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The FAL of Linear Relationships: Simple and Complex Word Problem Scenarios with Two Variables

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Introduction

"Math has never been my favorite subject. I've always hated it and that's not gonna change!" These statements often come from students who are repeating a math course for the second and third time. Then, it does not take but a few minutes at the start of the first day of school every school year for the age old question to be asked, "So why do I need to even be here? When am I ever going to use math in my life?" With all eyes on me, I proudly proclaim, "Math *is* your life! Math surrounds you every day!"

While my response is full of excitement and love for math, my students do not share the same sentiment and are not sure I am even in my right mind. They stare at me with puzzled looks on their faces and wonder how anyone could possibly be this excited about math. As I share my passion for math and my reasons for choosing to teach math as a profession, the puzzled looks fade into inquisitive eyebrow raises as the discussion flows into students reflecting upon their own math journey thus far. Wow, a glimpse of the inner mathematician in my students has been awakened! "Seriously? A mathematician? Ain't happenin'! No disrespect, Ms. Brown, but I don't wanna be an old man walking around with a pocket protector, calculator and funky glasses! That's geek status and I'll never be that crazy about math!" This, coming from a repeater student who is in an Algebra 1 level course for the third time, indicates why I need to know my student audience and their misconceptions about math prior to moving forward with any type of instruction. I share with them that using the term mathematician to describe my students does not imply I have an expectation they will be experts and math geniuses by the time they complete this course. It does imply, however, that I expect my students to respect and appreciate the unique challenges of solving real life scenarios while seeing the advantage of recognizing mathematical patterns and concepts [1] that contribute to their daily lives and help them determine solutions to real life scenarios.

Rationale and Objective

In life, we notice many personal characteristics that change with time. Age increases at a constant rate, beginning on the day of one's birth. Height increases at different rates depending on many factors: family genes, childhood health, and environmental factors. Weight changes at different rates due to health reasons, eating habits, active vs. inactive lifestyle, exercise, and much more. Rate of change can be used to analyze these and many other everyday life situations. This presents the potential for the use of math that I hope will become consciously recognized during this curriculum unit.

The main objectives of this curriculum unit are to have my students read word problem scenarios involving linear relationships, and determine the relationship between independent (x) and dependent (y) variables. The problems will gradually increase in complexity as the unit progresses. The class will discover that a fraction in the form of a ratio identifies a repeated pattern in a given scenario. Class discussions and related assignments will help lead to interpretation of this pattern as a rate of change and eventually relate that rate of change to the slope of a line in a graph describing the scenario. Students will progress to recognizing rate of change in everyday situations and will eventually be able to interpret that rate of change using multiple forms: word problem scenarios, tables of values, graphs, and linear equations in slope-intercept form. Emphasis will be placed on understanding the relation between dependent and independent variables for linear functions. I especially want my students to understand why the corresponding graphs are straight lines.

Both the contextual and the geometric meaning of the parameters of a linear function given in slope-intercept form $y=mx+b$ will be studied. From the contextual point of view, we will stress the importance of knowing that the parameter (b) specifies the initial or beginning value of the dependent variable (y), when the independent variable (x) is zero. Then the parameter m describes the constant rate of change of y, as x varies from 0. Equally, students should learn that, from the geometric viewpoint, the parameter b tells the y-intercept of the line that is the graph of the function, and m specifies the slope, or steepness, of the line. I want my students to understand how to move from that beginning point, and use the constant rate of change to find the value of x that achieves a desired value for y in any given scenario. By the end of this curriculum unit, students will use the variables and their linear relationship to understand the rate of change and its relation to slope. They will interpret problem scenarios involving linear functions, both simple ones and complex ones. They will create tables of values for a given function, draw graphs and write linear functions describing a scenario in slope-intercept form.

The target population for this curriculum unit will be students in the 9 th and 10 th grade levels at William C. Overfelt High School (WCO) located in San José, California. The WCO school enrollment is about 1500 students consisting of 80% Latino and 40% English Language Learners. Almost 9 out of 10 of our students come from low income families [2], which means the majority of WCO students come from working class or low to no income households in the east side of San José. While the east side of San José is considered a community within the heart of the Silicon Valley in the third largest city in California, access to all that the Silicon Valley has to offer is not readily available to them. Some of the school wide goals for students are to increase their use of critical and creative thinking skills, their ability to work in collaborative groups and teams, and their resilience in all aspects of their academic work. WCO students will need support and guidance in delving into word problem scenarios, to access their prior knowledge, build upon their mathematical skills, and move their learning forward.

As part of the transition to Common Core State Standards (CCSS) and with the introduction of a new Common Core Integrated Mathematics I (CCIM-I) course, mathematics instructors have been required to complete a three day workshop training on the newly adopted Mathematics Vision Project (MVP) curriculum. This workshop training is to provide mathematics instructors throughout the district the time to collaborate on how teaching and instructional strategies will need to change in order to successfully teach the new curriculum and also, how to encourage student engagement. Our goal is to have students shift from traditional teacher-directed instruction to seeing their teachers as facilitators, and to take more ownership of their learning. This drastic shift for students also includes having to work collaboratively with their peers to delve into the learning cycle of inquiry with real life scenarios involving mathematics. The Formative Assessment Lesson (FAL) process will be a means for me as an instructor to use a process to measure student learning and re-engage students in their learning process. I will work hard to weave the Common Core State Standards for Mathematical Practice (CCSS-MP) into each lesson as students become familiar with the practical implications of these standards.

Background

Far too often my students become intimidated by, shy away from and struggle with the sight of fractions in any form. They have trouble with decimals and knowing place values, especially within word problem scenarios. Herb Gross [3] describes this as "math phobia": my students have such a fear of math that they deny themselves the upward mobility that would come with success in this subject area. In this unit, I am seeking ways to make the content of mathematics appealing to students through practical applications of math concepts.

