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What an Expression Expresses

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by Sarah Kingon

Objective/Rationale

My students regularly ask "Why do I need to know this?" and "When am I ever going to use this in life?" I think the most pertinent answer to these questions is that the content in my classes teaches them to be logical and critical thinkers. However, I think it is also important to acknowledge that a lot of my students are asking these questions because they do not understand the math. If my students are getting the mathematics education they deserve, they should have an in-depth understanding of the topics involved in algebra and students should be able to justify why the mathematics they did worked and how to interpret what the mathematics means. By frequently incorporating word problems in my classroom, I want to help my students start to see situations in which basic algebra is used all the time. If the word problems are accessible, my students will be less discouraged and they will be acquiring more in depth knowledge of the content.

It has been my experience that, when students enter my Algebra I class, either as 9 th graders or 10th graders repeating Algebra, they seem to have been programmed to think that, in every problem, they need to find an answer or to solve for x. They usually approach a problem by solving for x (or any available variable) regardless of what the question directs them to do. I believe that this is due to the fact that they do not have a clear idea of what a variable is or the ways variables are used. In addition, they do not have an adequate understanding of the equals sign. For so many years, the work needed to develop the understanding has been ignored and step-by-step processes have been emphasized. Most of the time I find that my students have memorized "shortcuts" that were given to them without justification or explanation. A lot of these shortcuts come directly from the bold or boxed terms in the textbook and students think they are important to memorize because they have been emphasized in this way. My students are memorizing the bold without understanding why it is bold. This leads to my students guessing what they should be doing without thinking why. Students learn all these ways to compress and make things more concise and simpler without having meaning attached, so that when they are asked to decompress or "unpack" what they did, they are clueless and left behind. My goal in this unit is to give my students the foundation and tools they need to be critical thinkers and be successful at justifying their process. The main tools I will focus on in this unit are:

a) a sense of what variables are and ways in which they are used;

b) an understanding of how to work with expressions, including

i) what expressions are,

ii) what they are used for,

iii) how expressions are formed, and the grammar of expressions, including reading, interpreting and writing expressions

iv) what it means for expressions to be equivalent, and how to manipulate expressions to produce equivalent ones, especially to simplify them;

c) what equations are and how they are used, including

i) manipulating equations,

ii) solving equations, and

iii) writing equations that represent situations presented verbally (i.e., translating word problems into equations).

I also want them to understand that solving an equation is a process of logical reasoning.

As their teacher I intend to emphasize that shortcuts, that they may have seen before entering my class, are a privilege for the experts and that they will be permitted to use the shortcuts once they have proven to me that they are truly experts. I believe once a student has been asked and expected to justify his steps throughout a process of simplifying or solving by decomposing/justifying, and has shown that he is clearly an expert, he should no longer be required to show all the steps. Students should be encouraged to use a "shortcut" once they have an understanding of why it works. Not only will this help them as they go through high school, but my students will become more familiar with the two column proof context. In Geometry my students are presented with the two-column proof and asked to explain their reasoning process, which has never been expected before. Therefore, requiring my students in Algebra to follow the justification process will be serving a dual purpose: teaching them to always think critically and ask why they are doing what they are doing; and preparing them for a format for later by making connections.

When my students are presented with a problem set, they rarely read what the problem is asking them to do, nor do they understand why they are doing the processes they are doing. Specifically, I believe that my students do not have an understanding of variables and the equals sign. Without such basic understandings, they cannot know how to approach a problem or explain it clearly. This superficial understanding of the mathematics involved and the inability to "unpack" carries throughout Algebra 1 and into all of my higher-level classes. At my school this poses a huge problem as they progress through the math courses, but are unable to build on understanding because they had no foundation to begin with. The majority of my students have to retake a math class sometime in their high school career due to the fact that they are lacking the fundamental knowledge. In the state of California, students now need three years of different math, and at my school that means they must complete Algebra 2. Failing a class puts a student at serious risk of not graduating from high school on time. Therefore it is imperative that a student is engaged in the content and can demonstrate her understanding clearly or her success in high school and her ability to graduate is at stake.

The major problem for my Algebra 1 students is their confusion between expressions (recipes for computation) and equations (answers to particular questions). I chose to focus on expressions and equations because this is where I first start to see my students struggling with the concepts. I think this is because it is when they are asked to use the tools they have previously been taught. I see a fundamental misunderstanding between what an expression is and what an equation is. I think this is because we often take an expression and when we simplify an expression we use an equals sign to create an equivalent expression. Because there is no concrete understanding of what an equals sign represents, my students see an equals sign and immediately want to solve for the variable or find an answer. This different contexts of the equals sign causes a lot of confusion. Often times my students "do the problem correctly" except at the end add an equals sign and solve an equation that never existed. When I say, "do the problem correctly" I mean they took the correct steps to simplify an expression, but misinterpreted what the question was initially asking or what the expression represents. Therefore it is my specific objective in this unit to decompose and break down the fundamentals of an expression, equivalent expressions and the differences among as well as connections between expressions and equations.

Background

Variables

At the start of the 20th century, variables were defined as a quantity that could assume an infinite number of values or a generalized number. An example of a situation in which a variable is used in this way is an expression such as

$3x - 5x - 24$

because the *x* in the equation can represent an infinite number of values. However, between the late 1950's and 1980's there was a move to refine this terminology. Variables began to be defined as a symbol that represented an element of a set [1] . The set could be the real numbers, or the rational numbers, or the integers, or the whole numbers, or something else, according to context. I will define a variable as a symbol standing for any element of some set. If we were to be completely correct when defining a variable, the set should be defined, but in reality it usually is not. Sometimes the set can be determined from the context of the problem but other times, if vague, the set can be unclear. The shift in definition is not a radical change but rather a refinement.

