and algebra.

For a content activity, an ideal sort of word problem is the following: "If it takes Isa 2 hours to paint a room and it takes Rick 3 hours, how long will it take if they work together?" The first trick of this question is that it does not visually introduce the fractions that are inherent in the scenario context, yet in order to respond to such proportional reasoning challenges, students must make significant mental, written, and verbal efforts, all involving fractions. And, in order to show understanding of a solution; students must exhibit clear comprehension of the dualistic, inverse identities of multiplication and division inherent in the fraction form.

We must consider which units students use as an "anchor" for their understanding: length of time or quantity of rooms. Since there is only one room, and because the difference in the number of hours that Isa and Rick each need to paint the room is a more obvious feature of the problem statement, students are likely to attempt working with the hours first. Unfortunately, students who struggle with proportional reasoning may well falter at the first steps of "what to do with the numbers." These are times when we get patently incorrect answers like "5 hours."

After giving students several minutes to read the problem and get started on "private think time," I will ask them to work in groups (the ability to flip from individual work to small group work throughout a class is easy with the seating arrangement I use) and begin to circulate among them to listen to and observe their work. Guiding questions I will ask might include: How long would it take Isa to paint 2 rooms of equal size? How much of a room can Rick paint in 1 hour? Should it take "more" time or "less" time if Isa and Rick work together?

Students might need to work out many sub-steps to form their solutions. Conversion of units can be helpful: it takes Isa 120 minutes to paint one room it and it takes Rick 180 minutes to paint one room. Some of the more helpful sub-steps might be based on breaking the problem down to unit fractions: Isa can paint 1/2 of a room in 1 hour; Rick can paint 1/3 of a room in 1 hour. And, conversions to general fractions can be helpful: in 1 hour Isa and Rick together can paint 5/6 of a room. Graphic organizers might also be used, such as an array drawing of a rectangle to represent the 1 room with subdivisions for the portions of the room Isa and Rick paint in units of time.

All of these statements together allow students to employ their understanding that a general fraction is merely "so many copies" of a unit fraction and therefore if Isa and Rick are able to paint 5/6 of the room in 1 hour, they can paint the remaining 1/6 of the room in 1/5 of an hour. Thus, together Isa and Rick will take 1 hour and 12 minutes to paint 1 room. It is the comprehensive understanding of unit, unit fraction, and general fraction, in their interrelated roles, connected to the idea of fraction as division and multiplicative inverse, which makes the final step of such a solution possible.

For the quizzing or assessment of fractions in an algebra courses, my main thought is to err toward caution in making valid measures of student understanding. I break somewhat from my edict to keep all fraction work situated in algebra and recommend pre-quizzes of fraction management skills to be given prior to quizzes of algebra content that is reliant upon students' prior knowledge of primary fraction rules. A strong example of this is when a quiz might be given for students to determine relationships based on the slopes of linear functions. For instance, a quiz might ask: "Given points A(2, 5), B(8, 17), C(6, 3), D(12, 15), are line AB and line CD parallel, perpendicular, or neither?"

As stated the correct answer is that lines AB and CD are parallel (you might also notice that had the question asked about lines AB and BD the correct answer is perpendicular), however that determination is second to finding the slopes, which, since it is based upon the analysis of the x and y values of the points into ratio relationships, relies on student work with fractions. So, knowing that such questions were to appear on a quiz, I would present them with a pre-quiz with questions such as: "Given the complex fractions (8-5/11-5) and (12-6/23-11), simplify and determine whether the fractions are equivalent, reciprocals, or neither." In this simple case, they are of course equivalent.

Changing the higher level task from the upcoming quiz (of analyzing the points into slopes, to define linear terms), to make a more accessible task of fractional management for a pre-quiz, can give students the opportunity to practice the skills they need while still working toward the new skill set to be assessed. This also gives the teacher a formative opportunity to identify where mistakes are occurring, and can then be used as reference for students to see their fraction management skills at work in algebra.

## Notes

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1. Ivan Niven, 22
2. Ivan Niven, 3
3. Roger Howe, in private communication
4. John Allen Paulos, 99
5. John Allen Paulos, xiii
6. Roger Howe, from "Arithmetic to Algebra"
7. Susan J. Lamon, 3
8. Liping Ma, 11

9. Liping Ma, 7
10. Susan J. Lamon, xiii
11. Roger Howe, from seminar reading "Developing and Interpreting Multiplication and Division with the Number Line"
12. The series is Connected Mathematics 2 from Pearson
13. Roger Howe, from "Arithmetic to Algebra"
14. Susan J. Lamon, 3
15. Liping Ma, 7

## Annotated Bibliography

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Roger Howe, "Arithmetic to Algebra," Mathematics Bulletin, Chinese Mathematical Society and Beijing Normal University, (May 2010): 13 - 22.