Students need to be guided to do math, and it should be presented in a non-threatening way that is easy to internalize. Perhaps it would be helpful for students to view learning math as a game in which they are being exposed to math concepts using video lectures, PowerPoint presentations, animation and even something as simple as user friendly text that they can easily read and digest through graphic organizers and other means of organizing content knowledge. The seminar discussions and readings from Roger Howe's seminar "Place Value, Fractions, and Algebra: Improving Content Learning through the Practice Standards" brought forth the misconceptions and challenges students face when working with these mathematics topics in math courses at all levels. While many conversations occurred throughout the two weeks of the seminar, the topic discussed most was how to teach fractions in a way that students will understand. I automatically connected this with the difficulty my students have with recognizing slopes of linear functions, or rate of change from one point on a straight line to another, as a fraction.

Many of the fraction skills that are reviewed in the first year of high school mathematics, typically an Algebra 1 course, should have been first learned in third through the fifth grade and then solidified in the sixth through eighth grade math courses, so that high school students would be confident in working with fractions. However, this is often not the case with my students. My hope for this curriculum unit is that students accept the challenge of identifying rate of change as a fraction and apply this knowledge to linear functions in word problem scenarios. I want students to delve into real life scenarios with confidence and ease and to recognize that the process needed to arrive at a solution is just as important as the final answer.

The completion of Modules 1, 2 and 3 of the MVP curriculum will precede this curriculum unit. These modules will require students to define quantities and interpret expressions, to use units to describe variables, to understand problems, and to explain each step in a process to solve an equation. These modules will also include work with ratios, unit fractions, fractions with fixed denominators and fractions with different denominators. Students will have used the number line model to interpret unit fractions and general fractions, and how to use fractions to measure distance from zero to a given point on the number line. Students will have seen a coordinate plane and will understand how to place whole numbers, integers and fractions on the *x*-axis and *y*-axis. All this content knowledge is needed for the activities of this curriculum unit.

Fractions

Throughout the Common Core State Standards for Mathematics (CCSS-M) courses from elementary and middle school, students will have been exposed to the process of understanding the practical application of fractions in their everyday lives to provide a context for processing simple word problem scenarios. Upon entering the CCIM-I course, students should have a basic understanding of multiple representations of fractions and ratio. This curriculum unit will relate this prior knowledge to rate of change in everyday life scenarios.

I will expect my students to start this unit with the understanding that, in everyday life, all numbers come with units attached. In other words, all numbers can be considered adjectives that relate to a noun [4] . A fundamental convention of arithmetic is, that when we add numbers, they all must refer to the same unit. An example would be to use coins, such as nickels, dimes, and quarters. Consider that 3 dimes and 4 nickels are equivalent to 2 quarters. However, we would never write this as 3 + 4 = 2. Instead, we would refer each coin to its value in a common unit, for example cents: 5 cents per nickel, 10 cents per dime and 25 cents per quarter. By using coins as the noun, their unit value and the number of coins becomes the adjectives: 3 coins worth 10 cents each for a total of 30 cents combined with 4 coins worth 5 cents each for total of 20 cents is equivalent to 2 coins worth 25 cents each for a total of 50 cents. This unit consciousness will be extended in class discussions whose goal is to identify the more complicated units of quantities being used for ratios as unit fractions and general fractions.

Ratios in fraction form express parts of a certain quantity (numerator) in comparison to a specific unit (denominator). This is illustrated in unit fractions and general fractions. The CCSS-M approach to fractions is based on unit fractions. The unit fraction *1/d* of some unit is a quantity such that it takes *d* copies of *1/d* to make the original unit. Unit fractions are considered as *1/d* pieces of a whole unit. General fractions in the form of *n/d* are multiples of unit fractions: *n* copies of *1/d* produces *n/d*. For example, we use unit fractions in our system for measuring time:

1/60 of a minute = 1 second → *It will take 60 seconds to create 1 minute.*
1/60 of an hour = 1 minute → *It will take 60 minutes to create 1 hour.*
1/24 of a day = 1 hour → *It will take 24 hours to create 1 day.*
1/7 of a week =1 day → *It will take 7 days to create 1 week.*

Although it is not part of the CCSS-M definition of fractions, it is important in general, and especially for my curriculum unit, that students understand that a fraction also represents the result of a division. Thus, ¾ is defined as 3 pieces of size ¼ each, but it also is equal to the result of dividing 3 into 4 equal pieces. Realizing this equivalence will be necessary for my students to understand rates of change expressed as fractions.

The notion of units of measurement should be addressed within the ratios produced by fractions. Students will need to interpret the relationship between values in the numerator and denominator of a fraction: the numerator will be dependent upon the unit being used in the denominator. This establishes the independent and dependent relationship between the units being used to create the ratio. Students will identify these units to interpret the meaning of a ratio with relation to the scenario or context with which the ratio is being used. Consider the coins example in which 3 dimes and 4 nickels are equivalent to 2 quarters. The value of all three coins is a multiple of 1/00 of a whole dollar: each dime is 10 times 1/100 or 10/100 of a dollar, each nickel is 5 times 1/100 or 5/100 of a dollar and each quarter is 25 times 1/100 or 25/100 of a dollar. When illustrating the equivalence of these values using the same units we see that:

$$\begin{gathered}3\ (\text{dimes}) + 4\ (\text{nickels}) = 2\ (\text{quarters})\\ 3(10/100) + 4\ (5/100) = 2\ (25/00)\\ 30/100 + 20/100 = 50/100\\ 50/100 = 50/100\end{gathered}$$

The unit conversion relating all coin values to a dollar in the form of a ratio helps to compare the resulting fractions using the same unit of measurement. This understanding of ratios will translate into understanding how constant rates of change in the form of a ratio with specific units relate to slope.

Slope and Straight Lines

Given two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ in the Cartesian coordinate plane, the *slope between P_1 and P_2* is the quotient or ratio

$$s(P_1, P_2) = \frac{y_2 - y_1}{x_2 - x_1}.$$

Since $y_2 - y_1$ is the vertical change, or "rise" in carpenter's terminology, in going from P_1 to P_2, and $x_2 - x_1$ is the horizontal change, or "run", slope is also referred to as "rise over run", or rise divided by run. Sometimes, the difference or change in y-values in going from P_1 to P_2 is written as $\Delta y = y_2 - y_1$, and in similar fashion, the difference or change in the x-values is written as $\Delta x = x_2 - x_1$. The slope between P_1 and P_2 can then be represented with the notation $\Delta y/\Delta x$. We will use this notation for this curriculum unit.