Variables can be used in many different contexts. Some examples of specific contexts that I will discuss with my students are as follows:

- A variable can represent a quantity, such as area. When discussing area we might use the letter A for the variable. However, A can represent something completely different in different situations-it could be the area of a rectangle, or of a hexagon or of a circle, or whatever shape is being discussed.
- Variables may be used in equations to express a relationship between quantities. For example the area of rectangle (A) can be computed as A=bh where b is the base and h is the height of the rectangle.
- A variable is used to form equations or expressions in which one is representing a specific situation. This is the context of a variable mostly used in this specific unit (examples to follow).
- Often textbooks and many teachers define a variable as an unknown that we are looking for, but in reality here also, the variable is a quantity that can vary in some set. What distinguishes the variable as

unknown from other uses of variables is, that instead of just using the variable to represent a quantity, we are asking a question and some value of the variable will be the answer. For example in $2x - 8 = 0$, we are asking, is there any value of x in the set that makes this equation true. So x is varying in a set, and we are asking: can it take a value that makes the equation valid. For this particular equation, if we assume that we are in the set of whole numbers or integers or rational numbers, then x would equal 4. However, if we defined the set to be numbers between 15 and 20 there would be no value that x could be to make that equation valid.

Equals sign

The other content-related issue when approaching expressions is the misconception of the equals sign. Several mathematics education researchers remark that there is an operational view of the equals sign as well as a relational view [2] . Most of our students have been overexposed to the operational view, which states that students see the equals sign as representing an action needs to be performed and something needs to be written as an answer. The adoption and prevalence of this in the classroom, is partially due to the fact that most elementary students learn arithmetic in this manner and the equals sign becomes associated with finding an answer from a very early age. However, to be successful in middle and high school algebra, and conceptually understand the equals sign as meaning equivalence, students need to be exposed from an early age to the relational view [3] . This view presents the equals sign as something signifying both sides having the same value, or in other words the equals sign meaning "the same as." Because I am a high school algebra teacher I have no control over how my students have been exposed to the equals sign in earlier grades. However, because I do not know their background and conceptual knowledge of the equals sign, it is even more imperative to present it as a topic of discussion. I will talk with my students about the meaning and how we can learn to accept that one does not always want a computational answer, while simultaneously talking about when it does signify a computation.

Strategies/Structure

Variables

In order for my students to genuinely grasp the concept of algebraic expressions, I first want to discuss what variables are. I will present them with multiple contexts in which variables occur in mathematics situations and we will look at how different contexts translate to different definitions of a variable. We will cover why we use variables and what they represent, using the information provided in the background section above. It is imperative to provide examples (see above). For the context of this particular unit it is important to emphasize an example such as $2x - 8 = 0$, discuss what the variable represents, present the issue of the defined set and how that determines the value the variable takes on. Along with discussing the different contexts and representations of variables, we will discuss how to define variables properly and specifically so that they develop good hygiene when defining and using variables. For example defining a variable *x* as the number of cookies Brian ate or the cost of a box of cookies, in cents, versus defining *x* as cookies, is considered good hygiene This will not only help my students develop better habits, but also they will begin to understand and put in context the use of variables. When asked to create/translate/interpret equations or expressions there will be meaning associated, stemming from the understanding of variables. This will encourage the students to start to look at the context and not just the answer. In their future math classes this will assist when dealing with systems of equations, when it is necessary to be clear about what each variable represents.

Expressions

Because there is no definitive line between variables and expressions these two ideas should not be taught as isolated bits, but the discussion on variables should smoothly transition into expressions in ways that seem appropriate for you particular groups of students. We will start with numerical expressions, developing the idea of an expression as a recipe for computation, in which we are manipulating numbers and the variable(s) have not yet been introduced. For example presenting a problem in which the students are asked to write the numerical expression that I say: take 3 add 4 to it then multiply that by 2 and then subtract 1 corresponding to (2(3+4)) -1. Students will become more familiar with manipulating numbers, using parentheses, and learning why we use parentheses. Understanding that what is inside parentheses corresponds to a completed calculation (in this specific example taking 3 and adding 4 to it as a completed calculation) and the operation inside stays intact when manipulating the rest of the problem is crucial. Hearing the numerical expressions orally and seeing them written with the parentheses that accompany them (as shown above) will require students to think conceptually about what the parentheses stand for and for what purpose we use them. Following the translation of basic numerical expressions from words to numbers, I will challenge them with some number tricks, wherein I ask the students to choose a number and then give them a prescribed procedure to follow. There are two particular types of number tricks that I will choose to use to exhibit specific characteristics. The first type of number trick simplifies so that you are left with a simple expression. An example of this is take your number add 2 to it, multiply by 2, add 5, multiply by 2, add 6, divide by 4 and then subtract 3. This complicated expression will always simplify to your number plus 3, as shown below.

$$\frac{2(2(x+2)+5)+6}{4}-3=x+3$$

So if I ask my students to follow the steps and tell me their results, I will always be able to calculate what number they started with by working backwards or subtracting 3 from their final result. Students will be shocked that I can swiftly derive their starting numbers. If students choose different numbers, they will get completely different results. Seeing all of the different results will lead to a discussion about why if we followed all the same steps we all got different results.

