Yale National Initiative
to strengthen teaching in public schools®

Curriculum Units by Fellows of the National Initiative
2011 Volume VI: Great Ideas of Primary Mathematics

Strong Foundations = Success In Equations

Curriculum Unit 11.06.10, published September 2011
by Aimée MacSween

Rationale

All 8 th grade students at August Boeger Middle School take Algebra 1, regardless of their failure or success in kindergarten through 7 th grade mathematics. This aggressive approach leaves the math teacher a challenging task of building foundations while maintaining rigor and adequate pacing to complete the required standards. The challenge must be addressed by focusing on conceptual learning of the foundations so students have the ability to think and apply their skills accurately and fluidly to the wide variety of math problems they will encounter.

August Boeger Middle School is located in East San Jose. Seventy-four percent of our students receive free or reduced lunch. Gang life is prevalent on campus as well as in the homes of many students. The school faces many challenges in a difficult neighborhood. We serve a high percentage of English language learners, making English language development a key issue across all subject areas. The common thread I find in mathematics, if students are given clear steps and the support necessary to successfully complete a problem – they do it! The difficulty with Algebra is many students have fallen so far behind in math they are unable to even begin.

"Show your work." "No work. No credit." "Why are you turning something in without work?" My students jokingly mock me with the quotes they hear more times than they can possibly count. I have often heard that your greatest strength can also be your greatest weakness. This is true for me when it comes to teaching math. My strength is teaching steps and procedures. I have spent many hours agonizing over ways to teach Algebra to students who don't know how to do basic operations. I found that clear, detailed steps allowed all students regardless of ability to do the work. It eliminated excuses and made errors straightforward and easy to catch. Initially I was delighted with how well the steps worked and allowed students to successfully complete problems. I realized I had a problem when I saw students couldn't figure out which set of steps to use for which problems. It was like they had this big library of steps memorized, but they always needed me to tell them which one applies. I also saw they were unable to see the connection between the different methods. I had trained them to solve problems a certain way and they believed that using another method was not ok.

So now I am stepping out from my comfort zone in teaching to find ways to build conceptual knowledge in my students. I recognize the limits I place on them by allowing steps and procedures to be a main focus of their learning.

Objectives

The purpose of my unit is to give Algebra students the necessary background to successfully decipher a multi-step problem, formulate an equation, and solve the equation. In order to achieve this goal students will be taught to read, write, and interpret expressions. They will use the rules of arithmetic to simplify expressions and identify equivalent expressions. My intention is that their knowledge of the rules of arithmetic will be deep and conceptual allowing them to accurately apply the rules with confidence. Students will broaden their understanding of the equals sign as they learn to formulate and solve equations. They will be able to interpret information in word problems, formulate equations, and find solutions based on their understanding of the question.

California State Standards set forth the idea that students in Algebra should be able to "*…solve multi-step problems, including word problems, involving linear equations in one variable and provide justification for each step.*" [1] I believe students will achieve success on this standard if given a firm foundation. My unit will take approximately 6 weeks to teach and assess. As in the past, I expect over 50% of the students will enter Algebra below grade level. The challenge will be to build up necessary skills as well as teach the new content.

Background and Teaching Strategies

Translating Expressions

Students love a single number answer, so the idea of working with expressions is always a challenge. Expressions need to be taught with an emphasis on the values and meaning in order for students to be able to work with them accurately and effectively in the Algebra classroom. Though most Algebra classes are set up to focus on algebraic expressions, it is important to help students make connections to what they already know and understand. The study, done by Subramaniam and Banerjee, asked students to find the perimeter of a figure with k sides, each with a length of 5. There were "near zero" correct responses. However when the students were given the task of finding the perimeter for a shape with 10 sides, each having a length of 4, over 60% got the correct answer. [2]

An expression is recipe for a mathematical calculation. An introduction to expressions should begin with conversations about relationships between different values. I will begin with low numbers and simple ideas that allow all students to easily grasp the idea of the expression and to learn the standard way of writing basic expressions. Challenging manipulations can be used once students are clear on the components of the expression. As suggested by Roger Howe, "I advocate a gradual approach, steadily upping the ante, and checking for mastery at each stage."

A question to begin the process, "Ana has three more pencils than Carmen, how many does Ana have?" Most students will recognize the question cannot be answered, though some will attempt a solution. Those that jump for the solution are the ones who suffer from the need to always find a simplified answer. They will be a focus group later when I talk about the challenges of changing the way students think about the equal sign.

The majority will realize that they are unable to answer the question because there is not enough information. This question gives them a minute to ponder the idea of an "unknown amount." Before taking it to the variable expression, it is helpful to give students some amounts for Carmen. First Carmen has 1 pencil, then 5 pencils, then 10… allowing students to respond with how many pencils Ana must have. This dialogue helps students recognize the fact that each time I give them an amount for Carmen they are able to add 3 and find the amount for Ana. Numerical expressions are often clear and second nature. It is important to make the expression concrete by writing it down and connecting each part of the expression to the given definition.

Many students lose meaning and context with expressions when variables are introduced. In order to make a smooth transition I will write "Number of Carmen's Pencils + 3", as the expression. While the discussion seems very elementary for 8 th grade, the purpose is to avoid the disconnect that happens for most students with the variable and operation. Now is a good time to re-introduce the variable. I say, "re-introduce" because students have learned about variables in previous grades, however my concern is they have not been given a conceptual understanding and therefore don't use it with a clear awareness of its purpose. In the article, "Arithmetic to Algebra" Roger Howe makes a key point about variables. The unit must always be included in the definition along with the words "number of". [3] The variable is not identifying an item, but rather the number of items in the given problem. This "good mathematical hygiene," as Roger would say, prevents problems later on for many algebra students when they begin to work with systems of equations.

