Curriculum Units by Fellows of the National Initiative
2011 Volume VI: Great Ideas of Primary Mathematics

Boosting Number Sense in High School Students

Curriculum Unit 11.06.11, published September 2011
by Nancy Rudolph

Overview

"Our classrooms are filled with students…who think of mathematics as rules and procedures to memorize without understanding the numerical relationships that provide the foundation for these rules." [1] In other words, they are lacking number sense. Number Sense is included in the Delaware standards, as well as the Common Core State Standards beginning in Kindergarten; however, I continue to have Precalculus students tell me they cannot do basic arithmetic without a calculator. The goal of this unit is to boost my students' conceptual number sense and build their confidence in mental calculations. According to John Threlfall, mental math is worth teaching because most calculations in adult life are done mentally. He believes that mental math develops problem-solving skills, as well as increases success in later written calculations using algorithms. [2] In Rheta Rubenstein's opinion, mental math liberates students from calculator dependence, builds confidence, makes them more flexible thinkers, and makes them more able to use multiple approaches to problem solving. [3]

I have compiled the key components from Roger Howe's seminar entitled "Great Ideas of Primary Mathematics" along with the recurring weaknesses that I see in my classroom to create this curriculum unit. The great ideas we worked with are place value, measurement and the number line, and operations. This unit will encompass all of these ideas in the form of *calculator-free*, daily warm-up/bell-ringer activities for one semester, focusing on one topic per week, and spiraling some concepts back throughout the semester. Throughout the unit, I will expect students to describe and defend their thought processes to their classmates, so everyone is exposed to multiple strategies and learns from each other. Although I am writing this unit for upper-level math students, many of the ideas are basic enough to be used in upper elementary or middle school.

I teach in a vocational school district consisting of four high schools that draw from middle schools in all five districts in New Castle County, Delaware so that my students have varying mathematical experiences. We separate the highest- and lowest-performing students based on 8 th grade state test results and a placement test, then group the rest of the incoming freshman heterogeneously. We use the Core Plus integrated math curriculum. With the exceptions just noted, students take four semesters (including two semesters in their freshman year) of Core Plus before having the option to take Intermediate Algebra and then Precalculus – both traditional math courses. Students in Precalculus are all college bound, and typically the strongest students in the school. I plan to test this unit with Precalculus students, and if it is successful, I will pass the problem sets along to the rest of my department to begin in ninth grade.

Background

I read dozens of articles about Number Sense before formulating my own working definition of it. I break it down into three major categories: Number Systems, Mathematical Operations, and Flexibility in Mathematical Situations. The Number System refers to recognizing different types of numbers, place value, relative magnitude of numbers, and different representations of them. Mathematical Operations encompasses addition, subtraction, multiplication, division and the relationships between them. It also includes understanding the effect of these operations on different types of numbers, and the Rules of Arithmetic that govern operations. Flexibility in Mathematical Situations is the ability to recognize multiple ways to work with numbers/data and identify the most efficient method with which to proceed. It also includes the ability to recognize the reasonableness of answers and when an answer requires an estimate versus an exact answer. [4] Gaining flexibility in mathematical situations is really the ultimate goal of this unit because it is what will allow students to perform mental arithmetic.

Number Systems

I think nearly every high school math book begins by identifying and naming sets of numbers. There are the Natural (or Counting) numbers starting at one and increasing forever. Whole numbers are the Natural numbers and zero. Integers are the set of all Whole numbers and their opposites, so now we have positive and negative numbers. Simply put, Rational numbers are all those that can be written as a fraction (i.e. a quotient of two integers). Since Integers can be written as a fraction with a denominator of one, or in any equivalent form, they are included in the set of Rational numbers. Irrational numbers are all other Real numbers than cannot be written as a fraction. Well known examples are π and Sqrt(3). The Real numbers consist of the Rational numbers together with the Irrational numbers. Real numbers can be represented as points on the number line. When students reach Algebra 2 or Precalculus they learn about Complex numbers that are comprised of a real and an imaginary component, and Complex numbers can be represented as points in the plane.

Students with good number sense recognize multiple representations of numbers - pictures, symbols, graphs, or number lines can be used to represent numbers - and there are multiple symbolic ways to represent a number. For example, the symbols 3/2, 1.5, 1x1/2, and 1 + 5/10 all represent the same number.

Place value

We spent a considerable amount of seminar time discussing the remarkable characteristics of our base 10 number system. It is a compact method in which whole numbers are represented as sequences of digits, and the position of each digit determines its value. Students can identify the hundreds, tens and ones places in a three-digit number; however, as Roger Howe said, the names for the positions are often treated merely as vocabulary words. We considered a progression of representations for the expanded, or decomposed, form of numbers. Initially, young students decompose the number 532 into 500 + 30 + 2 showing the value of each digit. Later, students recognize place value as powers of ten. Each "single-place number" can be written as a digit-times-a-power-of-10, and any number can therefore be written as the *sum* of digits-times-a-power-of-10. For example, 745 can be written as the sum: 745 = 7x100 + 4x10 + 5x1 which becomes 7x10x10 + 4x10 + 5x1. For my high school students who have experience with exponents and scientific notation, each single place number can also be written as a digit-times-a-power of ten in exponential form. Thus, 745 = $7x10^{2}$ + $4x10^{1}$ + $5x10^{0}$. The pattern of decreasing the power of 10, going from left to right, continues for numbers having decimal places; the number 45.16 can be written as $4x10^{1}$ + $5x10^{0}$ + $1x10^{-1}$ + $6x10^{-2}$. Decomposing numbers in this way, will illustrate to my students why 10^{0} is defined as one, and why, according to the Laws of Exponents, negative exponents represent reciprocals (i.e. $10^{-1} = 1/10^{1}$ and $10^{-2} = 1/10^{2}$ =1/100). Expanding decimal numbers in terms of reciprocals of powers of 10 also illustrates the relative size of each place value. In this form, I can emphasize to students the fact that the fraction 1/10 is larger than 1/100 because they both have the same numerator, but the fraction with the smaller denominator represents a larger number because the "whole" is being divided into fewer, and therefore larger, pieces. In fact, 1/100 is only 1/10 of 1/10. It's not that I think my Precalculus students don't understand place value, but I'm sure they have not discussed it in many years of math classes, and they have not explicitly written numbers in expanded form using exponential form, or what they have learned in science classes as scientific notation. I think practice in decomposing numbers in this way will help them internalize the meaning and size of numbers.