The second type of number trick I will use in the introduction of numerical expressions is another long process that the students have to go through, however in this type of number trick you are always left with the same result (a constant). An example of this type of number trick would be to tell the students to pick a number add 4 to it, multiply by 2, add 6, divide by 2 subtract the original number and record the result. No matter what number the students choose, it will always result in 7 because the number (or variable cancels out).

$$\frac{2(x+4)+6}{2}-x=7$$

The discussion of the results in number trick type 1 (different results that I could guess fairly easily) and number trick 2 (the same result for everyone) will serve as a segue to ask the students instead of using a different chosen number for everybody, what can we substitute into that numerical expression to represent any number? Through these examples I will accentuate that an expression is a recipe for computation (the prescribed steps), or in other words an expression is a series that tells you how to manipulate/what to do with any given number.

The substitution of all of their different chosen numbers into the number tricks provided, will lead us into the heart of the unit, algebraic expressions, starting with the formation of algebraic expressions. It is important to start with scenario problems that challenge the students to think about expressions more deeply. For example asking the students questions such as, "If Louis has 6 more apples that I do how many apples do I have?" These scenario problems are a clear example of the blurred line between variables and expressions. This will prompt my students to say that they do not have enough information to figure it out, demonstrating the significance of an algebraic expression as a recipe for computation and not as a math problem with one answer. Simultaneously, students will be exposed to the fact that these expressions tell us to do something to numbers. In addition the variable tells you that one is able to follow the recipe for computation with any number they choose. I will also be sure to emphasize that the number of apples Louis has could be any number of apples from one to infinity and that we once again need to use a variable *a* to represent the *number* of apples Louis has, not *a* = apples. This is reinforcing the correct way of defining variables in context. The next step is to take one of those first algebraic expressions drawn from the scenarios, have them form another related expression (another scenario) and ask them to combine them. This allows them explore how they would combine two expressions to be one, before covering the nine rules of arithmetic. An example of combining algebraic expressions based on scenarios would be: Let x= the number of apples that Sean has. Louis has in his backpack 4 more apples than Sean (x+4), and at home Louis has 2 times as many apples as Sean has (2x). How many total apples does Louis have (x+4) + 2x. This example illustrates writing algebraic expressions based on given relationships, as well as how to combine expressions based on given relationships.

Since the students have now learned to form expressions based on different scenarios, they will practice reading and writing algebraic expressions with a partner, to build their comfort with expressions. In this situation partner A will read a problem such as "take a number and multiply it by 2, add 3, and divide by 4" while partner B is writing his numerical interpretation of it on paper : $(2x+3)/4$. The students will switch off, giving all the students a chance to practice with reading and writing expressions and letting them play with the numerical form as well as the spoken form. Students should become comfortable with the idea that the words that tell us to do something to a chosen number, can be written as well but the spoken and the written form serve as a recipe for computation.

Following reading and writing, students will do problems in which they are asked to evaluate expressions by computing when x is equal to specific values. This is to emphasize the fact that the variable could be any number (we can plug in any number). For example giving them an expression such as $2x$-5 and asking them to evaluate the expression if $x = 4$ as (2(4)) - 5 and then the same process if $x =5, 6, 7, 8, 9, 10$. Having the students compute the same expression at different values of x will accentuate the idea of being able to plug in any value for x, and reiterating that we are not looking for any specific value for x.

At this point, the students have been introduced to forming, reading/writing, and evaluating algebraic expressions and they should be comfortable with algebraic expressions and the meaning of them. The next step in the algebraic expression progression is to challenge the students with more complicated and longer expressions in which many more steps are involved, many more parentheses, as well as many more calculations to perform when evaluating. The goal is to have the expressions start to look messy and complicated so the students have the urge to begin to simplify them and make more manageable problems. For example giving them an expression such as

$$\frac{6(4(2(x-1)+3)+6)}{2}$$

The parentheses have become so cumbersome and confusing that students will start to complain about the length and steps involved in the lengthier problems or want to give up, giving us an opportune time to talk about the reasons why we would want to simplify these expressions and whether it is possible to simplify them. I will give them complicated expressions similar to the example above accompanied by the simplified versions (i.e. the simplified version of the example above would be 24x + 30) and ask my students to evaluate both the long version and condensed version at several values. For example evaluating the expression above at x=2 would be:

$$\frac{6(4(2(2-1)+3)+6)}{2}=\frac{6(4(2(1)+3)+6)}{2}=\frac{6(4(2+3)+6)}{2}=\frac{6(4(5)+6)}{2}=\frac{6(20+6)}{2}=\frac{6(26)}{2}=\frac{156}{2}=78$$

and if you evaluated the simplified version you would get:

24(2)+30=48+30=78

Students will hopefully start to see that the expressions are equivalent, but one is much simpler to work with than the other. The students will then be guided to explore different complicated expressions, such as the number tricks we used in the beginning of the unit, challenging them to simplify the expressions however they think they can. After exploration, we will have a discussion on the different methods they tried, with the hope of starting to develop an informal list of the nine rules of arithmetic that will lead to a formal list. I will use one of the complicated expressions as an example to exhibit many of the nine rules, going into depth with them, and supplement the other properties not covered. I will also accentuate the importance of the nine rules as properties that have been proven, that justify everything we do in in a lot of high school mathematics particularly the arithmetic of numerical and polynomial and rational expressions. It is imperative for them to understand what the rules are and how we use them, and once they do they can use the shortcuts to solve or simplify the problems. Again I will emphasize to the students that these shortcuts are the privileges of the experts and if they want to use them they will have to be experts on the nine rules of arithmetic or else they have the keep using the complicated form.