I want my students to understand that slope is exactly the right quantity to look at when dealing with straight lines in the coordinate plane. The intuitive idea of a straight line is precisely and elegantly captured by the condition of constant slope. When a set of points is given, whether or not they are in numerical order with respect to the independent (x) value of the point, and the slope between any two of the given points is the same as between any other two of the points, the whole collection of given points will all lie on a single straight line. While it is often helpful to list coordinate points in numerical order, it is not necessary when determining the slope between the points. The set of given points can be in any order and any two points can be used to find the slope, so long as the resulting slope is exactly the same every time for any pair in that particular set of points. The straight line passing through all of the given points can then be described by a simple equation. If $P_1 = (x_1, y_1)$ is a selected one of the points, and $P= (x, y)$ is any other point in on the line, and if m is the slope, then if we multiply both sides of the equation

$$s(P_1, P_{\square}) = \frac{y_{\square} - y_1}{x_{\square} - x_1} = m$$

by $x - x_1$, we get the equation

$y-y_1=m(x-x_1).$

By adding y_1 to both sides and expanding the product on the right hand side, this equation can be rewritten as

$y=mx+(y_1-mx_1)= mx+b.$

We will interpret the constant $b = y_1 - mx_1$ later on. This underlines the fact that there is a constant linear relationship between the dependent (*y*) and independent (*x*) values of points on a straight line. I will have my students work out many numerical examples of this remarkable characterization of straight lines, some purely numerical, and others arising in real-world scenarios, until they become convinced of its truth.

The slope (*m)* between any two coordinate points from the same straight line will always be exactly the same. This can be verified by choosing a set of points from a straight line, then calculating and comparing the resulting slopes from any two points on the straight line (Figure 1). To calculate the slope, determine the change in *y*-values (Δ*y)* in the numerator in relation to the change in *x*-values (Δ*x)* in the denominator:

$m = \Delta y / \Delta x =(y_2-y_1) / (x_2-x_1).$

I will have my students check the following key fact in many cases, with the hope that they will make the generalization: Given any three points, P_1, P_2, and P_3, if the slopes between P_1 and P_2 and between P_2 and P_3 are the same, then the slope between P_1 and P_3 is also equal to the common value of the first two slopes. This is illustrated in Figure 1.

Points on a Straight Line
(0, 0) (1, 5), (2, 10), and (3, 15)

Slope between (1, 5) and (2, 10)

$$slope = \frac{y_2 - y_1}{x_2 - x_1} = \frac{10-5}{2-1} = \frac{5}{1}$$

Slope between (1, 5) and (3, 15)

$$slope = \frac{y_2 - y_1}{x_2 - x_1} = \frac{15-5}{3-1} = \frac{10}{2} = \frac{5}{1}$$

Figure 1: Slope between selected pairs of Points on the Line

Related quantities

There are many situations in which two quantities are related in some way. For example, if someone has a bank account at a bank and makes regular deposits, the amount of money in the account will grow with time. Or if someone has some money, and uses it to buy things, the amount they have left will depend on the purchases. If what is needed is multiple copies of a single product, the amount left will depend on the number of copies bought. Or if someone has an exercise program that includes running, and runs a fixed amount each day, the total distance run will depend on the number of days the program has been followed.

Usually in these situations, we think of one quantity as determining the situation, and the other as depending on the first. In first and third examples above, time runs along (whatever we do), and the quantity we are interested in (bank account value, or distance run), depends on the time. In the middle example, we decide how many copies of a given product we need, and then the money we have to spend, and the money we have left, depends on the number of copies. We use the terminology *independent variable* to refer to the variable that controls the situation, and *dependent variable* to refer to the other quantity, whose value is determined by the value of the independent variable.

Table of Values

If we have a pair of related quantities, we can describe the relationship by creating a table of values. The table of values serves as a means to organize and identify the relationship between dependent (y) and independent (x) variables in a sequential order. Since it is frequent practice for the x values chosen to make a table of values are consecutive whole numbers, it is common practice to set up a table of values by using a two column table and placing the value of the independent (x) variables in numerical order in the first column and then listing the corresponding values of the dependent (y) variable in the second column. In Figure 2 a table of values is used to identify the relationship between two variables from a specific scenario. In the situations we will study, which involve only linear functions, the y-values will also be in numerical order of increasing or decreasing value in the table. Note that the table of values will eventually be used to graph the coordinate points and develop a linear equation, so it will be helpful for students to become familiar with identifying the value of y when x is zero in their table of values. Students will later identify this coordinate point as the y-intercept.

Scenario: Kyarra loves to collect stickers and currently has 4 stickers. Every day she plans to collect 4 more stickers. After 1 day she will have a total of 8 stickers. After 2 days she will have a total of 12 stickers. After 3 days she will have a total of 16 stickers.		
(Days, Stickers) Points On a Straight Line	**Days** Independent Variable (x)	**Stickers** Dependent Variable (y)
(0, 4)	0	4
(1, 8)	1	8
(2, 12)	2	12
(3, 16)	3	16

Figure 2: Table of Values

Rate of change

When two quantities are related, one is often interested in knowing how much a change in the independent variable affects the dependent variable. It turns out to be useful to look, not just at the change in the dependent variable, but at the *rate of change*. This is defined as follows: Given two values, x_1 and x_2, of the independent variable, and two corresponding values, y_1 and y_2, of the dependent variable, the rate of change of y between $x = x_1$ and $x = x_2$ is the ratio or quotient 777

$$r(x_1, x_2) = \frac{y_2 - y_1}{x_2 - x_1}.$$

It might not be obvious to my students why we should look at the rate of change. The formula may look complicated. I want to get them to understand that the rate of change is the key to understanding the wide variety of scenarios that we study. The remarkable thing about these scenarios is that the rate of change turns out to be constant, and completely independent of the two values of x that are chosen. Thus, in our scenarios, if we know the rate of change, and the value of y for a single value of x, we can compute the value of y for any value of x. The entire relationship can be encoded with just two numbers!