Once students can interpret and use the basic expressions, I will work with progressively more complex expressions, using a similar strategy. The original expression was "Ana has three more pencils than Carmen, how many pencils does Ana have?" The algebraic expression being p + 3, where p represents the number of pencils Carmen has. The next expression, *Max has 3 times as many pencils as Carmen.* I will repeat the same amounts for Carmen as I used in the previous example. Carmen has 1 pencil, 5 pencils, and 10 pencils. Students will be asked to determine how many pencils Max has. They will also have to decide what operation is being used to calculate the number of pencils Max has. Writing the arithmetic problems down is helpful, especially when making a connection to the algebraic expression. To create a visual example we will write down the arithmetic. Carmen has one pencil, how many does Max have? 1 x 3 = 3. Carmen has 5 pencils, how many does Max have? 5 x 3 = 15. Carmen has 10 pencils, how many does Max have? 10 x 3 = 30. So our new expression is "Number of Carmen's Pencils" x 3. Again being clear that our variable p does not represent pencils rather the number of pencils, we can write the second expression as 3p.

The next expression is a little tricky. *Sam has 3 times as many pencils as Ana.* The tricky part is, that we want to express the number of pencils that Sam has in terms of the number of pencils Carmen has. The increase in complexity here is a pivotal point. Students begin to disconnect from the original p + 3. It is crucial that the students again identify what the variable is representing as well as what the entire expression represents. Ideally the white board will be organized with each set of expressions. Therefore, we can go back and look at the previous expression for Ana, which is represented by p + 3. This moment is a great opportunity to identify students who track correctly with the expressions. I ask students, "If Sam has 3 times as many pencils as Ana, and Ana has 5 pencils, how many pencils does Sam have?" Students easily answer 15. The next question, "If we know how many pencils Carmen has, can we figure our how many pencils Sam has?" After students have had a moment to think the question through, I suggest that Carmen has 6 pencils, how many does Sam have? The three most common answers are 18 (because the student multiplied 6 x 3), 21 (because the student multiplied 6 x 3 and then added 3), and then of course the correct answer 27 (because the student added 6 +

3 and then multiplied the sum by 3). As students share their answers I also have them share the process. To clear up the confusion we begin with Carmen having 6 pencils and work our way through Ana, Max, and Sam. At this point we discuss the importance of parentheses. Students need to know that, when we put an expression in parentheses, we mean for the result of the whole expression inside to be multiplied by the number outside. Parentheses mean, "treat whatever is inside as a single number." The final expression for Sam will be 3(p + 3).

Even though students understand how to calculate the amount of pencils for Sam, some make a mistake and write the expression 3p + 3. Rather than just dropping this suggestion (if it is made), it is important to take this expression and define it in context of the situation. If 3p + 3 were the algebraic expression, then the verbal expression would have to be something like, *Sam has three times as many pencils as Ana, and three more besides.* Students need to be re-directed when they make an error, but it is equally important that they are aware of why a solution is not correct. Discussion about how Sam has 3 times as many pencils as Ana and therefore 3 must multiply to the whole quantity of p + 3 will reinforce the idea of parentheses. The precision with vocabulary here is also important. Students should be able to speak a verbal expression that represents 3(p + 3) using vocabulary such as "three times the quantity of p plus 3" or "three times the sum of p plus 3."

To continue practicing the precision of vocabulary students will take turns creating expressions and telling them to the class. The class writes them down and then compares to check for accuracy. I will give the students a few examples to get them started. See appendix A.

Simplifying Expressions

Students will learn to simplify complex expressions using the rules of arithmetic. The Associative, Commutative, and Distributive rules are not new to Algebra students. However, even the high achieving students see the properties as a vocabulary issue rather than rules to be internalized and applied. In her article *Teaching Arithmetic and Algebraic Expressions*, Subramaniam states, "Rules must be connected to concepts in order to enhance their learning and retention. Since concepts occur as referents in the statement of rules, conceptual misunderstanding may lead to incorrect learning of rules. Pupils need to be flexible in their application of rules and conceptual understanding mediates such flexibility." [4]

As an introduction to simplifying expressions I will begin this section with some number tricks. Students will fold a piece of notebook paper into three columns. They will do the work for each trick in its own column. This will allow them to go back and analyze their calculations to see if they can determine how the number trick works. Once we go through the number tricks and look at the visual diagram, students will go back and re-write expressions for the following three number tricks.

Trick 1: Choose a number between 1 and 10. Add 5. Double that. Subtract 6. Cut it in half. Subtract your original number.

I will then have students compare answers with a neighbor. Everyone should have a result of 2. I will give students two minutes to discuss why this happened with their group.

Trick 2: Choose a number between 1 and 10. Add 12. Multiply by 2. Subtract 18. Add the opposite of your original number. Subtract 6.

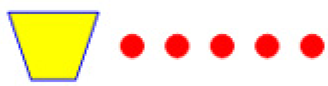
Everyone should now be at the original number they chose. Again I will have students discuss this with their group. I will encourage students to discuss why this trick is different from the previous trick.

Trick 3: Choose a number between 1 and 10. Add 7. Multiply by 4. Subtract 14. Divide by 2. Subtract the original number.

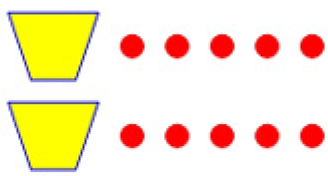
This trick will allow me to tell the student the number they chose by subtracting 7 from their final answer. Again students will be given some time to look at and discuss the three tricks. I will encourage students to share the different numbers they chose as well as the steps they took with each direction. It is my hope that a few students will look at the different papers within their group and begin to see the similarities in the process and work. Each group will be given the opportunity to share why the tricks worked.