The Nine Rules of Arithmetic (See Appendix B for a list of the properties)

The first of the nine rules that we will cover is the commutative rule for addition. From experience, this property seems very intuitive to the students, so no deeper discussion about why it works is called for. Therefore I intend to present the commutative property using specific numbers to start (see Appendix A, Fig.3). Before looking at the general form that we typically see bolded or boxed in textbooks, it is important to express the property as a geometric representation of length. Students will be able to see conceptually why this property is justified, without going into too much detail. From the diagram, students will be able to see that 3 + 10 = 10 + 3 and challenging them to find an addition problem that it doesn't work for. Because this applies to all addition problems we are able to generalize for any values and write $a + b = b + a$ because the total of the sum does not depend on the order we add the elements.

After the commutative property is clearly understood, we will discuss the associative rule of addition. Before presenting the property we will once again look at a physical representation of the property and discuss how and why the students think it is justified (see Appendix A, Fig. 4), reiterating the idea of length and distance. Students can visualize from the diagram that (3+5)+6=3+(5+6) and that can also be applied to the general form $(a+b)+c=a+(b+c)$ of.

In both the commutative and associative properties the order in which we added the elements did not matter. The combination of these rules multiple times that leads to the any-which-way rule. For example (2+4)+(7+6) = (2+4) +(6+7) = ((2+4)+6)+7 = (6 + (2+4)) +7 = ((6+2) +4) + 7. As the applications of these rules gets more and more complicated and garbled we can then apply the any-which-way rule. The any-which-way rule allows students to cut out the tedious steps of moving the parentheses and serve as a justification that we can add as many numbers as we choose in any order that we choose. It is one of the shortcuts of the experts!

When subtraction or negative signs become involved my students begin to shut down, so I think it is necessary from the beginning to make clear the connections between addition and subtraction and how we can also apply these properties to subtraction as well. Even though my students have been previously exposed to negative numbers, I will take the time to reintroduce negative numbers using the number line and asking students to explain what the negative numbers on the number line represent. One example of a problem that clearly demonstrates subtraction/negative numbers is: Amy, Blair and Chanel live on Elm Street. Blair lives three blocks from Amy, and Chanel lives 4 blocks from Blair. How far does Chanel live from Amy? Students should be puzzled because the problem does not explicitly state direction. It is the direction or orientation of distances that is captured by signed numbers. After a discussion about what they perceive the negative numbers to represent, I will emphasize that when we use subtraction of positive numbers or addition of negative number to represent going to the left, but subtraction of negative numbers or addition of positive numbers to represent going to the right. Students will be presented with a question such as: does (3-5=3+(-5)? And because we have previously discussed negative numbers students will be able to say yes. Therefore, because we have now created an addition problem, we can apply the properties of addition. A discussion needs to occur that challenges students to think about whether any subtraction problem can also be written as an addition problem, and out of that discourse, the idea of addition and subtraction being inverse operations, emphasized. From there, students should apply the properties when a subtraction problem is given, for example 3 – 5 = 3 + (–5)= –5 + 3. It also needs to be emphasized that the middle step is necessary or else the negative sign gets lost along the way and the students begin to develop incorrect and sloppy habits.

Continuing with this idea that $a-b=a+(-b)$, I will ask my students to apply it to the problems and ask what they think that can be simplified to. Most students will be able to say 0, but it is also important to guide my students to see it as $a+(-a)=0$, to reiterate the idea of inverse operations and introducing this as the inverse property of addition. This will bring the class to another property: the additive inverse property. This property is also intuitive to most students. I think it is important to show a visual representation on a number line of this, accentuating the idea of moving *a* units forward and then moving *a* units back, leaving us in the same position as we started or showing that 0 displacement has occurred. The students need to have a clear understanding of this property because it is critical when solving equations and understanding why we do that when solving.

An extension of this property on the number line is the identity property of addition. A number line is a good tool to accompany $a+0=a$ because it shows that we haven't gone anywhere from where we started. Using the number line also serves as a consistent structure used to demonstrate the properties of addition in a geometrical way.

Now that the properties of addition/subtraction have been covered, it is time to prompt the students by asking them what other operations do we know and developing some properties for those too. Once again it is imperative to stress the inverse relationship between addition and subtraction and make a parallel connection to multiplication and division, particularly because it comes back into play when simplifying and solving. To introduce the commutative property of multiplication I will provide an array model (see Appendix A, Fig.5) which graphically shows that *a* x *b* = *b* x *a* . Having the students count the dots for the specific examples and making sure students understand why we are able to put the values in any order, as in addition. Once students have seen this property work in many examples and are comfortable with it, I then intend to extend the property to division, as I did with the inverse operations above. So showing that 3 divided by 4 is the same as 3/4 or 3-(1/4). Or in general to apply the property to division *a* divided by *b* is equal to *a*/*b* or *a* - (1/*b*) which is equal to (1/*b*) - *a* because now we have created the multiplication, once again emphasizing the intermediary step of using multiplication.