Graphing a relationship

Even though the variables in a relationship (money, time, number of items purchased) may have no geometric significance, once we have recorded the relationship in a table of values, it is tempting to graph the pairs of values as points in a coordinate plane. I want my students to understand that, when we do this, we get a relationship between geometry and numerics: the rate of change that was a key to understanding linear relationsh ips turns into the slope between points, and the graph of a relationship with constant rate of change is a straight line!

When the structure of a table of values is set so that the rate of change between any two values of the independent variable is the same, then, when the table is graphed, this relationship will be seen as the slope of a straight line through any two points chosen from the table. This can be seen in Figure 3 where any two points are chosen from the table of values and when the slope is calculated, the result is exactly the same each time. The recognition of this slope in relation to the difference in dependent (*y*) and independent (*x*) variables will translate into plotting points from the table, in any order, on a coordinate plane to create a visual representation of the linear relationship between two variables.

The constant rate of change of 4/1 in Figure 3 indicates that when the coordinate points are placed on a coordinate plane, beginning with the coordinate point (0, 4) the next point will be four units up and one unit to the right from the beginning point. In essence, the resulting slope that is determined by finding the "rise over run" of the dependent (*y*) and independent (*x*) variables indicates that when we want to determine the value of coordinate points on a straight line, begin with one coordinate point, usually the first point from a table of values in numerical order, and use the slope to move the specified units to the next coordinate point. Continuing this pattern will result in the creation of a graph of a straight line.

Slope between Points on the same straight line (in numerical order by x-values)	Slope of Any Two Points on the same straight line
(0, 4) and (1, 8) $slope = \frac{y_2 - y_1}{x_2 - x_1} = \frac{8-4}{1-0} = \frac{4}{1}$	(0, 4) and (2, 12) $slope = \frac{y_2 - y_1}{x_2 - x_1} = \frac{12-4}{2-0} = \frac{8}{2} = \frac{4}{1}$
(1, 8) and (2, 12) $slope = \frac{y_2 - y_1}{x_2 - x_1} = \frac{12-8}{2-1} = \frac{4}{1}$	(1, 8) and (3, 16) $slope = \frac{y_2 - y_1}{x_2 - x_1} = \frac{16-8}{3-1} = \frac{8}{2} = \frac{4}{1}$
(2, 12) and (3, 16) $slope = \frac{y_2 - y_1}{x_2 - x_1} = \frac{16-12}{3-2} = \frac{4}{1}$	(0, 4) and (3, 16) $slope = \frac{y_2 - y_1}{x_2 - x_1} = \frac{16-4}{3-0} = \frac{12}{3} = \frac{4}{1}$

Figure 3: Slope between Any Two Points from a Table of Values

Graph of a Linear Relationship

The coordinate plane consists of all the possible values of the independent (*x)* variable along the horizontal, or *x*-axis and all of the possible values of the dependent (*y)* variable along the vertical, or *y*-axis. The units labeled along each axis represents the value of the variables. The points from a table of values are called coordinate points in the form (*x, y)*. The values of each variable within the coordinate point determines the location of that point within the plane. For example, we see in Figure 4 that when a table of values is created for the coordinate points (0, 0), (1, 5), (2, 10), and (3, 15), the independent (*x)* values are in order from least to greatest and all dependent (*y)* values are adjacent to the corresponding *x*-values. All four points are then drawn onto the coordinate plane and connected in numerical order to create a straight line.

Coordinate Points On a Straight Line	Independent Variable (*x*)	Dependent Variable (*y*)
(0, 0)	0	0
(1, 5)	1	5
(2, 10)	2	10
(3, 15)	3	15

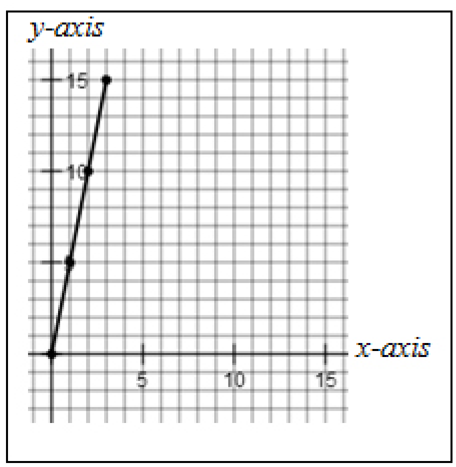


Figure 4: Graph of a Straight Line using Coordinate Points from a Table of Values

Linear Functions in Slope-Intercept Form

A straight line can also be described as the graph of a linear function. There is a constant slope between any two coordinate points, and every independent (*x)* value has exactly one dependent (*y)* value related to it. The slope between these coordinate points can be verified by determining the rate of change in a table of values. It is important to note that the previous examples of linear functions have all included an *x*-value of zero as the first coordinate point in the table of values and on the graph. This is intentional since that specific coordinate point is the location where the graph of the straight line intersects or crosses the *y*-axis. This coordinate point, known as the *y*-intercept, is the beginning point of the graph of a linear function and the slope is then used to move at a constant rate of change from that beginning point to all other coordinate points to create the graph of a straight line, thus forming a graphical representation of a linear function.

The coordinate points *(x, y)* become values that can be substituted, or replaced, for the *x* and *y* variables in the linear equation to validate that the given coordinate points will appear as points on the graph of the linear function. This validation also proves that the given coordinate points are now solutions to the linear equation representing the linear function. Consider the coordinate points (0, 6), (1, 13), (2, 20), and (3, 27) which can be organized into a table of values and then graphed in a coordinate plane to create a straight line. The slope between any two points from this set of given coordinate points would be 7/1, indicating that as the *x*-values increase by 1 unit, the *y*-values increase by 7 units. To illustrate the relationship of taking any coordinate point (*x, y*) and moving with the slope (*rise/run*) to determine the continuous coordinate points in a linear function, we use the notation *(x, y) —> (x + run, y + rise)*, where *(x + run, y + rise)* represents the next numerical coordinate point in the sequence of coordinate points for the linear function. For example, since we already know the slope of this specific linear function is 7/1, by following the notation we see a relationship developing between continuous coordinate points, using the slope between two points and beginning with the *y*-intercept:

Beginning Point		→ (0, 6)
(0, 6) →	(0 + 1, 6 + 7)	→ (1, 13)
(1, 13) →	(1 + 1, 13 + 7)	→ (2, 20)
(2, 20) →	(2 + 1, 20 + 7)	→ (3, 27)

The result of using this notation has produced continuous coordinate points that can now be graphed on a coordinate plane in a straight line to produce a linear function.