Thinkmath.edc.org offers a simple visual explanation. I've included the drawing they would use to represent the first number trick.

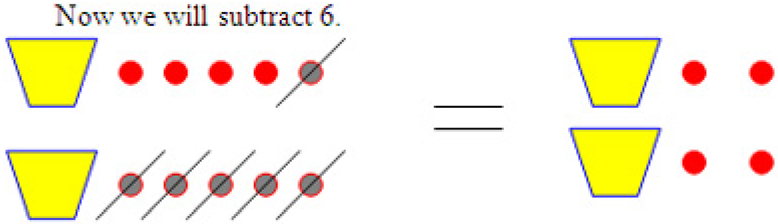
In the bucket is the unknown amount that was chosen, along with 5 additional dots.



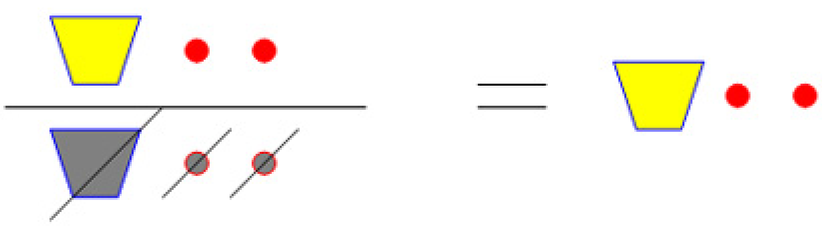
Here, the total amount has been doubled



Now we will subtract 6.



Next, we will cut it in half.



Finally, we will subtract the original amount, which is the unknown amount inside the bucket. This will leave everyone, regardless of the number they chose, with a final answer of 2.

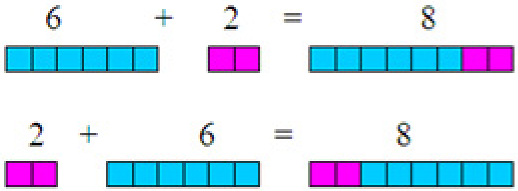
Once students understand the visual model, we will go back and replace the pictures with variables and numbers. I will have students look at the connection between the "unknown bucket" and a variable. Students will then work in groups to draw both visual models and write the expressions for the first three number tricks we did. Students will see how helpful variables and expressions are for explaining these number tricks. Students should then be given time to make up their own number tricks as well as the explanations for why the trick works.

Rules of Arithmetic

Arithmetic Rules will be an important focus of this unit. We will look first at the rules and how they apply to addition and then go back through to see how they apply to multiplication.

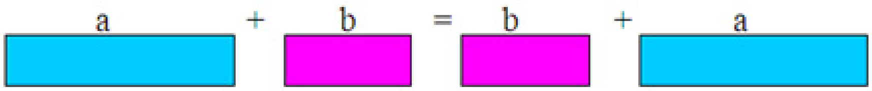
The Associative Rule of Addition allows us to group numbers in any order, (a + b) + c = a + (b + c). However, I find most students identify any expression with parentheses as the Associative Property regardless of the grouping. In order to avoid this misconception I want to give students scenarios to group the numbers in different ways but to see that the solution remains the same. What I find interesting is the concept seems elementary in a problem like *Jay received $5 from his dad and $3 from his mom. How much money does he have? A couple hours later Jay's sister gave him $4. How much does he have now?* Then to reword the same problem to *Jay received $5 from his dad. Jay went into the kitchen and his mom gave him $3 and his sister gave him $4. How much money did Jay receive from his mom and sister? How much money does Jay have altogether?* This type of discussion is obvious to most students. I feel it is important at this stage of clear numerical thinking to make the connection to the Associative Property using variables. Once students grasp the concept, the difficulty level will be raised. Students will be asked to find all the different ways to group the addends. Students will begin with 3, then 4, then 5 addends. They will work to come up with all the ways the addends can be grouped. Students will then extend this to writing word problems. They will keep the information the same in each problem, however write an explanation for the different groupings. This will be extremely challenging for many students especially the English Language Learners. In order to scaffold the activity, I will work with small groups to write the first problem. Then we will orally discuss how we would change the problem in order to change the grouping only. It will be important to check with struggling groups each step of the way, to be sure they are understanding both the concept of grouping as well as how to explain it in word format. I have never done an activity like this, but it makes me smile as I think about it. Students will struggle, but enjoy the challenge of it. I look forward to seeing what they come up with.

The Commutative Rule of Addition allows for numbers to be added in any order. For example 6 + 2 = 2 + 6.



The numerical and visual representation eliminates unnecessary confusion when variables are used to describe the generic rule of the property.

Addition:



Most students are comfortable with this straightforward explanation. However as expressions expand there is some difficulty with the movement of terms so as to collect like terms. I will address this difficulty later when I talk about collecting like terms.

Since the Associative and Commutative Rules of Addition are fairly obvious, I plan to deal with them first. Once students are clear on the addition rules we will move to the rules of multiplication.

The Commutative Rule of Multiplication allows numbers to be multiplied in any order. It states that ab = ba just as 6 x 2 = 2 x 6. An array model is a clear way to show students this rule. Using graph paper to draw a box that is 2 units long by 6 units wide and then flipping the same rectangle so it is now 6 units long by 2 units wide. To help students generalize their understanding of this rule I will ask them if this same thing would be true if the numbers were 5 and 8, 9 and 4, 32 and 14, 25 and 25, x and 4? As students begin to realize it doesn't matter what whole numbers we use they will see it is a matter of multiplying the number of rows by the number of columns or the number of columns by the number of rows, regardless of what those values are. A couple of examples using larger numbers is important as well as drawing the rectangles using variable expressions to represent length and width. It is also a good idea to acknowledge different types of units. For example, 6 CDs at $8 each and 8 CDs at $6 each both cost $48. 8 pizzas with 10 slices and 10 pizzas with 8 slices both have 80 slices (however, size of slice might come into discussion here.)