The next property is the associative property of multiplication, which is much harder to see visually. I will have my students play around with computations in which I switch the order of the multiplication, prompting my students to discover that the associative property also applies to multiplication. For example, giving my students

(3•4)•5=12•5=60 and 3•(4•5)=3•20=60

emphasizing the meaning of parentheses and showing that the order of the multiplication doesn't matter, hence we are left with the same result. After students have experimented with different combinations and are confident with using the rule as justification, I will then increase the number of terms in the problem. An example of a more complicated application of the associative property of multiplication (also an application of the any-which-way rule) would be

$$(4\bullet 3)(2\bullet 2)(5\bullet 1)=(12)(4)(5)=(48)(5)=240$$
$$\text{or}$$
$$(2\bullet 4)(2\bullet 5)(1\bullet 3)=(8)(10)(3)=(80)(3)=240$$
$$\text{or}$$
$$(5\bullet 3)(2\bullet 1)(4\bullet 2)=(15)(2)(8)=(30)(8)=240$$
$$\text{etc.}$$

Finally to bring back the idea of the inverse operations again and to emphasize the relationship, we will look at the progression to derive the inverse property of multiplication. I will show that a - 1/*a* = *a*/*a* which is equal to *a* divided by *a,* and anything divided by itself is 1. This will be an opportune segue into the another property, the identity property of multiplication in which any number multiplied by 1 just equals itself because any number one time is just that number.

The final of the nine rules of arithmetic to be covered in this unit will be the distributive property because it is a culmination of the other properties and used as a tool very often in algebra. Therefore it is imperative to understand how and why the distributive property works. We will start to discuss the distributive property by making the connection between distributing and a tennis tournament. More specifically, if a high school (Oceana) tennis team is playing another high school how is the game typically structured? The students will be able to give me some idea of how it works. However, we will adapt the game to work in our example by explaining that in this tennis tournament every player on the team has to play every player on the other team.

To connect this mathematically, we pose a problem such as (2)(3+5) and explain that in order for the game to be completed player 2 (on team Oceana) and has to play player 3 and has to play player 5 (on Terra Nova) because every player on Oceana has to play every player on Terra Nova. It is important to emphasize that the rules of the game state every player has to play every player on the opposing team. Giving more complicated problems, where more numbers are involved in both expressions, can then extend this farther. For example posing a problem such as, (2-4+6)(3+7+8) where there are many more ways to approach and simplify this problem. I will ask my students to find the nine different ways in which they can simplify this problem. Finally this can be extended to the use of algebraic expressions with variables. Once students are familiar with the game/the distributive property, a geometric application can be applied by using area models (also known as the box method). In this method the terms are arranged as the lengths and widths of the boxes (see Appendix A, Figs.1, 2) and then the students are asked to find the area of each inside rectangle (something the students are fairly confident with) as well as the area of the entire rectangle. The area model emphasizes the geometric connections as well as the fact that we have to multiply everything to everything and add it together in order to get a complete area. Now that the students are familiar with the distributive property, I will prompt the students to look back at one of the number tricks we looked at before (i.e. ((4((2(*x*-1))+3))+6)/2) and identify all the places in the complicated number trick that this property could be applied. In this specific example it can applied three times, once when the 2 is distributed to the x and the –1 and once when we distribute the 4 to the expression inside. We will follow with a discussion about which place, of the ones identified, do they think we can start with and what those parentheses tell us about the order in which we can apply the property, making sure to articulate that those operations have to be carried out in certain places before we can move outwards in the expression.

Simplifying To Create Equivalent Expressions

Since the students are now somewhat familiar with the nine rules of arithmetic that govern a lot of what we do in Algebra 1, we will transition back to the application of these properties to simplifying algebraic expressions. We will start with more basic applications of these properties: (2*x*+4)+(5*x*+2) in which we can apply the any which way rule in order to collect the like terms together (also referred to as regrouping) to be (2*x*+5*x*)+(4+2) and making students write the reasons they are able to do this, to emphasize the use of the properties. As the expressions become more complex, students will still be required to write the properties they are using as they do each step and the idea of a equivalent expressions will be discussed. Students here need to understand that when we put the equals sign it represents "the same as" and that we are justified in saying that they are the same because all we have done is used the properties to make the problems more manageable. An example of the extension to equivalence of the first example is (2*x*+5*x*)+(4+2) =7*x*+6. From there we will talk about ways to ensure their simplification process was correct by plugging in any value for *x,* we should get the same value on each side of the equals sign because they are balanced and the same.

Equations

Here is where it is imperative to show the connection between expressions and equations. An equation is a statement that two expressions are equal. Equations represent different relationships (as discussed in the background on variables). For example if I give the problem 4*x*+6=10 that means we are taking some number *x* multiplying it by 4 then adding 6 and the result is 10. This is the chance to explain that in this situation *x* is standing in for a number or a set of numbers and it is our job to find that number, which differs from the way in which we were looking at *x* before. The students will be asked to use the nine rules as the justification for each step. Students should understand the goal of solving an equation as finding what *x* has to be to make the equation true. That is, the equation is asking a question: for which numbers is this true? Also students should have to brainstorm how we would find a solution, and come to the conclusion that we need to find *x* because that is the missing part. They also need to see that in order to do that we need to use our properties to isolate *x*, while maintaining balance by applying whatever we do to both sides of the equals sign. The metaphor of an equation as a scale that we must keep balanced is valuable at this point. Also an introduction of the property (Equals added to Equals Makes Equals or Equals multiplied by equals are equals-see examples below) will help students transition into balancing equations. The format that I will use is that next to each equation is the rule/principle that permitted me to deduce the given equation from the one that came before it (similar to a two-column proof). An example of a solution is:

If $4x+2=14$, then	
$4x+2-2=14-2=12$	Equals Added To Equals Makes Equals
But $4x+2-2=4x+0=4x$	Additive Identity Property
Therefore 4x=12 and	Two things equal to the same thing are equal to each other.
$\frac{1}{4}\bullet 4x=\frac{1}{4}\bullet 12=\frac{12}{4}=3$	Equals Multiplied by Equals are Equal
But $\frac{1}{4}\bullet 4x=1\bullet x=x$	Multiplicative Identity Property
Therefore x=3	Two things equal to the same thing are equal to each other.