Slope-Intercept Form of a Linear Equation

The relationship between the slope, *y*-intercept, and coordinate points to create the graph of the linear function can be represented in the form of a linear equation *y=mx+b*, known as the slope-intercept form (Figure 5). The slope (*m*) is identified as the rate of change between any two coordinate points (*x, y*). The y-intercept (*b*) is a specific point of interest where the graph begins. The *y*-intercept will be identified as the coordinate point (0, *b)* to denote that the *x*-value is always equal to zero, and the *y*-value will indicate the *y*-intercept, or the location where the linear function will intersect with the *y*-axis. Returning to the previous example of a linear function whose slope is 7/1 and *y*-intercept is (0, 6), these values can be used to create the slope-intercept form of this linear function by substituting 7/1 (or just 7) for the slope (*m)* and 6 for the *y*-intercept (*b): y = 7x + 6.* This linear function is telling us that the graph of this function would begin at (0, 6) and move or rise 7 units up for each run of 1 unit to the right, so when x changes from 0 to 1, we would arrive at the point on the graph (1, 13). From this new point, the slope would continue to rise 7 units for each run of

1 unit to arrive at other points.

Given that…
m: the common slope between two of the points on a straight line
(0, b): a point on the y-axis
(x ,y): any other point on the straight line

We can prove that a straight line satisfies a linear equation…

$$m = \frac{y-b}{x-0} \rightarrow m = \frac{y-b}{x} \rightarrow (x)m = (x)\frac{y-b}{x} \rightarrow$$

$$mx = y - b \rightarrow mx + b = y - b + b \rightarrow mx + b = y \rightarrow y = mx + b$$

Figure 5: Straight Line Satisfies a Linear Equation

Coordinate Points as Solutions to Linear Equations

Linear equations can be created and used to understand, interpret and determine solutions to simple and complex word problem scenarios involving two variables related by a linear function. The process to create these linear equations is to identify the dependent (*y*) and independent (*x*) variables from the word problem scenario and create a table of values to organize the relationship between the variables. By computing the rate of change for each pair of adjacent values of the independent variable, students can recognize whether the relationship is linear by checking whether all the computed rates of change are the same. Using the slope and *y*-intercept derived from a table of values or a graph, a linear equation can be created to represent the linear relationship between the variables. Once this linear equation has been created, all the coordinate points from the table of values can be verified as being solutions to the equation by substituting the coordinate points into the linear equation, resulting in a true statement. There is a sequence of steps to create a linear equation from a word problem scenario and prove that coordinate points are solutions to the linear equation (Figure 6). Although when graphed, the relationship appears to have turned into pure geometry, it is important to remember its real world origins. To make sure that my students stay in touch with the real world meaning, I will ask them always to label the x- and y- axes to show the quantities they represent, including units.

An extension to the word problem scenario in Figure 6 may require an extension of the table of values and the use of the linear equation to determine another solution. The extension question shows how the linear equation could be used to identify and verify the solution to prove its relevance to the extension question. The ability to determine the solution to the extension question will assess student understanding of the relationship between the two variables, slope and *y*-intercept.

Word Problem Scenario	***Linear Equation***
Bianca currently has \$6 in her savings account and she wants to save all of her weekly allowance to build her savings account with more money. After the first week, Bianca now has \$13 in her savings account. Using a table of values, determine how much money Bianca would save after three weeks.	Given the coordinate point (0, 6), we know the y-intercept is 6. Given at least two coordinate points form the table, we know that the $slope = \frac{y_2 - y_1}{x_2 - x_1} = \frac{13 - 6}{1 - 0} = \frac{7}{1} = 7$ Therefore, the slope-intercept form of the equation is $y=7x+6$

Table of Values

Independent (x) Number of Weeks	Dependent (y) Total Savings \$
0	6
1	13
2	20
3	27

Coordinate Points as Solutions

Verify $y=7x+6$ *when* $(x, y)=(0, 6)$
$6=7(0)+6 \rightarrow 6=0+6 \rightarrow 6=6$

Verify $y=7x+6$ *when* $(x, y)=(1, 13)$
$13=7(1)+6 \rightarrow 13=7+6 \rightarrow 13=13$

Verify $y=7x+6$ *when* $(x, y)=(2, 20)$
$20=7(2)+6 \rightarrow 20=14+6 \rightarrow 20=20$

Verify $y=7x+6$ *when* $(x, y)=(3, 27)$
$27=7(3)+6 \rightarrow 27=21+6 \rightarrow 27=27$

Extension Question

Determine the total amount of money (in dollars) Bianca will have saved after 5 weeks. In other words, determine the value of y when x=5.

Linear Equation: $y=7x+6$ *when* $(x, y)=(5, \_\_\_\_)$ $\rightarrow x = 5$ *weeks, so* $y = \$\_\_\_$

Substitution: $y=7(5)+6 \rightarrow y=35+6 \rightarrow y=41$

Verification: $y=7x+6$ *when* $(x, y)=(5, 41)$
$41=7(5)+6 \rightarrow 41=35+6 \rightarrow 41=41$

Figure 6: Verifying Coordinate Point Solutions to a Word Problem Scenario

Strategies

This curriculum unit will be completed in three to four weeks, depending on the school schedule of regular, collaboration or block scheduling of classes, within an 18-week semester schedule at WCO. The course that is most suitable for this unit is CCIM-I, although it can be used in Common Core Math 8 (CCM-8), Summer Bridge Programs for incoming ninth graders, and in Common Core Traditional Algebra I. I will be teaching this curriculum unit to ninth and tenth grade students in the CCIM-I course in a comprehensive high school setting. The class periods range from 40 to 55 minutes each day and this course will be one of at least four academic courses students will take within a complete school day. Many of the lessons in this unit will incorporate common teaching strategies such as whole-class discussions, graphic organizers and composition books to maintain notes on content knowledge and skills, think-pair-share, collaborative group work, and individual tasks. These teaching strategies will help students be engaged in lessons and activities that are guided through the FAL process and the implementation of the 5 Practices Model [5] within the FAL process.