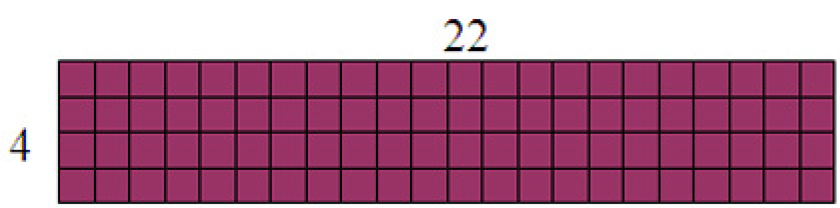
The Associative Rule of Multiplication is a little tricky. To demonstrate I will use a shoebox. It is important to stop here and review the formula for volume. To clearly show the Associative Rule I will explain to students that the volume of the box is the area of its base times the height of the box. So the question becomes, will the volume change if we put the shoebox on a different side, giving it a different base? Once students agree that it doesn't change, we will look at the measurements and different ways we can group the factors.

The Distributive Rule is a unique rule, which connects addition and multiplication. This rule is a favorite among 8th grade students. I find they prefer using the rule when variables are involved. Something like 4(3x +5) causes less of an uproar than 4(3 + 5). The tragic frustration stems from the fact that students have no depth in their understanding of the distributive property. They have never learned to break numbers apart (like 8 x 42 is equal to 8(40 + 2)). Therefore they don't see the connection between the order of operations and the distributive property. They recognize the need for order of operations in a problem like 4(3 + 5) but miss the connection that 4(8) is equivalent to 4(3) + 4(5) because 12 + 20 = 32. Time will need to be spent developing mental math skills to solve a problem like (8)(63) by breaking it up into 8(60 + 3). This will be a great opportunity to work with number manipulations and see the Rules of Arithmetic at work. It is crucial to take the time to break down their desire to compartmentalize everything they learn. Students must realize these rules are used because they work – always! The Distributive Rule is a phenomenal tool for calculation and we really don't teach students how to use it.

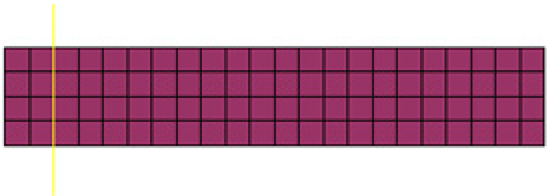
Since the Distributive Rule can be so valuable for mental calculations, I will use some mental math to introduce it in the classroom. I always place a high emphasis on the importance of showing work. Introducing an activity that requires mental math will spark interest in some students as well as anxiety in others. It is also an opportunity to see which students have some background with the Distributive Rule. I will ask students to mentally solve the following problems: 8 x 32, 4 x 39, and 12 x 26. Students will be given an opportunity to share the strategies they used to compute the problems mentally. This will be the first time I have done an activity like this, so I look forward to hearing the different approaches.

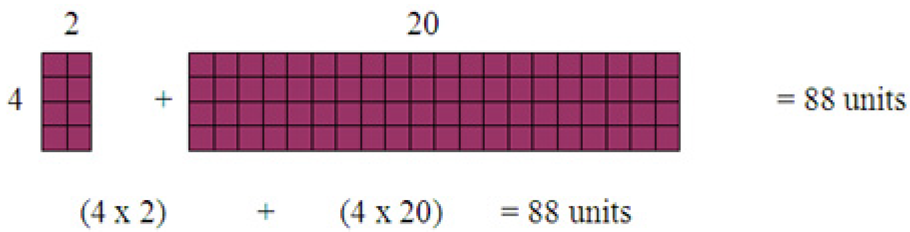
I anticipate that some students will have used the distributive property to expand the numbers, for example 8 x 32 will be broken down into 8 x 30 + 8 x 2. This observation will allow us to look at other problems and see why it works. Students will also discuss why we choose to break 8 x 32 into 8 x 30 + 8 x 2. Again this addresses the importance of students learning these rules conceptually so that they can be flexible and confident in the way they solve them. 8 x 30 + 8 x 2 is thought to be the easiest way to mentally solve the problem 8 x 32, however 8 x 15 + 8 x 17 and 8 x 25 + 8 x 7 are other forms of the distributive property, all of them creating equivalent expressions.

After the mental math and work breaking apart the multiplications problems, students will move to working with area figures to deepen their conceptual understanding of breaking these numbers apart. Take a problem like 4 x 22, I will show students a rectangle with a length of 22 and a width of 4.



I will ask students to find the area of the given rectangle by using the formula Area = length x width. Once students calculate the area of the rectangle to be 88, I will have a student draw a line to cut the rectangle into two rectangles. Which might create a figure like the one below.





Now rather than 4 x 22 representing the area we have broken the rectangle into 4 x 2 and 4 x 20. Students will then see visually the area of the rectangle has not changed, even though it has been broken apart. This idea can be extended into two-digit by two-digit multiplication as well. For example we will use a rectangle with a length of 47 and a width of 22. Now, the rectangle will be cut both horizontally and vertically and will create four new rectangles. Though many cuts could be made, to begin, we will cut the rectangle's width of 22 into 20 and 2. The rectangles length will then be cut into 40 and 7. The area of each new rectangle will be added together to get the total area of the original rectangle. So, 22 x 47 = 20 x 40 + 2 x 40 + 7 x 20 + 7 x 2 = 1,034.