After time, persistent and patient students will become much more fluent with the idea of balancing equations, and have been presented with many different contexts, including situations in which simplification has to happen first:

$2x - x - 3 = 1 + 2$

$x - 3 = 3$ Two things equal to the same thing are equal to each other.

$x - 3 + 3 = 3 + 3 = 6$ Equals Added To Equals Makes Equals

$x + 0 = x = 6$ Additive Identity Property

When there are variables on both sides of the equation:

$3x - x + 4 = x + 5$

$2x + 4 = x + 5$

$2x + 4 - x = x + 5 - x$ Equals Added To Equals Makes Equals

$(2x - x) + 4 = (x - x) + 5$ Any Which Way Rule

$x + 4 = 0 + 5 = 5$ Additive Identity Property

$x + 4 - 4 = 5 - 4 = 1$ Equals Added To Equals Makes Equals

$x + 0 = 1 = x$ Additive Identity Property

And equations in which elements need to be rearranged:

$3x + 2 - x + 4 = x - 1 + 2x + 3$

$(3x - x) + (2 + 4) = (x + 2x) + (-1 + 3)$ Any Which Way Rule

$2x + 6 = 3x + 2$

$2x + 6 - 2x = 3x + 2 - 2x$ Equals Added To Equals Makes Equals

$(2x - 2x) + 6 = (3x - 2x) + 2$ Any Which Way Rule

$0 + 6 = 6 = x + 2$ Additive Identity Property

$6 - 2 = x + 2 - 2$ Equals Added To Equals Makes Equals

$4 = x + 0 = x$ Additive Identity Property

Finally, we will come back to the number trick idea that began the unit. I will provide my students with a number trick equal to a specific number and instead of letting the students choose a number this time I will have students attempt to guess the number I chose to get the number trick to equal a certain number or working the number trick backwards. Students will have to translate the number trick from spoken word, back into a numerical problem, which now is equal to a specific value, chosen by me. They will then solve the equations, making sure to use their nine rules along the way as justification (using the format shown above). At this point I will see the progress my students have made and their understanding of the rules, and decide whether they are experts and allowed to stop the justification and use the shortcuts.

The final portion of this unit is to expand on the idea of the number trick, by presenting my students with equations that are not written in the traditional way. This will require my students to take word problems, interpret what is being asked in the problem, write the problem by defining variables (in the clean and correct way), and setting up the equation so that it logically makes sense. This will be a major focus, as to get students comfortable going from written or oral to numerical and back again, without losing sight of the connectivity between them all. An example problem would be: The local commuter train has three passenger cars. When it is full each car holds *p* people. In addition to the passengers, the train has 8 workers. Write an equation to represent the total number of people the train can hold if when completely full it holds 176 people. How many passengers fit in each car? Students will redefine the variable p as the number of people each car holds. They will write equations based on the given information, just as they did with expressions. So 3p= the total number of passengers on the train, 3p+8=the total number of people on the train and 176 is also the total number of passengers, therefore we can say that 3p+8=176. Once students are able to set up the equations based on the scenarios they will follow the steps using justification, just as they did above to solve the equation. Finally students will be expected to write the final answer as each car can fit 18 passengers

Classroom Activities

Lesson: Number Tricks

Objectives

-To explore numerical expressions and to discover the freedom to choose any number to plug into a number trick.

-To show that the original number can be easily guessed based on the result if you know what was the recipe for computation in sufficiently simple form.

-To emphasize a number trick as an expression and make a clear connection to an expression as a recipe for computation.

-To increase comfort with the equals sign in the problem and what that means.

-To display there are infinite results to some number tricks.

-To display the different types of number tricks and their results.

-To learn how to translate computation in words to computation with mathematical symbols (i.e. parentheses, operations and variables).

Procedure

The lesson begins with a number trick (Appendix C, number tricks problem 1). The teacher asks students to choose any number and do the performed steps that are asked and the students are asked to record their results. The results are recorded on the board, and the teacher begins to guess what different students starting numbers were using the results. Students are asked to try and brainstorm ways in which the teacher was able to guess the numbers so easily. Another similar number trick is used (Appendix C, number tricks, problem 2) and the same process is repeated, results and process are discussed as a class. Finally a different kind of number trick is presented in which all students get the same result (Appendix C, number tricks, problem 3). This second type of number trick is discussed as a class and the students are asked why the result was always the same. Teacher and students come up with a list of ways in which to guess the result of a number trick.

Now as a class we discuss what can represent any chosen number (a variable) and how we can use a variable to write the steps in the number trick as an expression. As we are writing a mathematical statement to represent the steps, the idea of an expression as a recipe for computation is emphasized. Students will then be asked to write a number trick using words, as done in the previous problems and then translate them into mathematical expressions.

Lesson: Evaluating Expressions

Objectives

-To see the connections between an expanded expression and its simplified form.

-To understand the idea of equivalence and the equals sign in this particular context.

-To evaluate the expanded and simplified form at any given value in order to check simplification.