Formative Assessment Lesson

There are many types of skills used to acquire knowledge that students will need to focus on for 21st century college and career readiness and navigate through life, one of which is critical thinking skills. This entails interpreting information and drawing conclusions as well as seeking relationships and patterns [6] . The sequence of lessons in the FAL will address these skills, introduce students to a new curriculum through the CCSS-M, including the application of the CCSS-MP that require students to expand on their creativity, take ownership of their own thinking and learning process both inside and outside of the classroom. Students will become an integral part of small collaborative groups and whole class discussions to defend and support their thinking process. The FAL process will encourage students to be active participants in their learning and actively engage in the process of solving problems to produce and make sense of solutions to problems.

The FAL process will guide students through the mathematical thinking and learning needed for success in this curriculum unit. Research has shown that effective teaching strategies that assess students and require they apply their prior learning assist teachers in the process of adapting their teaching to the needs of their students [7] . There are also some instructional strategies that are used in the FAL process that help improve instruction: clarify learning intentions for student success, facilitate effective whole-class discussions and collaborative learning tasks, provide feedback to move student learning forward, encourage students to be instructional resources for each other, and encourage students to take ownership of their own learning [8]

The FAL process is used to measure student learning and as a means to re-engage students in their learning process. Students take ownership of their learning through assignments, tasks, activities, and assessments. It is an important skill for students to determine mistakes in their own thought process and have the resiliency to fix those mistakes using prior knowledge and new knowledge. Instructors are able to use results from class discussions, activities and assessments to improve their teaching strategies and re-engage students by addressing common misunderstandings.

When viewed as a process, the FAL is a structure that lends itself to a gradual change leading to a particular result, a series of actions or operations leading to an end result, and the opportunity to receive information in order to generate an action or response. Successful implementation of the FAL process will take time, energy, planning and a deep understanding of student needs for successful implementation, all the while being student-centered in nature. The FAL process allows for meeting CCSS-M and promote the CCSS-MP throughout each step of the process. The result will be gradual changes in the way students think. It will encourage students to take actions based on the integration of their prior knowledge while receiving new knowledge, and allow them the opportunity to respond and take action leading to solutions.

A pre-assessment task is provided to each student to complete individually prior to the start of the FAL. This individual task is typically administered the last 15 minutes of a class period at least a day or two before the lesson is to be taught. This task is designed to reveal current understandings and misunderstandings of each student that will need to be addressed at the start of and throughout the lesson. The instructor must collect and review student work on the assessment task to create a list or chart of questions addressing the most common student difficulties and misunderstandings. This list or chart becomes a tool for instructors to prepare for whole-class discussions using the results from individual pre-assessment tasks. Whole-class discussions occur at the start of the first day the FAL is to be implemented. Research has shown that classroom discussions led by instructors are, "…one of the most universal instructional practices." [9] Instructors will address the common difficulties and misunderstandings in the form of questions or prompts within short tasks. "Questions solicit both knowledge and processing of knowledge at a designated cognitive level. Further, the effective question-answer exchange [will] …prompt students to provide answers that are either more correct, more complete, or at the desired cognitive level." [10] Students will be given processing, or wait time to develop a response and then answer questions to improve their knowledge and contribute to whole-class understanding of a concept or skill. The purpose of this wait time is to provide students with time to develop longer responses and answer questions with confidence while teachers have the time to ask fewer probing questions so they can develop more thoughtful responses to students [11] .

Students will then transition into cooperative learning groups, also referred to as collaborative groups, for further discussions and work collaboratively to discuss and complete a task or activity to further address the common issues and misunderstandings from the individual tasks. "Sensible, sense-making mathematics makes extensive use of high-quality instructional and assessment tasks to introduce, develop, reinforce, connect, apply, and assess understanding of key mathematical concepts." [12] Instructors should explicitly state the specific mathematical goals of the lessons and task so students can remain accountable for their mathematical learning and remain engaged in the lesson [13] . Specifying explicit learning goals lends itself to evidence of student learning [14] . The CCSS-MP are embedded throughout the collaborative group discussions and activities to promote student engagement and accountability. The activities within the collaborative groups range from simple tasks where students are completing skill based activities to more complex tasks where students will need to organize manipulatives in order to make sense of a situation, often given in a word problem scenario. During the collaborative group time, the instructor is tasked with implementing the 5 Practices Model [15] to identify and monitor student thinking throughout activities to select student work that will engage students in sharing their thought process and new knowledge that connects to the key mathematical ideas from the task. Instructors may find it useful to create and complete a chart [16] to monitor students' work on a specific task related to the strategies students use to complete their task or activity.

To bring the FAL to a close, there is a whole-class discussion to debrief on the task in which the instructor starts the discussion by helping students draw connections between their approaches to the task in relation to other students. This is where the chart is useful to begin calling upon students to present their findings and solutions to the task. The instructor also revisits the common difficulties and misunderstanding to the whole class in the form of questions. Students are given wait time to process and answer the questions with their new knowledge and better understanding about mathematics. The hope is students will improve their solutions on the post-assessment task with knowledge acquired through collaborative discussions and completing the tasks as well as keeping common misunderstandings in mind while processing through the collaborative task.

Activities

FAL Pre-Assessment and Post-Assessment Tasks

The pre-assessment and post-assessment task for the FAL (Appendix B) in this curriculum unit involves the exact same skills and mathematics knowledge to be completed in its entirety within 15 to 20 minutes for each task. The pre-assessment task will be administered the last 15 minutes of the class period two days prior to the start of the FAL. This provides me with time to review all student responses to the task and develop the list of common difficulties and misunderstandings from the task. The post-assessment task will be administered to students the last day of the FAL. I will compare the growth in skills and knowledge between the pre- and post- assessment results in anticipation of reviewing the results with the students.