Solid understanding of the Distributive Rule is necessary to combine like terms with consistency and confidence. Students will be able to distribute numbers or variables into the parentheses and then use the commutative and associative rules to re-order terms. Students struggle with being able to identify a complete term as well as keep the correct sign with it. Once students simplify using the distributive property they will box each term with its sign in the expression. The students begin with an expression, 3(2x – 7) – 8x + 12. The students should start by distributing the 3 into the parentheses, so the new equivalent expression will be 6x + (– 21) + (– 8x) + 12. At this point, students box the terms so they can easily identify each individual term.

6x	-21	-8x	+12

Students will then use the commutative property to re-order the terms. The re-ordered expression will be written 6x + (– 8x) + (– 21) + 12. In the past I have allowed students to identify and combine like terms, however doing this prevents them from seeing the distributive rule still at work. One of my favorite Roger Howe quotes is, "Shortcuts are the privilege of the expert." Therefore, students will need to show the distributive property as the next step. Taking the new expression and grouping together the like terms so (6x – 8x) – 21 + 12, and then recognizing that the x can be pulled out of both terms x(6 – 8) -21 + 12. Students continue to simplify x(-2) – 21 + 12 = -2x – 21 + 12, and finally the simplest form of the expression -2x – 9.

Once students learn the properties and how to use them to simplify the expressions they will need time to practice the manipulations. Students will practice by setting up a two-column paper. On the left, they will write the expression and work the computational part of the problem. On the right, they will list the property that is being used.

Example: Simplify 5(3x -7) + 4x

5(3x – 7) + 4x	Given Expression
5(3x – 7) + 4x = 15x – 35 + 4x	Distributive Property
15x – 35 + 4x = 15x + 4x - 35	Commutative Property
15x + 4x - 35 = (15x + 4x) – 35	Associative Property
(15x + 4x) – 35 = x(15 + 4) – 35	Distributive Property
x(15 + 4) – 35 = x(19) – 35	Simplify
x(19) – 35 = 19x – 35	Distributive Property

The expression in simplest form is 19x – 35.

The manipulations of the expressions will make a direct connection to the concept of equivalent expressions. Students apply the properties to the expressions in order to identify which expressions are equivalent. Students will need to be able to explain why expressions are equivalent by identifying which property they are using.

Equations

The background taught for the expressions will be the foundation upon which solving equations will be built.

The equal sign has become a stumbling block in mathematics curriculum, mainly because the education system has pushed so hard for students to memorize steps, arrive at an answer, and bubble it in on a scantron. We have lost the sense of process and understanding. J.J. Madden illustrates this problem in the article, "What Is The Equals Sign?" "Another dysfunctional thought-model connects the equals sign we write on paper with the equals key on a calculator, which we press when we want the calculator to show us the answer?" [5] I hope to alleviate this issue by spending quality time looking at equivalent expressions. In his article Madden points out that students who come into Algebra with the notion that there should be a, *math problem = answer*, are not incorrect. The equation 7 + 4 = 11 is a true statement and an accurate use of the equal sign. However, students need a broader interpretation of the equals sign. They need to see the relationship between values and expressions in order to grasp the meaning of the equals sign in all contexts.

To solve equations students must learn two more Rules of Arithmetic, the Identity Rule and the Inverse Rule. However, before directly teaching these rules I want to help students get a clearer idea of what an equation is. A clear explanation of the change between expressions to equations is required. Students have now spent a considerable amount of time learning that an expression is a recipe for calculation. The equation now takes two expressions and states that they are equal, and then asks us to find the value of the variable that makes this true.

When describing the goal for solving the equation students will use the words, "What value of (given variable) makes the equation true?" I will then give students equations they are able to compute mentally. Some examples are 5x = 30, x + 7 = 25, 3x + 4 = 10. Rather than following the usual rules and procedures for solving equations, I want students to make the connection that x has a value. Then I will put up the problems ¾ x – ½ = 17 ½ and 2.7x + 13.08 = 26.58. Two problems students are not able to solve mentally. How do we figure out the value of x now? After some discussion we will look at the Identity Rule and Inverse Rule.

The Identity Rule can cause some confusion for students if its purpose is not made clear. It tells us that there is a number such that, when it operates on any other number, it does not change it. It is clear to most students that The Identity Rule for Addition states that a + 0 = a. The rule allows us to build the foundation for understanding inverses.

The Inverse Rule for Addition states, for any number a, there exists a number, -a, such that a + (-a) = 0. After a few samples, students can easily answer a questions like, 7 plus what number gives you 0? Or, what is the additive inverse of – 9? Regardless of their ability to answer these questions I think it is important that the idea is continually reinforced with a number line or visual model. This concept is easy to memorize in isolation, but can be difficult to apply when working with equations. Students will need to practice applying the rule first to simple equations and then continue with more advanced equations. For example x + 7 = 9, students will add -7 to both sides, (x + 7) + (-7) = 9 + (-7). Students will then simplify using the Commutative and Associative Rules, x + (7 + (-7)) = 2, x + 0 = 2, x = 2.

A little more difficult for students to understand is the Multiplicative Inverse Rule, which tells us that a rational number "a" multiplied by its reciprocal equals 1, (a)(1/a) = 1. Few students grasp multiplication of fractions in the primary grades and therefore it is a challenge to understand (a) (1/a) = 1. Some review of multiplying fractions is necessary for them to see the connection. This will provide an excellent opportunity to review the unit fraction and the idea that multiplying by 1/d is the same as dividing by d. Since this is not a new topic it is a great place for group discussion. Students will work in groups to discuss why (5)(1/5) is the same as 5 divided by 5 and why (25)(1/5) is the same as 25 divided by 5. Though this type of computation should be second nature by 8 th grade, it is not. Extra time spent here will prevent confusion when solving equations, especially those containing fractions. To see how well the students have mastered the multiplicative inverse concept I will ask questions like these: Why does zero not have a reciprocal? What is the reciprocal of 0.001?