Number Line and Order of Magnitude

The next stage in building number sense is getting a feel for the relative size of numbers. For smaller numbers, the number line can connect quantity and measurement. This idea was also discussed in our seminar. In order to visualize the relative size of place values, the first step is to define one unit on the number line as the distance between zero and one. Then the number two on the number line is twice that same distance from zero, and the number ten is located ten times that distance from zero; the distance from zero to 100 is 100 times longer than one unit and also ten times longer than ten units. The distance between zero and one can be divided into ten equal spaces to show tenths on the number line. A meter stick is a number line that illustrates the relative sizes of three orders of magnitude: 1 meter = 10 decimeters = 100 centimeters = 1000 millimeters (the smallest marking on the stick).

The power of ten determines the Order of Magnitude of a number; each order of magnitude is related to the next by a factor of ten. Some familiar scales report only the Order of Magnitude. The numbers on the Richter scale, describing the magnitude of earthquakes, are the powers of ten. A magnitude 7 ($=10^{7}$) earthquake is actually ten times stronger than a magnitude 6 ($=10^{6}$) earthquake. On the pH scale, a solution with pH = 3 is 100 times more acidic than one with pH = 5. Likewise, decibels are the powers of ten used when measuring the intensity of sound. These are all logarithmic scales (pH is a negative logarithm), which I can use as examples of applications for my Precalculus students. I present more Order of Magnitude problems in the Activities section and in the sample problem sets in Appendix C.

Mathematical Operations

Once students are adept at expanding numbers based on place value, they can operate on them. They can add and subtract using the process they previously learned as "combining like terms." The "like terms" are those that have the same place value! They will add each place separately. Since each place contains a single digit, the addition proceeds quickly and easily. Even in cases that require regrouping, students at this level should not feel the need to use a calculator. As an example, to find the sum of 4,289 and 5,176, the first step is to expand each number: 4289 = 4x10^{3} + 2x10^{2} + 8x10^{1} + 9x10^{0} , and 5176 = 5x10^{3} + 1x10^{2} + 7x10^{1} + 6x10^{0} . To find the sum, we add like terms: (4x10^{3} + 5x10^{3}) + (2x10^{2} + 1x10^{2}) + (8x10^{1} + 7x10^{1}) + (9x10^{0} + 6x10^{0}). Then, the sum becomes 9x10^{3} + 3x10^{2} + 15x10^{1} + 15x10^{0} . We recognize that 15x10^{0} (aka 15) must be rewritten as 1x10^{1} + 5x10^{0} making the tens place 16x10^{1} (aka 160), which in turn is rewritten as 1x10^{2} + 6x10^{1} . Our final sum becomes 9x10^{3} + 4x10^{2} + 6x10^{1} + 5x10^{0} , which is put back together as 9,465. At first this process may seem cumbersome, but as Roger Howe told us repeatedly, "Shortcuts are the privilege of the expert." It is well-worth spending the time to reinforce understanding of place value and require students to expand the single digit numbers so they can move towards mental calculations. In fact, I would expect students to add two or three 2-digit numbers mentally by adding the digits having the same place value.

Within the above example for finding the sum of two four-digit numbers, there is a perfect opportunity to address some of the Properties of Arithmetic, namely the Associative and Commutative Properties of Addition, and the Distributive Property of Multiplication over Addition/Subtraction. (All nine Rules of Arithmetic are defined in Appendix B.) Students have been taught these properties, but typically in isolation, so they have very little carryover for when they might be useful. Once each number was written in expanded form, combinations of the Commutative and Associative Properties allowed us to rearrange the terms to place like terms next to each other and group them together. Although, it wasn't illustrated precisely the way textbooks state it, the Associative Property allowed us to group the like terms to work with them as four pairs of sums. It is the Distributive Property that allowed us to combine the like terms to form a single term with the same power of 10. When adding 4x10^{3} + 5x10^{3} , what we actually did was factor out the common factor of 10^{3} which gave us (4 + 5)x10^{3} . We then performed the arithmetic in parenthesis, giving us 9x10^{3} .

We perform subtraction the same as addition, using the expanded form of multi-digit numbers. We subtract single place numbers with the same power of 10. If we need to borrow, we convert one unit of the next higher place value, to make 10 units of the place that needs it, since the next place to the left is ten times larger than the given one. For example, to find the difference between 17.2 and 9.8 we begin by writing each number in expanded form: 1x10^{1} + 7x10^{0} + 2x10^{-1} - 9x10^{0} - 8x10^{-1} . Rearranging and regrouping, we get 1x10^{1} + (7x10^{0} - 9x10^{0}) + (2x10^{-1} - 8x10^{-1}). We will need to borrow twice (1x10^{1} = 10x10^{0} and 7x10^{0} = 6x10^{0} + 10x10^{-1}), so the difference becomes 0x10^{1} + (16x10^{0} - 9x10^{0}) + (12x10^{-1} - 8x10^{-1}) which is equal to 7x10^{0} + 4x10^{-1} , or 7.4. Later, in the Flexibility section of this unit, I will discuss methods that would make such a subtraction problem easier to do mentally, without pencil and paper. However, as mentioned earlier, students with good number sense need to be flexible in their mental calculation strategies, and there will be times that it is simplest to subtract single digit numbers place by place.

We can also perform multi-digit multiplication by expanding each number into its single digit components and multiplying each component from one number by each component of the second number and adding the resulting products. Essentially we are applying the Distributive Property multiple times. For example, multiplying a two-digit number by a three-digit number requires using the Distributive Property twice: $(a + b)(c + d + e) = (a + b)c + (a + b)d + (a + b)e = ac + bc + ad + bd + ae + be$. This process can be extended to any size numbers, and leads to what Roger Howe and Susan Epp refer to as the Extended Distributive Property, or the "Each with each" rule.[5] In practice, however, I find it easier to set up the multiplication as we did in the seminar, using a grid to ensure I don't miss any products. I will demonstrate the process by finding the product of 482 x 736. First, expand 482 = 4x10^{2} + 8x10^{1} + 2x10^{0} and 736 = 4x10^{2} + 8x10^{1} + 2x10^{0} . Next, label three columns of a box with the components of 482 and label three rows with the components of

736, as shown below. (The Commutative Property of Multiplication allows us to multiply in any order, so there is no significance to putting 482 on the top and 736 on the side; they could be reversed.) Using what is called an area model, although the boxes are not drawn to scale, multiply the two "dimensions" (base = $4x10^2 + 8x10^1 + 2x10^0$ and height = $7x10^2 + 3x10^1 + 6x10^0$) to compute its area. The total area of the shaded region is the composite sum of all nine individual shaded boxes.