The content of the pre- and post-assessment tasks will be for students to be given a word problem scenario for which they must clearly identify the dependent (*y*) and independent (*x*) variables of the scenario, create a table of values of coordinate points, graph the coordinate points on a coordinate plane and write a linear function in slope-intercept form from either the table of values or the graph. These tasks will require students to identify the slope and y-intercept in the table of values, identify and label the slope and y-intercept in the graph, and identify the slope and *y*-intercept in the linear function in slope-intercept form. Students will determine a solution to the word problem scenario using the table of values, graph and the linear function in slope-intercept form. Students can extend their learning by determining how the slope and *y*-intercept help to validate what they believe to be the solution to the word problem scenario.

FAL on Simple to Complex Word Problem Scenarios

There will be five lessons within the FAL process in which students will engage in collaborative group tasks and activities on several concepts relating to word problem scenarios. The reason behind using word problem scenarios rather than just providing students with data is because CCSS is expecting all subject areas to teach and support literacy, including math courses. [17] With this in mind, the first three lessons are designed to have students read and interpret simple word problem scenarios, identify academic vocabulary, and establish a symbolic relationship between and independent (*x*) and dependent (*y*) variables within the scenario. The process continues with students creating a table of values relating any numerical values and variables presented in the scenario. By including a value of zero for the independent (*x*) variable within the table of values, students will recognize this as the y-intercept of the coordinate points when used in the graph of a linear functions, and eventually when creating a linear equation to represent the function. The Plot and Draw program [18] and the use of the graphing features on the TI-Nspire graphing calculators supplied by the district math coordinators will be useful technology to provide visuals of this process to go from word problem scenarios into tables of values, graphs and linear equations. The last two lessons in this FAL are similar to the above mentioned sequence, however students will transition into using more complex word problem scenarios to increase the level of text complexity or difficulty to promote literacy in this mathematics course. [19]

Multiple Representations of Linear Functions

Prior to students taking the post-assessment for the FAL, students will be placed in collaborative groups of no more than four students to complete a group tasks. This task (Appendix C) will require students to complete four rounds of activities related to the FAL goals of interpreting word problem scenarios into tables of values, graphs of coordinate points and linear functions in slope-intercept form. In round one, collaborative groups will be given a word problem scenario and will need to create a table of values, graph the function and write a linear function in slope-intercept form to determine a solution to the scenario. Round two will begin with a table of values in which collaborative groups will need to create a word problem scenario, graph and linear function in slope-intercept form. Round three will provide a graph of a straight line with coordinate points clearly labeled, with which the collaborative groups must create a word problem scenario, table of values and linear function in slope-intercept form. Round four concludes the task by providing the collaborative groups with a linear function in slope-intercept form and will be required to create a word problem scenario, table of values and graph. In all four rounds it is important for students to always identify the dependent (y) and independent (x) variable, including the units of both variables, find the slope and y-intercept as well as a solution to the word problem scenario. Collaborative groups will be required to display their process for completing all four rounds on poster paper for students to do a gallery walk and see the work from all the collaborative groups. Students will use post-it notes to pose comments or questions on the posters as they conduct the gallery walk. Those comments and questions will serve as a whole-class discussion debrief to facilitate a conversation surrounding the goals of the task and the mathematical skills needed to be successful in completing the task.

Appendix A

Implementing State Standards

Number and Quantity

Quantities (NQ-Q) (1, 2, 3): Reason quantitatively and use units to solve problems. [Foundation for work with expressions, equations, and functions]

Algebra

Creating Equations (A-CE) (1): Create equations that describe numbers or relationships

Functions

Interpreting Functions (F-IF) (6, 7, and 9): Interpret functions that arise in applications in terms of the context.

Building Functions (F-BF) (1): Build a function that models a relationship between two quantities.

Statistics and Probability

Interpreting Categorical and Quantitative Data (S-ID) (7): Interpret linear models.

Common Core State Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them
2. Reason abstractly and quantitatively
3. Construct viable arguments and critique the reasoning of others
4. Model with mathematics
5. Use appropriate tools strategically

6. Attend to precision
7. Look for and make use of structure
8. Look for and express regularity in repeated reasoning

Appendix B

FAL Pre- and Post- Assessment

Word Problem Scenario

Bianca currently has $4 in her savings account and she wants to save all of her weekly allowance to build her savings account with more money. After the first week, Bianca now has $8 in her savings account. Using a table of values, determine how much money Bianca would save after three weeks. Using the graph determine, determine how much money he would save after five weeks. Using the linear equation, determine how much money he would save after 12 weeks.

Linear Equation

Given the coordinate point ______, we know the *y*-intercept is _______.

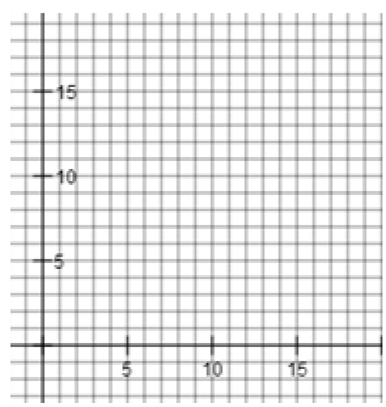
Given at least two coordinate points form the table, ________ and _________ we know $slope = \frac{y_2 - y_1}{x_2 - x_1}$

Therefore, the slope-intercept form of the linear equation is _____________.