While conceptual understanding is crucial for students to be successful in math, I also value procedures and organization. To practice using the properties to solve multi-step equations, student set up their work in a clear and organized manner. As we begin to look at some problems as a class, students will set up the two-column paper. The left is for the computation and the right is for identifying the property (same as we did when we simplified expressions). Students will write the equation and draw a line down from the equal sign to show the balance of the equation. It is useful to keep the equal sign working its way down the equation in order to continue to reinforce the idea of balance and equivalent expressions.

Once students are able to find a value for the variable, their solution needs to come in some form of a statement, such as "when the value of x is 7, this equation is true." It is important for students to stay connected to what this statement means. Therefore they must plug the value of the variable back into the original equation to be sure the statement is true. Lots of discussion is important at this point. If a student gets an answer that doesn't work, it is a great opportunity to discuss what the solution would need to be if that was the correct value. Also, as students check their answer by plugging it back into the equation, they should be asked to read expression as the recipe for calculation. For example, if a student solves the problem 2(3x + 8) = 40 and states the answer, "When the value of x is 4, this equation is true." They should then be able to go back and check by saying, "Three times 4 is 12, and 8 more is 20, then doubled makes 40."

As the problems increase in difficulty, students will find variables on both sides of the equations. This is one area it is so easy to allow students to just follow the procedure to solve rather than discussing what it means. I would like to see students gain the ability to explain the idea of equivalent expressions. Often, students are taught to get variables on one side only and solve. Yes, this works, however it is important that they understand that each side contains an expression. The goal is to find the value of the variable that can be plugged into both expressions to find equal answers. While students will learn to solve equations by getting the variable one side, I think it will be beneficial to reinforce this concept of the expressions being equal when checking the answer. See Appendix A for sample problems.

Word Problems

8 th graders see word problems as the last two problems on the page, optional and often not worth the effort. When asked, students feel justified in their answer, "I didn't understand, I tried but couldn't figure it out, etc…" The more ambitious student recognizes that the first 20 problems on the page were solved using single step addition equations and therefore the word problem must follow this pattern. The diligent, committed to completing homework, plug numbers and variables into the format and hope they chose location correctly.

My degree is in bilingual education. So my strength is working with English language learners in the content of the classroom. Math is often seen as a second language, so I have learned skills necessary to teach word problems to non-native math speakers. The word problem can be broken down and understood totally apart from the computation process. Time spent doing this will help break the habit students have of filling in random numbers and operations regardless of meaning. To begin, students will work with a wide variety of word problems that look at situations students can relate to and understand. Students practice by drawing pictures, charts, graphs, to represent the information given. Initially they will work on identifying what information is given, what information is the question asking, and what is the relationship between different knowns and unknowns in the word problem. Students will be formally assessed on their ability to simply understand all the components of the word problem. See examples in Activity Three.

Once students begin to develop the habit of reading word problems for meaning, we will move to writing the equations that solve the word problems. This is a difficult transition, however if students have mastered the previous sections of the unit, they have the background knowledge necessary to build the equations. As suggested in the article "Arithmetic to Algebra" by Roger Howe, students need to practice translating a wide variety of word problems into both algebraic and arithmetic expressions. The idea in this article is that students will benefit from translating the same types of problems into algebraic notation and arithmetic notation. The simple problems will teach students to set up equations and the more challenging problems will show them the advantage of using the equations. "By experiencing the strong connections between the two, they can come to appreciate the maxim that 'algebra is generalized arithmetic'. Three key benefits of teaching the problems in parallel are: all levels of problems work both arithmetically and algebraically, the arithmetic and algebra are comparable in structure, and as the problems advance in difficulty the advantage of the algebraic form becomes more obvious. [6]

Students with the ability to read the word problem, interpret the information, translate it into expressions that in turn create an equation, solve for the unknown value, and identify what the unknown value represents, have mastered the multi-step problem and are in an ideal place mathematically to move forward in their learning.

Classroom Activities

Activity One – Translating Expressions

An expression is a recipe for a calculation. Expressions can be represented in a verbal sentence as well as a mathematical sentence. In this lesson, students will practice translating back and forth between the two forms.

Students will begin by playing a number trick. Tell students to choose a number. Multiply the number by 2. Add 14. Multiply by 3. Divide by 6. Subtract the original number. All students should end up with a final answer of 7. Students will work in groups of 4 to discuss why everyone ended up with an answer of 7.

Vocabulary Background: Students will work with their assigned group to define important vocabulary. All members of the group should be able to give an accurate definition, use the word correctly in context, and give an example. The words students will define include: sum, total, increased by, more than, difference, decreased by, less than, product, twice, half of, third of, quotient, ratio, square of, cube of, consecutive, even, odd. Students who finish early will look for synonyms of given vocabulary as well as other relevant terms not mentioned on the list.

Once students have finished defining the vocabulary. I will put up problems to practice the use of the vocabulary. Students will volunteer different math sentences to express the given problem.

Example: 8 + 5 = 13. Possible Responses: The sum of eight and five is 13. Five more than eight is thirteen. Eight increased by 5 is thirteen etc…

Students will set up 2 columns in their notes. At the top of column one they will put Verbal Expression and at the top of column two they will put Mathematical Expression. If the given problem is in the form of a verbal expression, students will write the mathematical expression. If the problem is in the form of a mathematical expression, students will write the verbal expression. Students should not copy the given problem. For homework students will take their notes home and write the missing expressions, which will match the original problem set.

Example:

1. Take a number, multiply by 2, add 3.

2. 5(x + 4)

	Verbal Expression	Mathematical Expression
1.		2x + 3
2.	Take a number, add 4, multiply by 5.	