	$4x10^2$	$8x10^1$	$2x10^0$
$7x10^2$	$7x10^2x4x10^2$ $= 28x10^4$	$7x10^2x8x10^1$ $= 56x10^3$	$7x10^2x2x10^0$ $= 14x10^2$
$3x10^1$	$3x10^1x4x10^2$ $= 12x10^3$	$3x10^1x8x10^1$ $= 24x10^2$	$3x10^1x2x10^0$ $= 6x10^1$
$6x10^0$	$6x10^0x4x10^2$ $=24x10^2$	$6x10^0x8x10^1$ $=48x10^1$	$6x10^0x2x10^0$ $=12x10^0$

Notice that all of the multiplications involve only single digits, meaning students only need to know their basic multiplication facts from one to nine. Also, notice that like terms can be found on diagonals going from the lower left to upper right of the box. Thus, grouping like terms can be done quickly; the product of 482 x 736 = $28x10^4 + (12+56)x10^3 + (24+24+14)x10^2 + (48+6)x10^1 + 12x10^0$. After simplifying, the product becomes $28x10^4 + 68x10^3 + 62x10^2 + 54x10^1 + 12x10^0$. To finally get the number in standard decimal form, we do regrouping. Regrouping five times from right to left, we get $3x10^5 + 5x10^4 + 4x10^3 + 7x10^2 + 5x10^1 + 2x10^0$, or 354,752. This area model method can be applied to any number of digits, whether the two factors have the same number of digits or not. While I would not expect my students to multiply two three-digit numbers without paper and pencil, I would expect them to be calculator-free.

This multiplication example presents the opportunity to reinforce more Rules of Arithmetic - the Commutative and Associative Properties of Multiplication, as well as the Extended Distributive Property (aka "Each with Each" rule). The Distributive Property appears as the "Each with Each" rule, and then again when we sum the terms involving a given power of ten. In each box we arrange the four individual factors so we first multiply the single digits together and then multiply the powers of ten together. In the process, we are also demonstrating the Law of Exponents - when multiplying the same base (in this case the base is 10), we add the exponents. In other words, if we multiply $10^2 x10^1$, we are multiplying ten twice and then once more for a total of 10^3. The fact that this does not depend on the way the factors of 10 are grouped, only on the total number of factors, is another instance of the Associative Rule.

Since I am not allowing students to use calculators for this, multi-digit multiplication is a perfect opportunity to practice estimation to check the reasonableness of their answer. Performing the product of just the largest place value from each number gives us $4x10^2 x7x10^2 = 28x10^4 = 2x10^5 + 8x10^4$ = 280,000. Therefore, the order of magnitude of the estimated product and the actual product, 354,752, are both five. Estimation, along with the amount of error relative to the actual number, will be discussed in more detail in the Flexibility section.

There is a strong connection between multi-digit multiplication using the expanded form of a number and multiplication of polynomials. For students that have already studied algebra, the next example demonstrates the connection between algebra and arithmetic. If I convert the product of 482 x 726 into a polynomial in x, the product becomes $(4x^2 + 8x + 2)(7x^2 + 2x + 6)$. This product can be computed using the area model, also:

	$4x^2$	$8x$	2
$7x^2$	$7x^2 \cdot 4x^2 = 28x^4$	$7x^2 \cdot 8x = 56x^3$	$7x^2 \cdot 2 = 14x^2$
$3x$	$3x \cdot 4x^2 = 12x^3$	$3x \cdot 8x = 24x^2$	$3x \cdot 2 = 6x$
6	$6 \cdot 4x^2 = 24x^2$	$6 \cdot 8x = 48x$	$6 \cdot 2 = 12$

Looking at the diagonals, we can combine like terms to get the product: $28x^4 + 68x^3 + 62x^2 + 54x + 12$. Then, if we replace x with 10, this becomes the same product as the previous arithmetic example. Thus, students have been multiplying polynomials since elementary school; it's just that the variable always had a value of ten! I have taught students how to multiply polynomials successfully using the area model/box method. I think they will appreciate the connection between algebra and basic arithmetic, and it will take away some of the mystery of working with variables.

Operating on Rational Numbers

So far, my multiplication examples have used only whole numbers. Young students spend a great deal of time learning arithmetic with whole numbers that are all greater than one. As a result, many believe that multiplication always makes things bigger. Likewise, they believe that division makes things smaller. These generalizations must be replaced when they study rational numbers. Recognizing that operations have different effects on different types of numbers (i.e. multiplying natural numbers versus rational numbers) is a critical understanding within number sense.

To convince students that multiplication does not always make things bigger, I will present them with counterexamples such as 600 x 0.2. To find the product, we can rewrite each number showing place value: $6x10^2$ x $2x10^{-1}$ = $12x10^1$ = $1x10^2 + 2x10^1$ = 120, which is clearly smaller than 600. Multiplying multi-digit numbers containing decimal fractions can be done in the same way as whole numbers, with the area model/box method. To find the product of 654 x 0.55, first expand each factor and calculate the product $(6x10^2 + 4x10^1 + 3x10^0)$?$(5x10^{-1} + 5x10^{-2})$:

	$6x10^2$	$4x10^1$	$3x10^0$
$5x10^{-1}$	$6x10^2$ x $5x10^{-1}$ = $30x10^1$	$4x10^1$ x $5x10^{-1}$ = $20x10^0$	$3x10^0$ x $5x10^{-1}$ = 15 x 10^{-1}
$5x10^{-2}$	$6x10^2$ x $5x10^{-2}$ = $30x10^0$	$4x10^1$ x $5x10^{-2}$ = $20x10^{-1}$	$3x10^0$ x $5x10^{-2}$ = $15x10^{-2}$

Combining like terms on the diagonals, we get $30x10^1 + 50x10^0 + 35x10^{-1} + 15x10^{-2}$. Regrouping to get single digits times a power of 10, the product becomes $3x10^2 + 5x10^1 + 3x10^0 + (5 + 1)x10^{-1} + 5x10^{-2}$ = 353.65, which is, again, less than the starting number, 643. The next question becomes, how accurately do we need to know the product? If we were doing scientific calculations, and needed a high degree of accuracy, we would use a calculator. However, for typical everyday calculations, an estimate is probably sufficient.