Procedure

Students are given a long and complicated expression in expanded form (Appendix C, evaluating/simplifying, 2-5) and asked to evaluate these expressions at different values. Then the students are presented with the simplified versions of the expanded form and asked to evaluate those expressions at the same values. In groups students discuss results and are asked "Do you think the results will be equivalent for any value of x, why or why not?" The groups share their answers to the questions and the class debriefs why it works and how they think we can transform the expanded form to look like the simplified form. The idea of equivalence should be discussed here and this particular context of the equals sign should be emphasized. Students should be able to explain what equivalence is and that we are not looking for a solution. As a class, we will then draft an informal list of different ways we can simplify expressions, prior to the nine rules of arithmetic being presented.

Lesson: Solving Equations

Objectives

-To solve one-step and multi-step equations.

-To be able to justify all steps when solving equations.

-To understand and articulate what the solution to an equation represents.

Procedure

Students start solving one step equations and then multi step equations (Appendix C, Equations, problems 1, 2) and included in solutions are justification for each step using the nine rules of arithmetic (see Strategies) as well as our solving equations properties (equals added to equals and equals multiplied by equals). After students have practiced many problems, we will discuss equations that look different (i.e. equations with variables and constants on both sides and equations that need to be simplified first). In these problems students also need to provide their justification steps. After students are very comfortable with different types of algebraic equations, they will be allowed to solve equations without justification. Finally students will apply their equation solving skills to word problems in which they are required to define variables, set up equations, solve the equations and explain what the answer represents (Appendix C, Equation Word Problems, 1-7)

Appendix A: Figures

Fig. 1 (3+5) (6+10)

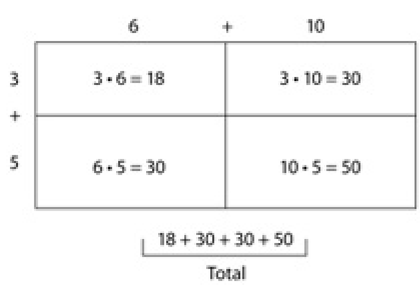


Fig. 2 (x+2) (x+1)

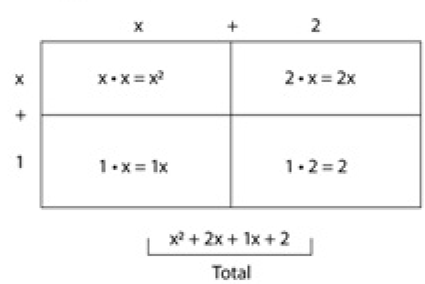


Fig. 3

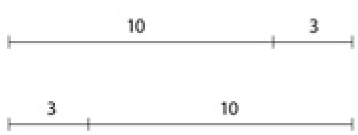


Fig. 4

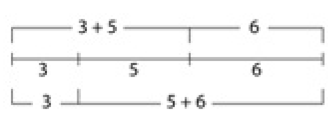
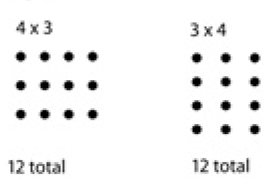


Fig. 5



Appendix B: Nine Rules of Arithmetic

Properties of Addition:

1. Commutative Property of Addition

$a+b = b+a$

2. Associative Property of Addition

$a+(b+c) = (a+b)+c$

3. Additive Inverse Property

$a+(-a) = 0$

4. Additive Identity Property

$a+0 = a$

Properties of Multiplication:

1. Commutative Property of Multiplication

$a \text{x} b = b \text{x} a$

2. Associative Property of Multiplication

$(a \text{ x } b) \text{ x } c = a \text{ x } (b \text{ x } c)$

3. Multiplicative Inverse Property

$a \text{x} (1/a) = 1$

4. Multiplicative Identity Property

$a \text{x} 1 = a$

5. Distributive Property

$a(b+c)=ab+ac$

Appendix C: Problem Sets

Numerical Expressions

Write the numerical expression that represents the computation

1. Take 5, multiply by 2, add 2, divide by 3 then add 6.

2. Take 10, divide by 2, multiply by 3, subtract 7 and double.

3. Take 7, subtract 1, multiply by 2, divide by 6, add 3 and triple.

4. Take 88, divide by 11, multiply by 2, divide by 4, add 3, subtract 6.

5. Take 63, divide by 9, multiply by 2, divide by 14.

Write the recipe for computation of the given expressions

1. $\dfrac{((2(6 \div 3))+4)}{3}$

2. $\left(3((9 \times 2)+4)\right)+4$

3. $3(\left(\left(2(64 \div 8)\right) \div 4\right)+2)$

Number Tricks

Students choose numbers, teacher guesses starting numbers from the given results.

1. Choose a number. Add 6. Multiply by 3. Subtract 10. Multiply by 2. Add 50. Divide by 6. What is the result?

2. Choose a number. Multiply by 3. Subtract 4. Multiply by 2. Add 20. Divide by 6. Subtract your starting number. What's your result?

3. Choose a number. Add 5. Multiply by 2. Subtract 7. Add 1. Divide by 2. Subtract 2. What's your result.

1. Choose another number and do the same trick from number 3. What is your result? What do you notice? Do you think this works for any number?

2. Choose a number. Add 3. Multiply by 2. Add 7. Subtract 15. Add 2. What is your result?

Scenario Problems

1. Ricky has 3 fewer apples than John. Let *j* stand for the number of apples that John has. Write an expression for the number of

apples that Ricky has.

2. Sara has 6 more dresses than Stephanie. Let *d* stand for the number of dresses that Sara has. Write an expression for the number of apples that Stephanie has.