Next students will work with a partner to write Recipes for Calculations. I will do the first one on the overhead with the students. Take a number and add 7. Multiply by 3. Decrease the number by 4. Multiply by 3 times the number. Students will write both the directions and the expression on their paper. Then with their partner they will write 3 – 5 Recipes for Calculations. Once most students have finished they will then partner up with another group of 2 students. Each group will take turns reading their recipe, while the other group writes the expression.

To close, I will give students two number tricks. Since these tricks are following the activity Recipes for Calculation, I think more students will be able to use the skills to identify why the trick works. Trick: Choose a number. Add 12. Multiply by 2. Subtract 18. Add the opposite of your original number. Subtract 6. This will leave students at their original number.

Trick: Choose a number. Add 7. Multiply by 4. Subtract 14. Divide by 2. Subtract the original number. I will be able to tell students their original number by subtracting 7 from their final answer.

Students will spend the rest of the period working with their assigned group to try to figure out the number trick.

Activity Two – Model Drawings of Arithmetic Rules

All students will be given graph paper for the drawings. The purpose of the activity is to visually show students why these properties work.

Commutative Property Model

Students will fold the graph paper into three sections. The first section will be labeled Commutative Property. Students will begin with the problem 4 + 6. The numbers will be colored so students will use their colored pencils to match the color of the box with the color of the number. They will begin by writing the problem across the top. Then the students will color 4 red boxes and 6 green boxes. Students will then represent the Commutative Property by writing the reverse 6 + 4 and coloring in the corresponding boxes. Students will then be given a blue strip of paper, an unknown amount, which will be represented with the variable x. They will then write the expression x + 7. Students will color in the x boxes blue by lining the strip of paper up on the graph paper to see how many boxes should be colored and then adding an additional 7 orange boxes. Then to show the Commutative Property students will write the expression 7 + x and color in 7 orange boxes and then line the blue construction paper up to color in the x unknown boxes blue.

Associative Property Model

Students will label the second section Associative Property. Again students will color to match both numbers and variables. They will begin with the problem (3 + 2) + 4. They will begin by coloring in the corresponding number of red, blue and green boxes. Then students will take their pencil and shade over the three red boxes and the two blue boxes to show the grouping of 3 and 2 in the parentheses. For the second problem, 3 + (2 + 4) students will color the boxes the same, however this time they will shade over the two blue boxes and the 4 green boxes to show the grouping of 2 and 4. Students will follow this same procedure when given the problem (5 + x) + 3 and 5 + (x + 3). Again they will be given a strip of paper to measure boxes for the unknown, x, amount.

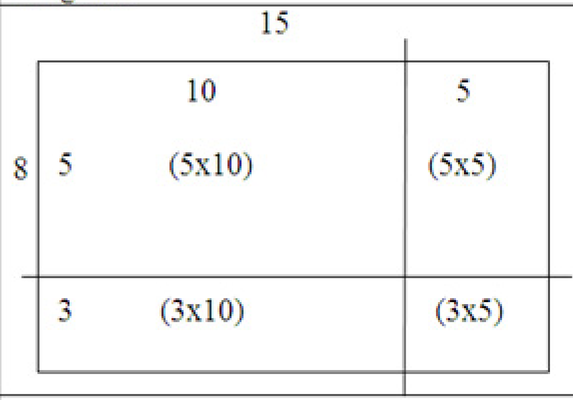
Distributive Property

The third section of the paper will be labeled as the Distributive Property. Students will start by sketching a 4 by 6 box on their paper. They will identify the problem as being 4 x 6 and the solution as 24. Then students will draw a line down the center of the box, breaking it up into two 4 by 3 boxes, each with an area of 12.

Students will see that the area didn't change. We will write the second problem as 4(3 + 3). Students will then draw a 12 by 8 box and look for different ways to break up the boxes. I will put several charts of the 12 by 8 box on the board. Different students will show how they divided the box and how they calculated the area. While students are working I will walk around and encourage some to try breaking the 12 and 8 up more than once, for example (4 + 5 + 3) (3 + 5). Students who present on the chart paper will explain the drawing as well as show the arithmetic they used.

Then we will move to two-digit multiplication. Students will draw a box that is 25 by 45. They will then break the box into 20 and 5 by 40 and 5, hence creating four different rectangles. They will find the area of each individual rectangle and then add the areas together.

Students will then complete the following problems by drawing a rectangle and breaking it into smaller rectangles. Each problem will need to be solved 2 different ways, both containing a diagram as well as the arithmetic to show the solution.

Problem	Diagram	Arithmetic
Example: 8 x 15		
5 x 22		
5 x 22		
18 x 6		
18 x 6		
22 x 62		
22 x 62		
75 x 85		
75 x 85		
58 x 125		
136 x 164		

Activity 3 – Word Problems

This lesson will focus on reading and understanding word problems. Students will be able to label each numerical value, identify the unknown, and write a complete sentence to answer the question.

Students will begin by working in their groups on the following two problems. They may solve them any way they choose. Each member of the group should be prepared to give the explanation.

Problem 1: The cost to rent a sailboat from Ocean's Rental Company is $120 plus $60 per day. If Marco paid a total of $360, for how many days did he rent the sailboat? [7]

Problem 2: The local cell phone company charges a monthly fee plus an additional $0.05 per text message. Margo sent 60 text messages this month. If her bill was $33, how much is the monthly fee?

Problem 3: Caper's Carpets charges $100 plus $0.25 per square foot to clean carpets. The carpet in Jackson's house covers 1,000 square feet. If Jackson has saved $200, how much more money does he need to get his carpets cleaned?

Students will then be given the opportunity to share their answers and methods for arriving at such answers. Student answers will not at this time be evaluated as right or wrong.

I will discuss with students the importance of analyzing and understanding a word problem before applying mathematical operations to calculate a solution. So as a class we will work through the information in the problems without calculating solutions. Students will be evaluated on their ability to describe the problem and the information in the problem. During this activity the final solution will not be required. Tomorrow's lesson will focus on building the equations to solve the problem.