Another characteristic of number sense is understanding the relationship between operations. When algebra students learn to solve equations, they learn to use inverse operations. Addition "undoes" subtraction and vice versa; therefore, they are inverse operations. Multiplication and division are also inverse operations. This concept can possibly be used to help students understand the "flip and multiply" algorithm they learned for dividing fractions. If $x=a/b \div c/d$, then, multiplying (the inverse of dividing) both sides of the equation by c/d gives us $c/d \cdot x = a/b$. (A division problem is the same thing as a missing factor problem – something I will discuss with my students to ensure they understand it.) To solve for x, we would multiply both sides of the equation by the reciprocal d/c, which gives us $(c/d \cdot d/c)x = (1)x = a/b \cdot d/c$. Thus, setting the two expressions for x equal to each other, $a/b \div c/d = a/b \cdot d/c$, we see that dividing by a fraction is equivalent to multiplying by the reciprocal of that fraction. [6] Another way to think about this process is in two steps: first divide by c, and then divide by 1/d, which is the same as multiplying by d. (These steps can be done in the reverse order because of the Commutative Rule for Multiplication.) Students that understand both processes will gain flexibility in manipulating expressions. Students can reinforce this process by working practice problems using inverse operations with numbers in place of the variables.

Flexibility in Mathematical Situations

The third major characteristic of number sense is having flexibility in mathematical situations. It involves recognizing when it is appropriate to estimate an answer and when a more exact answer is needed. It involves checking answers to determine whether they are reasonable. Flexibility is also the ability to consider multiple strategies and select the one that is most efficient for the situation. I think that learning and practicing techniques based on number sense will ultimately lead to improved confidence in performing mental arithmetic and less reliance on a calculator.

Estimation and Relative Error

In the first multiplication example I presented (482 x 736), I estimated the order of magnitude of the product to be five (10^5) by multiplying only the first digit (highest order of magnitude) from each number. By the same method, the order of magnitude of the product of the last example (643 x 0.55) is two because $6 \times 10^2 \times 5 \times 10^{-1} = 30 \times 10^1 = 3 \times 10^2$. In our seminar, we calculated relative error in terms of the desired quantity and discussed how the number of digits used in a calculation determines the accuracy of the answer. To calculate Relative Error, we use the formula $E = |Q' - Q|/Q$, where Q is the desired answer and Q′ is the calculated answer. In the previous example, we can calculate the relative error as $E = |300 - 353.65|/353.65 = 0.15 = 15\%$. In many everyday cases, 10 - 20% error is acceptable. If we were to use the first two digits and multiply the highest place value of each number with the first *two* digits of the second number (i.e. the 3 boxes that make up the upper left corner in the area model), we would get a product of $3 \times 10^2 + 5 \times 10^1 = 350$. The relative error would be reduced to $E = |350 - 353.65|/353.65 = 0.01 = 1\%$ which would definitely be acceptable for everyday use.

We considered the effect of estimating answers to the first one, two and three digits of a number. The "worst-case" for estimating with only the first digit is that the estimate has 50% error. The worst case occurs when the first digit is one because if the actual number is very close to two, the ratio of the difference to the actual is 1/2. For example, estimating the number 199 as 100 gives relative error as $E = (199 - 100)/199 = 0.497 \approx 50\%$, meaning we know at least half of the actual number. If we use the first two digits to estimate, the error is reduced to a maximum of 10%, meaning we know at least 90% of the actual number. Estimating with three digits gives at most 1% error, meaning we know at least 99% of the actual number. Therefore, when doing mental arithmetic, it is typically not necessary to carry out long multiplication problems; they can be estimated with a high degree of accuracy using the two, or at most three, single digits with highest place value in each number, even though the relative errors from each estimate are additive. Depending on the group of students, I may demonstrate how to calculate relative error, and lead a discussion about what is acceptable error in different situations.

Mental Computation Strategies

Lastly, I would like some of the activities in this unit to teach my students mental computation strategies that they may never have been taught formally. I want to build their confidence in performing arithmetic so they no longer feel the need to rely on a calculator. The strategies described below come from the book *Number Talks: Helping Children Build Mental Math and Computation Strategies* and from multiple articles about performing mental calculations.

Partitioning

Partitioning, in one sense, is decomposing numbers according to place value, as described earlier, and performing operations on them place by place. Partitioning makes addition and subtraction of two 2-digit numbers quite simple to do mentally. For example, to add 54 + 36, we add 50 + 30 = 80 and 4 = 6 = 10, then 80 + 10 = 90. To subtract 54 – 36, we do 54 – 30 = 24, then 24 – 6 = (24 – 4) – 2 = 20 – 2 = 18.

Even adding 3-digit numbers, or beyond, can be simple. For example, to add 116 + 118, 116 can be partitioned into (110 + 6) and added to (110 + 8). The sum becomes (110 + 110) + (6 + 8) = 220 + 14 = 234, which can be performed mentally. For younger students, partitioning can also refer to breaking a number apart to "make a ten." Using the same example, 116 + 118, 116 can be partitioned into (110 + 2 + 4) and added to (110 + 8). The sum becomes (110 + 110) + (2 + 8) + 4 = 220 + 10 + 4 = 234. The 'trick" was recognizing that 2 + 8 makes a ten, and the six ones in 116 can be partitioned into 2 + 4.

Doubling and Halving

Most people can double numbers easily in their heads. Looking for doubles and then adjusting as needed is another strategy that can be applied to find the sum of 116 + 118. If we double 116, we get 232, and can adjust for the 118 in the problem by adding 2 to get the sum 234. The actual thought process was probably (100 + 100) + (16 + 16 +2) = 200 + 32 + 2 = 234.

For multiplication and division, doubling numbers and cutting them in half is relatively simple for students. Therefore, dividing by four could be done mentally by halving twice. Likewise, multiplying by 8 can be done by doubling 3 times.