3. Louis has double the amount of red trucks that Maia has. Define a variable and write an expression for the number of trucks Louis has.

4. Sean has 14 more pieces of paper than Brian. Define a variable and write an expression that represents the number of pieces of paper Sean has. Write an expression for the number of pieces of paper Brian has.

Combining Expressions

1. Donald counts the number of quarters he has in his piggy bank. He has 25 more quarters than his brother Robert. If *r* is the number of quarters Robert has, write an expression that represents the number of quarters that Robert and Donald have together.

2. Alanna has 10 more pairs of shoes than her sister Greta. They have the same size feet so they like to share shoes. Define a variable and write an expression that represents the number of shoes Alanna has, write an expression for the number of shoes Greta has and write an expression for the number of shoes they have together.

3. Marcelo has eaten 10 fewer burritos then Andrew. Define a variable and write an expression that represents how many burritos they have eaten combined.

1. Challenge: Shannon has 3 fewer dogs than double Lesley's. Define a variable and write an expression that represents how many dogs they have combined.

Reading and Writing

Pairs switch off between reading and writing.

1. Take a number. Multiply by 2. Subtract 5. Multiply by 9. Subtract 3.

2. Take a number. Multiply by 2. Add 5. Multiply by 2. Add the number.

3. Take a number. Multiply by 2. Add 1. Multiply by 3. Add 13.

4. $2(x + 3) – 6$

5. $(4(3 – (x – 1))) – 2$

Evaluating/Simplifying

Evaluate the given expressions at the given values.

1. Evaluate $3(x+6)-10$ at $x = 2, 3, 10, 20$

2. Evaluate $\frac{4x+3x+2x+x}{x}$ at $x = 3, 17, -2, 11$

3. Evaluate $3x+(x-1)\bullet(x-2)\bullet(x-3)\bullet(x-4)\bullet(x-5)$ at $x = 1, 2, 4$

4. Evaluate $\frac{6(4(2(x-1)+3)+6)}{2}$ at $x = 2, 5, 10, 40$

5. Evaluate $\frac{2((3((3(x-2))-5))-6)}{2}$ at $x = 4, 8, 10, 11, 20$

6. Evaluate $24x+30$ at $x = 2, 5, 10, 40$. What do you notice about these results and the results in number 4? Do you think that those expressions are equal even though they look completely different? What can we do the complicated expression to make it look much simpler?

7. Evaluate $9x-39$ at $x = 4, 8, 10, 11, 20$. What do you notice about these results and the results in number 5? Do you think that those expressions are equal even though they look completely different? What can we do the complicated expression to make it look much simpler?

Nine Rules of Arithmetic

1. Does 3+10=10+3? What rule justifies your answer?

2. Does 3+(6+4)=(3+6)+4? What Rule justifies your answer?

3. What is 3+(-3)=? Explain why.

4. What is 3+0=? Explain why.

5. Does 3•2=2•3? Show your reasoning.

6. Does (4•5)•6=4•(5•6)? Show your reasoning.

7. What is 3x(1/3) = ? Explain why.

8. What is 3•1=? Explain why.

9. What is 3(4+*x*)=? Show your reasoning.

Simplifying Expressions

Simplify the following expressions. Make sure to give a reason for each step you perform.

1. $2(x+2)-(x+2)$
2. $4(x-7)-2(2-3x)$
3. $2z+7z-z+5$
4. $7(6t+2)+3-5(t+1)$

Evaluate the original expressions above AND the simplified form at x=1, 2, 3, 4, 5. If you do not get the same answer for the original and simplified form, you have simplified wrong and you must go back and fix your simplified form.

Equations

Solve for the variable. Make sure to give a reason for each step you perform. Remember you can check your solutions.

1. $x+10=11$
2. $2x+4=6$
3. $3x+4=x+8$
4. $4-2y=y+10$
5. $5x+4-2x=-(x+8)$
6. $-2-3k-2=-2k+8-2k$
7. $3x+7=-x-1$
8. $1-2x-5=4x+2$
9. $-3x=x-(6-2x)$

Equation Word Problems

Define your variables. Write an equation based on the problem and solve for the given variable.

1. John had some apples and today he bought 4 more apples. Now he has seven apples. How many did he have?

2. John bought 3 packages of donuts. He opened them up and counted them and there was a total of 24 donuts. How many donuts were in each package?

3. John has 4 packages of donuts and 5 leftover donuts not packaged. In total he has 21 donuts. How many donuts are in each package?

4. There are 33 students in the class. There are 7 more girls than boys. How many boys are there and how many girls are there?

5. The local commuter train has three passenger cars. When it is full each car holds *p* people. In addition to the passengers, the train has 8 workers. Write an equation to represent the total number of people the train can hold if when completely full it holds 176 people. How many passengers fit in each car?

6. Tony rode his bike some amount of miles. If Katie rode 10 less than twice the number of miles Tony rode. How many miles did Katie ride?

7. If Oceana has *x* students and Terra Nova has 200 more than 2 times the amount of students Oceana has, how many students go to each school?

Appendix D: Implementing District Standards

The California Mathematics Standards for Algebra I that support this unit are as follows:

Seeing Structure In Expressions

1.Interpret expressions that represent a quantity in terms of its context

2.Use the structure of an expression to identify ways to rewrite it.

Write expressions in equivalent forms to solve problems

3.Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.

Creating Equations

-Create equations that describe numbers or relationships

Reasoning with Equations and Inequalities

-Understand solving equations as a process of reasoning and explain the reasoning

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