Table of Values

Independent (x) *Units:* ______________	*Dependent (y)* *Units:* ______________

Graph



Solutions to Word Problem Scenario

Appendix C

Collaborative Group Task: Multiple Representations of Linear Functions

Word Problem Scenario

Simple – determine one solution

Complex – determine multiple solutions

Linear Equation

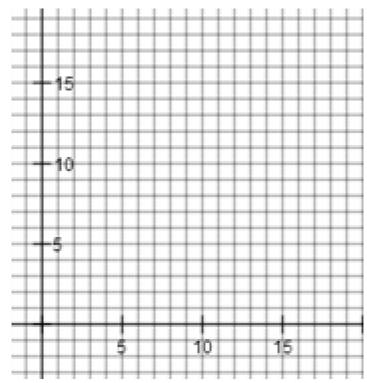
slope = ______ *y-intercept* = ______

$y = mx + b$ → $y = (\_\_\_\_)x +$ ______

Table of Values

Coordinate Point	Independent (x)	Dependent (y)

Graph



Coordinate Points as Solutions

$y = mx + b$ → _____ $= m(\_\_\_\_) + b$

Solution(s) to Word Problem Scenario

**Be sure to use units in solution.*

Appendix D

Simple and Complex Word Problem Scenarios for Formative Assessment Lesson

Simple (label all solutions using the proper units)

- Chantal runs 3 miles around her neighborhood each day. She runs 5 days a week (Monday through Friday). Create a table of values to show the number of miles Chantal will have run at the end of 5 days. If Chantal continues this pattern, how many days will it take for Chantal to run 45 miles?

- Ricardo went for a run last week and completed 24 miles. This week he will begin track practice and has to run 1 mile each track practice as a warm up exercise. Create a table of values to show the number of miles Ricardo will have completed after 5 days of track practice. If Ricardo continues this pattern, how many miles will he have completed after 10 days of track practice?

- Arnette has just started learning how to play the piano. Over the summer she learned to play 4 songs. Then she enrolled in a piano lesson class and is required to learn how to play one more new song every week. Create a table of values to show the number of songs Arnette will know how to play after 6 weeks. If Arnette continues this pattern, how many songs will she be able to play after 15 weeks?

- Dana is studying to be a female body builder and is required to eat meals each day containing 340 calories per meal. Create a table of values showing how many calories Dana will have consumed from meals after 5 days. If Dana continues this pattern, how many calories will she have consumed after 28 days?

- Dena is writing a research paper for her summer English class and has just completed writing 4 pages. She needs to finish her paper before she goes on vacation with her family. Dena has discovered she is able to write 2 pages every hour. Create a table of values showing how many pages Dena will have completed after 8 hours. If Dena continues this pattern, how many hours will it take for Dena to complete her research paper that needs to be 16 pages?

- Mr. Barrientez gives his physics students 2 quizzes every week. He has to be sure he grades all the quizzes and returns them to his students to place in their class binder. Create a table of values showing how many quizzes Mr. Barrientez gives his students after 6 weeks. If Mr. Barrientez continues this pattern, how many quizzes will he have given his students at the end of 15 weeks?

Complex Word Problem Scenarios

- Maura and Eric belong to different CrossFit gyms. Maura pays $25 per month and a one-time registration fee of $55. Eric pays only $15 per month but had to pay an $85 registration fee. After how many months will Maura and Eric have spent the same amount on their gym memberships? How much more will Maura pay than Eric when they both have belonged for 1 year?

- Trey and Kai are playing video games at the arcade in Eastridge mall. Trey has $20 and is playing a motorcycle racing game that costs 50 cents per game. Kai arrived at the arcade with $22 and is playing Mario Kart that costs 75 cents per game. (a) Create two linear equations (one for Trey and another for Kai) that gives the amount of money each boy has left in relationship to the number of games they have played. Let the number of games played = g. (b) After how many games will the two boys have the same amount of money left? (c) How much money do they have at this point?

- Makaela took a trip to London and Paris as part of a school field trip. While in Paris, Makaela went to a souvenir store to purchase Eiffel Tower key chains for everyone who helped make her trip possible and memorable. Each keychain costs $2 and she also purchased a souvenir t-shirt for herself that cost $17. Create a table of values showing how much Makaela spent in the souvenir store after purchasing key chains for family members, starting with her brother, 3 cousins, grandmother and mother. If Makaela continues this spending pattern, how much money will Makaela have spent if she needs to buy 30 Eiffel Tower key chains?

- A paddle boat rental service at Lake Cunningham Park charges a $20 transportation fee and $15 dollars an hour to rent a paddle boat. Write and graph an equation representing the cost, *y,* of renting a paddle boat for *x* hours. What is the cost of renting the paddle boat for 6 hours?

- The water tank at Branden's house already contains 55 gallons of water. Since there are 5 family members living in Branden's house, the family will need more water in the tank. Branden begins to fill the tank with more water that flows at a rate of 8 gallons per minute. Write a linear equation to model this situation. Find the gallons of water in the tank 25 minutes after Branden begins filling the tank.

- Ms. Gammons has a small business making dessert baskets. She estimates that her fixed weekly cost for rent and electricity are $200. The ingredients for one dessert basket cost $4.50. If Ms. Gammons made 40 baskets this past week, what was her total weekly cost? How many dessert baskets did she make if her total cost was $564.50?

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Notes

1. American Mathematical Society

2. Although this is roughly 87% of the student population, through anecdotal evidence this projection is likely closer to 92%.

3. Herb Gross, *A Dose of Gross* "*It All Adds Up Applying Math Lessons to Life*"

4. Herb Gross, *A Dose of Gross*

5. Smith & Stein, *5 practices for orchestrating productive mathematics discussions*.

6. Greenstein, *Assessing 21st century skills: a guide to evaluating mastery and authentic learning*., 24

7. Black & William, *Inside the black box: raising standards through classroom assessment*.

8. Wiliam, *Embedded formative assessment*., 46

9. Ibid., 78

10. Walsh & Sattes, *Quality questioning: research-based practice to engage every learner*., 97

11. Ibid., 81-83

12. Leinwand, *Sensible mathematics: a guide for school leaders*., 54

13. Smith & Stein, *5 practices for orchestrating productive mathematics discussions*., 13

14. Ibid., 14

15. Ibid., 8

16. Ibid., 47, Fig. 5.3

17. Calkins, Ehrenworth, Lehman, *Pathways to the common core: accelerating achievement*., 12

18. Plot and Draw program http://bit.ly/plotanddraw

19. Calkins, Ehrenworth, Lehman, *Pathways to the common core: accelerating achievement*., 96