Adding Up

"Adding up" works for both addition and subtraction when students understand how the two operations are related. For 116 + 118, start with 116 and add 100 (from 118) to get to 216. Then, add 10 (also from 118) to get 226. Next, partition the remaining 8 as 4 + 4 to add up to 230 and then 234. To use the same strategy for subtracting 123 - 59, begin at 59 and count what you need to add to reach 123. So, 59 + 1 = 60, 60 + 40 = 100, 100 + 23 = 123. Therefore, 123 - 59 = 1 + 40 + 23 = 64. In the process, I used benchmark numbers (60 and 100) that are multiples of 10 because they are easy to work with. Some students may choose to count backward when subtracting, but the idea is the same.

Adjusting to Create an Easier Problem

Sometimes it is beneficial to adjust the problem slightly to make it easier and then adjust for it at the end. For example, to find the difference 123 - 59, I could change it to 123 - 60 = 63, and add 1 to compensate for adding 1 to 59, so the difference is 63 + 1 = 64. Sometimes I have to think about whether I should add or subtract the 1 at the end. To make the decision, I look at just the ones digits to determine what the final answer should have in the ones place. I see 13 - 9 = 4, so I was correct to add the 1.

Compensation/Constant Difference

The compensation method allows us to shift an entire subtraction problem. It is a horizontal translation on the number line, maintaining the same difference between the numbers, but making the subtraction simpler (i.e. no borrowing required!). For example, 6002 - 2537 can be rewritten as 5999 - 2534 = 3465 by subtracting 3 from each number; the difference remains the same but there is no need to borrow as you subtract the digits with the same place value.

The compensation method can be applied in many different ways. We practiced another form of Compensation in our seminar that is a bit more challenging, but interesting, nonetheless. The process is to line the numbers up vertically, and then subtract the smaller number from the larger number *in each place*. This step leaves a zero in each place for one of the numbers. Next, "group" the top number into multiples of 10 and subtract within each group to find the difference.

Example 1: 40982 -> 40 700 because 900 – 200 = 700
–3295 -> –3 013 because 5 – 2 = 3 and 90 – 80 = 10
37,687 because $40 \times 10^3 - 3 \times 10^3 = 37 \times 10^3$ and 700 –13 = 687.

Example 2: 634,761 -> 40 20 10
–242,952 -> 1 02 01
39 18 09 = 391,809

Making Friendly Numbers

Similar to making tens for addition, it is sometimes possible to make friendly numbers for multiplication. For example, the product of 15x8 can be thought of as 3x5x2x4. By applying the Commutative and Associative Properties of Multiplication, the factors can be rearranged and recombined to become 10x12 or 6x20 to simplify the multiplication.

Partial Products

Partial Products is essentially a way of using the Distributive Property. The area model/box method demonstrated earlier is an example of finding partial products by place value and adding the pieces back together. When working with smaller numbers, it is possible to do the products and addition mentally. The product of 15x8 can be thought of as (10 + 5) x 8, giving 80 + 40 = 120. I hope my students will find it feasible to multiply two two-digit numbers using partial products. The area model for multiplication shows that multiplying two 2-digit numbers involves four partial products. Not all students will be able to organize this computation mentally, but some probably will enjoy doing so. Again, starting with the product of the tens will give the largest term. Then the products of the ones from each factor with the tens from the other gives the next in order of size. Finally, adding on the product of the ones will give the complete product. Thus, 43 x 27 = 800 + (60 + 280) + 21 = 800 + 340 + 21 = 1140 + 21 = 1161.

Strategies

Instruction of this unit will begin within the first one or two days of the semester. The first question I will ask my new students is, "Have you ever seen a number?" If they say yes, then "What did it look like?" I hope to promote a discussion centered around the idea that numbers are adjectives that modify nouns (the units). Written numerals are merely symbols that represent numbers, and we need to know what they describe to assign meaning to them. In order to operate on them (i.e. add or subtract, etc.) the numbers must be modifying the same noun. I may use PowerPoint slides created by Herb Gross and Richard Medeiros [7] to illustrate this adjective-noun relationship in mathematics. I expect Precalculus students to see the adjective-noun connection to algebra - adding "like terms." I plan to start with this adjective-noun relationship to be able to refer to it throughout the semester in different contexts.

Based on weaknesses that I have observed in my students, and supported by my readings and seminar discussions, I have selected weekly mathematical topics for Warm-up/Bell-Ringer activities. For the past several years, I have prepared my Warm-up activities using computer software so that they are displayed on the interactive white board when students enter the classroom. All activities in this unit will be *calculator-free*, and I expect students to attempt to solve each problem in more than one way. Wherever possible, I would like them to perform mental arithmetic, although they will record their mental thought process. These activities should be completed independently in the first five minutes of class, and followed by a full-class discussion. I will expect students to share their own strategies and listen and learn alternate strategies. I may demonstrate some strategies that are described in the Background section of this unit, if no students use them.

In my classroom, students receive a template every Monday that they fill in each day of the week and turn in on Friday. The template has space to copy the daily Essential Question (objective for the class period) and to copy and work out the daily Warm-up problems. I do put a grade on them (a very small percentage of their course grade), typically for completeness, and return them to be kept in their notebooks. I have found that this system keeps students accountable, organized, and provides a record of what they have studied throughout the semester. I will continue with this strategy with one addition - I will add a space on Friday's section for students to summarize what they learned from the weekly Warm-up activities. At the end of the semester, I will ask students to review all of the weekly Warm-up activities and, again, summarize strategies and concepts they learned throughout the semester.

Activities

Our school district operates on two 18-week semesters per year. Therefore, I developed 18 sets of sample problem sets that are included in Appendix C. I plan to present one to three questions per day, depending on the topic and type of question. The sample problem sets have between one and four days worth of problems. While I tried to arrange them in a logical order, they can be rearranged to suit the school schedule – shortened weeks, testing, etc. To complete a "normal" five-day week, I will simply model the remaining days' problems after the early ones, using different numbers.

Problem Set I focuses on place value. I will give instruction on all of the forms for expanding single place components of numbers. By the end of the week, I expect students to be able to write any number, including decimal fractions, as the sum of the digits-times-a-power-of-ten.

The objective of Problem Set II is to recognize that all three sums for each day are the same. By expanding each number to show place value, and adding the same place values, the digits are always the same, but in a different order. It also provides the opportunity to review or reinforce the Commutative Property of Addition.

Problem Set III is both a vocabulary check and a means for discussing the relative size of numbers, all based on place value. If students demonstrate proficiency with the vocabulary, I will supplement problems that practice the basic Rules of Exponents (Appendix B).