As a class we will work through the first problem answering the following questions.

1. What is the question asking?

Response – How many days did Marco rent the sailboat?

2. Write a complete sentence for the solution.

Response – Marco rented the sailboat for ________ days.

2. Identify all numbers and what they represent in the problem.

Response – $120 = Cost to rent the sailboat. $60 = cost per day to rent the sailboat. $360 = total amount Marco paid to rent the sailboat.

3. Explain a method that could be used to solve this problem.

Response – Marco started by paying $120 to rent the sailboat, then he had to pay an additional $60 for everyday he rented the sailboat. So starting at $120, I could keep adding $60 until I got to $360 and see how many times I had to add $60.

Students will answer the same questions for problems 2 and 3. The assessment for this assignment will be identifying key pieces of information, not a numerical answer.

Bibliography

"Algebra I Released Test Questions." CA Dept of Education. www.cde.ca.gov/ta/tg/sr/documents/rtqalg1.pdf (accessed August 1, 2011). Released test questions from the California State Test.

"Content Standards – Standards & Frameworks (CA Dept of Education). "CA Dept of Education Mobile. http://www.cde.ca.gov/be/st/ss/ (accessed August 3, 2011). Algebra I Standards for the state of California.

Cuoco, Albert. CME *Project.* Boston, Mass: Pearson, 2009. Excellent, straight-forward text, which gives clear, well-rounded examples.

Howe, Roger. "From Arithmetic to Algebra – Journal of Mathematics Education." China National Knowledge Infrastructure. http://ath.arizona.edu/~ime/2008-09/Algebra%20Essays/ArithtoAlg.pdf (accessed August 3, 3011). All Algebra teachers should read this article. It is a detailed explanation with examples about teaching word problems.

Howe, Roger. "The 'Rule of Signs' in Arithemtic." Essays for the Noyce-Dana project. https://www2.bc.edu/~friedber/essays.html (accessed August 1, 2011). Great teacher background for signs. Also, valuable examples dealing with signed numbers for students at the end of the article.

Ma, Liping. *Knowing and Teaching Elementary Mathematics: Teachers' Understanding of Fundamental Mathematics in China and the United States.* New York: Routledge, 2009. Convicting book, good for all level math teachers.

MacNeal, Edward. *Mathsemantics: Making Numbers Talk Sense.* New York: Penguin Books 19951994. This is an eye-opening book. Great thoughts to keep in mind while teaching.

Madden, J.J., "What is the Equals Sign?." *LATM Journal 5*, no.1 (2008) http://www.lamath.org/journal/vol5no1/vol5no1.htm (accessed August 1, 2011). The use of the equals sign and how student understanding of the equals sign needs to broaden.

Merzbach, Uta C., and Carl B. Boyer. *A History of Mathematics.* 3rd ed. Hoboken, N.J.: Wiley, 2010. This book is huge and dense. I've only read some small sections of it, but I really enjoyed the background information and I intend to keep working my way through it.

Subrmaniam, K., and Rakhi Banerjee. "Teaching Arithmetic and Algebraic Expressions." Eric World's largest digital library of education literature. http://www.eric.ed.gov/ERICWebPortal/search/detailmini.jsp?_nfpb=true&_&ERICExtSearch_SearchValue_0=ED489552&ERICExtSearch_SearchType_0=no&accno+ED489552 (accessed August 1, 2011). Three teachers' approaches to teaching algebraic expressions.

Appendices

Appendix A

Translating Expressions Practice Problems. For numbers 1-6 students write the algebraic expression. For numbers 7 – 12 students write the verbal expression in words.

1. A number multiplied by 4.

2. Twice a number decreased by 7.

3. 6 more than 3 times a number.

4. The sum of 5 times a number and 8.

5. The sum of 3 times a number plus 6, multiplied by 2 and then increased 4. The final result multiplied by 8

6. The difference of 4 times a number and 6 divided by 2 times the number.

7. Take a number, multiply it by 4 and then add 16.

8. Take a number, add 7 more, multiply by 3, divide by 8.

9. Take half of a number, square it, subtract 12.

10. $7x$

11. $x + 3$

12. $2x – 12$

13. $8(3x -3)$

14. $\frac{1}{2} x – 6$

15. $2[4 + 3(2x + 1)]$

Appendix B

Equations with Variables on Both Sides

1. $4x = 3x + 5$

2. $2x – 1 = 4 + x$

3. $6x + 3 = 2x + 11$

4. $10 – 3x = 2x – 8x + 40$

5. $5 + 4x – 7 = 4x + 3 – x$

Appendix C – Implementing California Content Standards for Algebra I

4.0 Students simplify expressions prior to solving linear equations and inequalities in one variable, such as $3(2x – 5) + 4(x – 2) = 12$

5.0 Students solve multistep problems, including word problems, involving linear equations and linear inequalities in one variable and provide justification for each step.

The focus of this unit is on standards 4.0 and 5.0. I believe that a firm foundation in understanding expressions and the rules of arithmetic used to simplify them will lead students to be successful when they take on equations.

Endnotes

- 1 "Algebra I Released Test Questions." CA Dept of Education.
- 2 Subrmaniam, K., and Rakhi Banerjee. "Teaching Arithmetic and Algebraic Expressions."
- 3 Howe, Roger. "The 'Rule of Signs' in Arithmetic."
- 4 Subrmaniam, K., and Rakhi Banerjee. "Teaching Arithmetic and Algebraic Expressions."
- 5 Madden, J.J. "What is the Equals Sign?."
- 6 Howe, Roger. "From Arithmetic to Algebra—Journal of Mathematics Education."
- 7 "Algebra I Released Test Questions." CA Dept of Education.