Curriculum Units by Fellows of the National Initiative
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Introduction

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The seminar on *Great Ideas in Primary Mathematics* had as its goal the study of the basic ideas that underlie much of the mathematics studied in elementary school, but which often do not get the explicit attention that might enable more effective learning. Because of the substantial representation of middle- and high-school teachers among the seminar Fellows, the overall level of the topics in the seminar was more advanced than originally planned. We appreciated that the Fellows showed substantial enthusiasm for discussing topics at many grade levels.

A substantial amount of seminar time was devoted to the idea of place value. This is frequently treated in the U.S. as a vocabulary item, but in fact, it is the basic principle by which we deal with numbers, and it is relevant to all grades, first through high school. Carl Friedrich Gauss, often described as the greatest mathematician since Newton, considered the place value system as a tool of unparalleled value. He is paraphrased by the mathematics historian Howard Eves as saying, "The greatest calamity in the history of science was the failure of Archimedes to invent positional notation."

The key idea of place value is summarized in the sequence of equalities

$$743 = 700+40+3 = 7x100+4x10+3x1 = 7x(10x10)+4x10+3x1 = 7x10^2 +4x10^1 +3x10^0$$

The first expression is our usual compressed way of writing seven hundred forty three. The second, often called "expanded form," reminds us that each digit in the compressed form represents a number of a special type, and that the whole number is implicitly a sum of these special numbers. In the seminar, we called these special numbers "single place numbers," because each has a non-zero digit in only one place. The third expression decomposes each single place number as a product of a digit times a "base ten unit." In this example, only the three units 1, 10 and 100 are involved. The fourth expression exhibits the fact that each base ten unit is a product of some number of 10s. The final expression uses exponential notation to write these products as powers of 10. Each way of writing 743 represents a conceptual advance over the one to its left. Arithmetic instruction should see to it that students learn all these levels of representation, so that they would leave middle school with a conceptual understanding of the place value system (including its extension to decimal fractions); but in the U.S., this typically does not happen.

The place value system governs not only how we write numbers; it also governs all the computations we do with them. It is not much of an exaggeration to say that the standard methods for computing (adding/subtracting and multiplying/dividing) with base 10 numbers are governed by computations with single place numbers, combined according to the Rules of Arithmetic. Computations with single place numbers in turn are controlled by the sums and products of single digits — the "number facts" — which is why these facts are so basic for computation. Estimation is also greatly facilitated by a grasp of the size relationships among single place numbers.

The last way above of writing 743 suggests to think of a number expressed in base ten form as a "polynomial in 10," and we spent some time exploring this idea. In fact, there are substantial parallels between computing with base ten numbers and computing with polynomials, and students might enjoy and benefit from explicit study of these parallels.

After whole numbers, fractions are probably the largest topic in the elementary curriculum, and in a lot of ways they are the most problematic. To deal successfully with fractions, it seems necessary to pay more attention to what numbers are and how we use them. We took as our working definition, that numbers express a relationship between a quantity and a unit or between two like quantities, with one effectively functioning as the unit. Less formally, one might say that a number is an adjective that modifies a noun (the noun being the unit). At any rate, it is clear that a number usually is incomplete in itself, and needs to be attached to a unit to have a definite meaning. We emphasized the point, that in dealing with fractions, it is vital to keep the unit in mind. Also, working with different units for a given type of quantity can be useful for getting acquainted with fractions. Thus, if a can (of say, soda)is the unit, then a six-pack is 6, and a case of four six-packs is 24. But if a case is the unit, then a can is 1/24 ,and a six-pack is 1/4.

We also discussed the circumstance that the common way to teach fractions in the U.S., by describing, say, 5/8 as "five out of eight" things may be an example of example insufficiency, meaning that it is too narrow a conception of fractions, and does not accommodate all the ways that fractions get used. If it gets established as a student's only way to think about what a fraction is, then it may limit the student's ability to work with fractions in contexts where it does not apply. We use fractions in many situations where the fraction is not part of the whole. We can talk about 1/2 gallon of milk, without the half-gallon being part of any specific gallon.

In the seminar, we took the point of view that the basic fractions are the unit fractions, 1/2 , 1/3 , 1/4 , 1/5 , ... Each unit fraction effectively defines a new unit, of which it takes several copies to make the original unit. Thus it takes 2x(1/2) s, or 3x(1/3) s, or 4x(1/4) s to make 1. Non-unit fractions are then multiples of unit fractions. Thus 3/2 gallons of milk would be 3 half-gallons of milk.

It is also important to have ways to visualize fractions. We discussed breaking rectangles into equal pieces to represent fractions, and also representing them using a number line. If the number line is thought of as being divided regularly into intervals of unit length, with the division points labeled by the integers, then halves give a regular division of the line that is twice as fine, with two half-intervals fitting into each unit interval. Similarly, thirds would produce a regular division in which each unit interval consists of three equal subintervals. It is an interesting exercise to imagine the unit interval being divided into, say thirds, and also into fifths, and to find the lengths of all the subintervals created in this way.

We also discussed some topics related to algebra, such as variables, expressions and equations, and word problems. Our basic point of view here was that a variable is a symbol that can stand for any member of some set, usually a set of numbers, such as the whole numbers or the rational numbers. We dealt with expressions as recipes for calculations. For example 3x - 4 would say, "Take a number x, and multiply it by 3, then subtract 4 from the result." We saw that word problems can be solved either algebraically or arithmetically (meaning, without the use of variables), showing a stronger relationship between arithmetic and algebra than is commonly apprehended by students.

The seminar units reflect all aspects of this wide-ranging discussion. The units of Emily Dentel, Autumn Laidler and Kishayla Payne-Miller deal primarily with whole number arithmetic, with an emphasis on place value. Michael Pillsbury's unit also focuses on place value, but from a more advanced perspective that emphasizes its connections to algebra, and especially, computation with polynomials. Troy Holiday's unit also utilizes place value ideas, to discuss scientific notation, especially how it emphasizes size and accuracy, making it useful for science. The units of Valerie Schwarz, Joseph Condon and Jonathan Fantazier present fractions from multiple perspectives. Sarah Kingon and Aimee MacSween have written units that discuss the beginnings of algebra based on working with expressions as recipes for computations, and using the Rules of Arithmetic to transform and simplify expressions. Finally, Nancy Rudolph's unit borrows ideas from all parts of the seminar to create warm-up exercises for her pre-calculus class, with the goal of increasing their number sense. We are sure that all the seminar Fellows join us in hoping that readers will find the treatments described in these units provide a more unified perspective on their respective topics than do typical textbooks.

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