The objective of Problem Sets IV, V, and VII is to practice and/or learn mental arithmetic strategies for addition and subtraction. It is especially important for students to be able to verbalize their strategies and share them with classmates. Throughout the problem sets, I expect students to expand numbers to show place value, look for "friendly numbers" such as 10 or 100, use compensation (keep a constant difference), or "add up" in chunks to multiples of 10. I do not expect students to have names for the strategies, only to be able to explain and justify them.

Before starting Problem Set VI, I will give instruction on the Compensation Method (described in the Background section) that is useful for subtraction of multi-digit numbers that requires regrouping. I think my higher-level students will find it intriguing and will understand why it works. It could be optional for lower-level students as it could turn out to be another algorithm without meaning to them.

Problem Sets VIII and XIV both connect arithmetic to algebra. First students are asked to add, subtract and multiply polynomials as they have done in previous algebra classes. They can only add or subtract like terms (having the same power of x), and they add exponents of x when multiplying terms. After finding the sum, difference or product in simplest form, students will substitute $x = 10$, and, hopefully, recognize that the polynomials represent the expanded form of numbers showing place value for base ten numbers. If they don't make the connection themselves, I will help them see it!

Problem Sets IX, XI and XVI are designed to practice the Rules of Arithmetic. Students will use the Commutative and Associative Properties of Addition and Multiplication, and the Distributive Property, to simplify an assortment of arithmetic problems. Again, students will be asked to explain and justify their strategies.

Problem Sets X and XV are strictly number sense practice. Students will identify the relative order of numbers. In Problem Set X, they will demonstrate understanding of relative size based on place value of decimal fractions. If students demonstrate understanding early in the week, I will again supplement with exponent practice exercises. In Problem Set XV, they will demonstrate understanding of the effect of repeated multiplication (exponents) on different types of numbers – positive, negative, and rational. Some problems also evaluate understanding of negative exponents.

There are two objectives for Problem Set XII. The first objective is to estimate the product of multi-digit numbers using only the first digit (highest place value) of each. The second objective is to translate the estimated product into its Order of Magnitude from the highest power of ten. I will emphasize the estimation technique in this problem set as a way to check the reasonableness of answers quickly.

The purpose of Problem Set XIII is to demonstrate the area model/box method for multiplying multi-digit numbers in the same way algebra students use the area model to multiply polynomials. When numbers are expanded to show place values, they form the dimensions of the box, and each component in the first number is multiplied by each component of the second number ("Each with each rule"). This model also provides the opportunity to review/reinforce the Commutative and Associative Properties of Multiplication, the Distributive Property to combine like terms (having the same place value), and Rules of Exponents when multiplying powers of ten.

Problem Sets XVII and XVIII address the relative size of numbers, estimation and order of magnitude through problem solving. The problems primarily deal with very large numbers and ask students to relate them to smaller units they know. My primary source for these problems will be Lawrence Weinstein and John A. Adam's book entitled *Guesstimation: Solving the World's Problems on the Back of a Cocktail Napkin*. [8] The problems in the book are arranged by topics such as Animals and People, Energy, Transportation, etc. Not only do these problems ask students to estimate answers to within a power of 10, they require students to recognize relevant information. I will provide them the opportunity to search for the information they need using computers in the classroom or at home. These problems may need to be spread out over more than one day – brainstorming one day to determine what they need to know to solve the problem, collecting information and doing the rough calculations later. I expect these problems to be dynamic and change according to the interests of my students and items in the news. I may also use them randomly at other times throughout the semester.

Appendix A – Implementing District Standards/Delaware State Standards

---

Content Standard 1 – Numeric Reasoning: Students will develop Numeric Reasoning and an understanding of *Number Sense and Operations* by solving problems in which there is a need to represent and model real numbers verbally, physically, and symbolically; to explain the relationship between numbers; to determine the relative magnitude of real numbers; to use operations with understanding; and to select appropriate methods of calculations from among mental math, paper-and-pencil, calculators, or computers.

This unit addresses all of the *Number Sense* and *Operations* Grade Level Expectations (GLE's) for grades 9-12, with the exception of those pertaining to absolute value, imaginary numbers, matrices, and expressions and equations with square roots and fractional exponents.

Throughout this unit students are asked to explain and justify their thought processes, listen and learn from each other, and apply new strategies in appropriate situations. In this way, the unit also addresses all of the Process Standards:

Standard 5 – Problem Solving: Students will... develop and apply strategies to solve a wide variety of problems; and to integrate mathematical reasoning, communication and connections.

Standard 6 – Reasoning and Proof: Students will develop their Reasoning and Proof ability by solving problems in which there is a need to investigate significant mathematical ideas in all content areas; to justify their thinking; to reinforce and extend their logical reasoning abilities; to reflect on and clarify their own thinking; to ask questions to extend their thinking; and to construct their own learning.

Standard 7 – Communication: Students will develop their mathematical Communication ability by solving problems in which there is a need to obtain information from the real world through reading, listening and observing; to translate this information into mathematical language and symbols; to process this information mathematically; and to present results in written, oral, and visual formats.

Standard 8 – Connections: Students will develop mathematical connections by… allowing the flexibility to approach problems, from within and outside mathematics, in a variety of ways.

Appendix B – Rules of Arithmetic and Rules of Exponents

Rules of Arithmetic

Rules for Addition

Commutative Property of Addition: $a + b = b + a$
Associative Property of Addition: $(a + b) + c = a + (b + c)$
Additive Identity: $a + 0 = a$
Additive Inverse: $a + -a = 0$

Rules for Multiplication

Commutative Property of Multiplication: $a \cdot b = b \cdot a$
Associative Property of Multiplication: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
Multiplicative Identity: $a \cdot 1 = a$
Multiplicative Inverse: $a \cdot \frac{1}{a} = 1$

Distributive Property

of Multiplication over Addition: $a(b + c) = ab + bc$

Rules of Exponents

The primary Law of Exponents: $a^m \cdot a^n = a^{m+n}$

These two statements are definitions that must be true in order to make the Law of Exponents true:

$a^0 = 1$

$a^{-n} = \frac{1}{a^n}, a \neq 0$

The remaining useful rules can be derived from the Law of Exponents, or directly form the definitions:

$(ab)^n = a^n \cdot b^n$

$(a^m)^n = a^{m \cdot n}$

$\frac{b^m}{b^n} = b^{m-n}, b \neq 0$

$\left(\frac{c}{d}\right)^n = \frac{c^n}{d^n}, d \neq 0$ (This is another form of the first rule since division is the same as multiplication by the reciprocal.)