The juxtaposition of different solution paths has been an essential tool for my instruction of algebra for years, the additional juxtaposition of arithmetic methods makes for a much stronger and comprehensive learning opportunity for students entering secondary study.

Roger Howe, "Developing and Interpreting Multiplication and Division with the Number Line," course bibliography of readings, Yale National Initiative, (June 2011)

This short paper (2 page) on number line modeling of multiplicative operations is not publicly available, the strategies above, using number line models, reflect main ideas.

Glenda Lappan, et al., Michgan State University, Connected Mathematics 2. Boston: Pearson, 2009.

This text series, designed for the 6th, 7th, and 8th grades, uses significant methodology akin to Singapore Mathematics. I believe it is invaluable for secondary teachers to be aware of the jargon, content, and format students have used in prior study.

Susan J. Lamon, Teaching Fractions and Ratios for Understanding: Essential Content Knowledge and Instructional Strategies for Teachers. New Jersey: Lawrence Erlbaum Associates, 2nd Edition, 2005.

Lamon's text for teachers is a very comprehensive, robust taxonomy of pedagogy of fractions. It can be valuable for high school teachers working with struggling students.

Liping Ma, "Three approaches to one-place addition and subtraction: Counting strategies, memorized facts, and thinking tools," Carnegie Foundation for the Advancement of Teaching, draft, June 21, 2011

Ma's work is focused on the content of early primary education, however the pedagogical observations she makes have ramifications that are enlightening for teachers at any level.

Ivan Niven, Numbers: Rational and Irrational. Washington, DC: The Mathematical Association of America, 1961

I found this book very helpful for an Algebra 2 curriculum that will have some focus on group theory and the taxonomy of numbers. Niven introduces rational numbers well.

John Allen Paulos, Innumeracy: Mathematical Illiteracy and Its Consequences. New York: Hill & Wang, 2001

I recommend the book more as a generally beneficial reading for math teachers than as a primer or guide specific to fractions.

## Appendix of State Standards

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The standards listed here are for Algebra 1 (containing "A1"), Algebra 2 (…"A2"), or grouped parallel sets of Algebra 1 and Algebra 2 standards for Pennsylvania, circa January 29, 2011. This relatively brief list does not touch on all standards to which fractions might apply. My comments reflect how I view each as highly applicable.

2.1.A1.C - Use ratio and proportion to model relationships between quantities: This basic hallmark of Algebra 1 is the most general notion of my strategies for relating patterns in number sequences to fractional reasoning.

2.1.A1.F. - Extend the concept and use of inverse operations to determine unknown quantities in linear and polynomial equations: As I illustrated above, understanding the fraction as division is essential to algebra students in applying inverse operations.

2.1.A2.D. - Use exponential notation to represent any rational number: Student use of this advanced notation in Algebra 2 is reliant upon comprehension of rational numbers.

2.3.A1.C. - Find missing quantities in measurement formulas by applying equation solving techniques/2.3.A2.C. - Solve a formula for a given variable using algebraic processes: As emphasized at the beginning of my strategies section, I most particularly wish to target issues with fractions as they arise in the central algebraic skill sets that students might otherwise find more success with, e.g. solving for unknown variables.

2.5.A1.B/2.5.A2.B: Use symbols, mathematical terminology, standard notation, mathematical rules, graphing, and other types of mathematical representations to communicate observations, predictions, concepts, procedures, generalizations, ideas, and results: The wording of this standard is identical for both Algebra 1 & 2. I hope that I have addressed to some degree the codependence between managing fractions and the central rules, notation, and terms, as well as multiple representations, in algebra studies.

2.6.A1.A. - Design and conduct an experiment using random sampling: If it is not too extraordinary to say such a thing, I am particularly fond of this standard. What is more, it exemplifies how implicit and taken for granted proportional reasoning is, as any numerically based experiment using samples will produce fractions and percentages as measurements that must be carefully and critically analyzed.

2.7.A2.C. – Compare odds and probability: This content of Algebra 2 relies on students having some sophistication in working with both part-to-whole and part-to-part reasoning simultaneously, while making valid observations of these two fraction types.

2.11.A1.B. – Describe rates of change as modeled by linear equations: The magic word is "slope."

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