Appendix C – Sample Problem Sets

ALL problems are *calculator-free*!

I. Write each number in expanded form showing place values.

1. 482
2. 20,315,037
3. 871,594
4. 0.2874
5. 0.09136
6. 3.045
7. 509.702
8. 4,006.018
9. 32,407.53

II. Write each number in expanded form showing place values, then find the sum.

1. 361 + 428
2. 368 + 421
3. 328 + 461
4. 2900 + 61
5. 2061 + 900
6. 2001 + 960
7. 155 + 238
8. 158 + 235
9. 135 + 258
10. 576 + 429
11. 526 + 479
12. 479 + 526

Explain any observations you made.

III. Base 10 relationships. *Be prepared to explain your method.*

1. How many thousands make a million?
2. How many millions make a billion?
3. How many thousands make a billion?
4. What is a tenth of a hundredth?
5. What is a hundredth of a hundredth?
6. What is a thousandth of a hundred?
7. What is a tenth of a thousand?
8. How many hundredths make a hundred?
9. How many thousandths make a thousand?

IV. Find the sum. *Be prepared to explain your method.*

1. 247 + 39
2. 2.47 + 3.9
3. 0.247 + 0.039
4. 516 + 702
5. 51.6 + 7.02
6. 51.6 + 0.702
7. 0.559 + 8.3
8. 559 + 83
9. 5.9 + 0.83

V. Find the difference. *Be prepared to explain your method.*

1. 204 – 87
2. 200 – 83
3. 199 – 82
4. 6002 – 3987
5. 5999 – 3984
6. 3003 – 1839
7. 4012 – 3456
8. 1104 – 985
9. 2100 – 946

VI. Use the Compensation Method to subtract. *Be prepared to explain your methods.*

1. 40,982 – 3,295
2. 634,761 – 242,052
3. 521,346 – 401,927
4. 12,364 – 9,186
5. 24,176 – 22,584
6. 746,982 – 573,687

VII. Find the most efficient way to add/subtract. *Be prepared to explain your methods.*

1. 158 + 221
2. 262 + 134
3. 267 + 268
4. 700 – 477
5. 151 – 98
6. 257 – 118
7. 500 – 367
8. 256 + 342
9. 619 + 236

VIII. Find the sum or difference.

1a. $(3x^2 - 9x + 5) + (6x^2 - 4)$ 1b. $305 + 600 - 94$

2a. $(2x^2 + 10x) - (4x^2 - 10)$ 2b. $210 - 400 + 10$

3a. $(5x^3 + x^2 - 2x - 1) + (-3x^3 - 4x^2 - 5x + 2)$ 3b. $5100 - 21 - 3450 + 2$

Let x = 10. *Be prepared to discuss your observations.*

IX. Find the product using multiple methods. *Be prepared to explain your methods.*

1. 20 x 46 4. 96 x 1.5 7. 4.5 x 36
2. 25 x 32 5. 2.25 x 72 8. 18 x 12
3. 16 x 35 6. 3 x 135 9. 9 x 45

X. State a number that is between the two numbers given.

1. 7.1 and 7.11 4. 0.47 and 0.477
2. 45.3 and 45.33 5. 0.056 and 0.0566
3. 60.5 and 60.55 6. 0.203 and 0.2033

XI. Use Properties of Arithmetic to calculate. *Be prepared to explain your methods.*

1. 42 + 87 + 58 4. 79 + 88 + 12 7. 49 x 22 ÷ 14
2. 4 x 31 x 25 5. (7x8) + (3x8) 8. 120 x 28 ÷ 21
3. 5 x 17 x 20 6. 29x14 – 4x29 9. 34x(-9) + 17x18

XII. For each product, state the order of magnitude and estimate the product to the first two digits. *Be prepared to explain your methods.*

1. 84 x 38 4. 620 x 0.3 7. 984 x 0.9
2. 73 x 109 5. 817 x 0.07 8. 75.3 x 4.86
3. 56 x 272 6. 1428 x 0.02 9. 0.48 x 0.276

XIII. Write each number in expanded form showing place values, then find the product using the area model/box method.

Example: $481 \times 208 = (4x10^2 + 8x10^1 + 1x10^0)(2x10^2 + 8x10^0)$

	$4x10^2$	$8x10^1$	$1x10^0$
$2x10^2$	$4x10^2 \cdot 2x10^2$ $= 8x10^4$	$8x10^1 \cdot 2x10^2$ $= 16x10^3$	$1x10^0 \cdot 2x10^2$ $= 2x10^2$
$8x10^0$	$4x10^2 \cdot 8x10^0$ $32x10^2$	$8x10^1 \cdot 8x10^0$ $= 64x10^1$	$1x10^0 \cdot 8x10^0$ $= 8x10^0$

$481 \times 208 = (8 + 1)x10^4 + (6 + 3)x10^3 + (4 + 6)x10^2 + 4x10^1 + 8x10^0 = 100{,}048$

1. 84 x 38 4. 620 x 0.3 7. 984 x 173.9
2. 73 x 109 5. 817 x 0.07 8. 75.3 x 4.86
3. 56 x 272 6. 1428 x 0.02 9. 0.48 x 0.276

XIV. Find the product.

1a. $(3x^2 + 9x + 5)(6x^2 + 4x + 2)$ 1b. 395×642

2a. $(5x^3 + x^2 + 2x + 1)(3x^3 + 4x^2 + 5x + 2)$ 2b. 5121×3452

3a. $(2x^2 + 10x)(4x^2 + 5x + 3)$ 3b. 210×453

Let x = 10. *Be prepared to discuss your observations.*

XV. Using > or <, identify which expression represents the larger number.

1. 33^8 32^8
2. $(-7)^{10}$ $(-8)^{10}$
3. $(0.85)^7$ $(0.085)^7$
4. $(-4)^5$ $(-4)^9$
5. $(1.4)^6$ $(1.4)^7$
6. $(-0.9)^{11}$ $(-0.9)^{12}$
7. $(-0.1)^{25}$ $(-0.1)^{31}$
8. $(24)^{-2}$ $(24)^{-4}$
9. $(-3)^{-5}$ $(-3)^{-7}$

XVI. Use Properties of Arithmetic to perform the operations in the most efficient way.
Be prepared to explain your methods.

1. $21^2 - 21$
2. $48 \times 52 + 52^2$
3. $399^2 + 399$
4. $98 \times 99 + 98$
5. $103^2 - 3 \times 103$
6. $48^2 + 96$
7. $5.9 \times 6.1 + (6.1)^2$
8. $9.8 \times 9.9 + 0.98$
9. $4.8 \times 4.9 + 1.48$

XVII. Relative size of numbers/Conversions

1. What was the approximate date 1 million minutes ago?
2. What was the approximate date 1 billion minutes ago?
3. How many gallons of water are in a cubic mile of water?
4. How many nanometers are in 100 centimeters?
5. If a millionaire left a $100 tip, how much would a billionaire leave, maintaining the same proportion of his/her wealth (i.e. same ratio)?

XVIII. Estimate to the highest single place value (Order of Magnitude).

1. How many cm^3 of water do you drink in your lifetime?
2. How many times does you heart beat in your lifetime?
3. How much air do you breathe in your lifetime?
6. What is the height of a stack of 1 trillion one-dollar bills? How does that compare to the height of Mt. Everest?
7. If 1 billion grains of rice were spread evenly over the state of Delaware, how deep would the layer be?
8. How many gallons of water are in the Great Lakes?
9. How many gallons of water are in all of the oceans?
10. How many miles do all Americans drive in America a year? How does that compare to the distance from the Earth to Mars? To Jupiter? To Pluto?
11. What is the volume of trash collected from American homes each year?

Be prepared to explain your solutions, including assumptions and calculations.

Resources

Davis, Sue. "Oral and Mental Mathematics." *Mathematics Teaching* 215 (2009): 45-48., http://vnweb.hwwilsonweb.com (accessed June 25, 2011).

Gardiner, A. *Extension mathematics Beta, Gamma*. Oxford: Oxford University Press, 200, Problem sets to promote number sense, especially for stronger students.

Gross, Herbert, and Richard Medeiros. "Mathematics As A Second Language.", Mathematics As A Second Language. http://www.adjectivenounmath.com/ (accessed July 20, 2011), Power Point slides available for arithmetic, algebra and calculus lessons.

Howe, Roger , and Susanna Epp. "PMET - Resources: Taking Place Value Seriously.", Mathematical Association of America, http://www.maa.org/pmet/resources/PVHoweEpp-Nov2008.pdf (accessed July 21,2011).

Mcintosh, Alistair, Barbara Reys, and Robert Reys. "A Proposed Framework for Examining Basic Number Sense." *For the Learning of Mathematics* 12, no. 3 (1992): 2-8., http://www.jstor.org/stable/40248053 (accessed July 17, 2011).

Parrish, Sherry. *Number talks: helping children build mental math and computation strategies, grades K-5*. Sausalito, Calif.: Math Solutions, 2010., Strategies and sample problems for mental math calculations.

Pitta-Pantazi, Demetra, Constantinos Christou, and Theodossios Zachariades. "Secondary school students' levels of understanding in computing exponents." *The Journal of Mathematical Behavior* 26, no. 4 (2007): 301-311, http://www.sciencedirect.com (accessed July 13, 2011), Comparison of relative size for exponential expressions.

Rubenstein, Rheta. "Mental Mathematics beyond the Middle School: Why? What? How?" *Mathematics Teacher* 94, no. 6 (2001): 442-446., Objectives and sample items for number sense, algebra, and Precalculus.

Threlfall, John. "Flexible Mental Calculations." *Educational Studies in Mathematics* 50, no. 1 (2002): 29-47. http://vnweb.hwwilsonweb.com (accessed July 20, 2011)., Purpose and strategies for teaching mental calculations.

Tirosh, Dina. "Enhancing Prospective Teachers' Knowledge of Children's Conceptions: The Case of Division of Fractions." *Journal for Research in Mathematics Education* 31, no. 1 (2000): 5-25., http://www.jstor.org/stable/749817 (accessed July 17, 2011), Sources of errors in dividing fractions and how to help children understand division conceptually.

Weinstein, Lawrence, and John A. Adam. *Guesstimation: solving the world's problems on the back of a cocktail napkin*. Princeton, N.J.: Princeton University Press, 2008., Sets of estimation problems and solutions organized by topic.

Yang, Der-Ching, Robert Reys, and Barbara Reys. "Number Sense Strategies Used by Pre-Service Teachers in Taiwan." *International Journal of Science and Mathematics Education* 7, no. 2 (2009): 383-403., http://vnweb.hwwilsonweb.com (accessed June 25, 2011).

Notes

---

1. 1 Sherry Parrish, *Number talks* (Sausalito: Math Solutions, 2010), 4.
2. 2 John Threlfall. "Flexible Mental Calculation," *Educational Studies in Mathematics* 50,
3. no. 1 (2002), 29-47, http://vnweb.hwwilsonweb.com> (20 July 2011).
4. 3 Rheta Rubenstein, "Mental Mathematics beyond the Middle School: Why? What?
5. How?" *Mathematics Teacher* 94, no. 6 (2001), 442-446.
6. 4 Alistair Mcintosh, Barbara Reys, and Robert Reyes, "A proposed Framework for
7. Examining Basic Number Sense," *For the Learning of Mathematics* 12, no. 3 (1992):
8. 2-8. http://www.jstor.org/stable/40248053> (17 July 2011).
9. 5 Howe, Roger , and Susanna Epp. "PMET - Resources: Taking Place Value Seriously."

10. Mathematical Association of America.
11. http://www.maa.org/pmet/resources/PVHoweEpp-Nov2008.pdf> (21 July 2011).
12. 6 Dina Tirosh, "Enhancing Prospective Teachers' Knowledge of Children's Conceptions:
13. The Case of Division of Fractions," *Journal for Research in Mathematics Education*
14. 31, no. 1 (2000): 5-25, http://www.jstor.org/stable/749817> (17 July 2011).
15. 7 Herb Gross and Richard Medeiros, "Mathematics As A Second Language."
16. http://www.adjectivenounmath.com/ > (20 July 2011).
17. 8 Lawrence Weinstein and John A. Adam. *Guesstimation: solving the world's problems*
18. on the back of a cocktail napkin. (Princeton: Princeton University Press, 2008).