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Developing Proportional Reasoning

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Context

"How could they think their answer is even close to being correct? What were they thinking? Did they seriously just add those two numbers together? I didn't teach it that way! Do they even know what a ratio is? Why are they cross-multiplying everything? Did they read the problem? Why are they not take the time to set up the proportion correctly?" Thoughts and questions like these ran through my mind when analyzing student work from my ratio and proportion unit last year. I have found that it is easy as a teacher to fall into this frustrating line of thinking, questioning whether or not I taught a certain skill correctly, if students are ever paying attention, or if some are even capable of applying sound proportional reasoning. Upon reflection, I concluded that the way I perceived my student's deficits was at the very least flawed, and counterproductive to my practice. In writing this unit I aim to teach ratios and proportions in a way that promotes the development of proportional reasoning and avoids the dependence on rules and procedures that lead to shallow understandings. In order to drive my unit in a productive manner, I kept the following guiding questions at the forefront. What has obstructed the development of my students' proportional reasoning? Why have students consistently made the same mistakes, employing similarly flawed reasoning in problem solving, and how are curriculum and instructional strategies contributing to these problems?

I teach at an elementary school on the southwest side of Chicago. We are a neighborhood school that is 95% low income and serves roughly 1,200 students in pre-kindergarten through eighth grade. This next year I will be teaching four sections of sixth grade math in a general education setting. According to district and school assessments, this upcoming class, by a significant margin, presents the most remedial needs in mathematics when compared to other grades at my school. They will be entering middle school with large deficits in the fundamental areas of fractions, operations and algebraic thinking. I chose to write my unit on the subject of ratios, proportions and proportional reasoning because the concepts overlap in almost all other areas in the middle grades math standards and is determined by the Common Core to be a major work the sixth grade (CCSS Initiative 2015). Developing a robust and conceptual understanding of ratios and proportions will be important in the effort of setting a solid foundation for the entire school year. Ideally, through this unit, students will become accustomed to learning conceptually before procedurally, setting the standard for attaining a deeper understanding of subsequent concepts and rational problem solving that utilizes sound reasoning. Besides its foundational nature, the subject of ratios and proportions offer venues to remediate gaps in my students' prerequisite skills such as fractions, operations and measurement. Throughout the unit's lessons, I will draw out these basic skills to demonstrate how these concepts extend to more advanced ratio and proportion problems.

Content Background

What Proportional Reasoning Is Not

Consider the following problem: A recipe calls for 3 lemon wedges to make 12 ounces of lemonade. How many lemon wedges would you need to make 20 ounces of the same lemonade? To analyze deficits in my students' proportional reasoning, I will offer three example responses to the lemon problem that represent some of the most common trends in the way past students ultimately approached ratio and proportion problems by the end of the unit.

Example A	Example B	Example C +8
$\frac{3\ L}{12\ oz} \quad \frac{x}{20\ oz}$	$\frac{12\ oz}{3\ L} \quad \frac{x}{20\ oz}$	$\frac{3\ L}{12\ oz} \quad \frac{x}{20\ oz}$ +8
$3 \times 20 \div 12 = 5\ L$ *x= 5 lemon wedges*	$12 \times 20 \div 3 = 80\ L$ *x= 80 lemon wedges*	$3 + 8 = 11$ *lemon wedges*

Example A shows the use of the most widely utilized algorithm to find a missing value in a proportion often called cross multiply and divide, also known as "rule of three". To utilize this algorithm, students need to find one value when given three. The student here arrived at the correct answer, but we can't definitively tell if the student utilized proportional reasoning. The only thing we can be certain of is their ability to substitute numbers into an algorithm.

Example B shows the same algorithm being utilized. Here they set up the proportion incorrectly, leading them to an incorrect comparison. This response gives cause for worry, because there is no evidence of attending to and coordinating the different units to make an accurate comparison. A multiple choice test question would likely include this answer as a trap choice due to the common nature of this mistake. The unreasonableness of this answer is also evidence of the student's narrow and procedural understanding.

Example C shows no evidence of proportional reasoning. Here addition is used to find the missing value; the rationale being that the ounces went up by eight, so the lemons must go up by the same amount. Instead of proportional reasoning, the student utilized additive reasoning by only considering the difference in magnitude of the two quantities. Until 6th grade this type of thinking is of greater focus. (The CCSS Writing Team 2011) To employ proportional reasoning, students need to compare multiplicatively.

What Proportional Reasoning Is and Why it Matters

In my research I came across a wide range of definitions, attempting to explain the complexity of proportional reasoning. The term encompasses a wide range of concepts including ratio, multiplicative comparison, co-variation, proportion, rate, unit rate, and constant rate of proportionality. Having a solid grasp of these concepts is indeed required in proportional reasoning. However to include all of these elements in its definition, clouds the essence of what proportional reasoning is. To simplify and to cut to the core of this term, I will use the following definition:

Proportional reasoning is reasoning about proportional relationships.

Being able to understand the presence or absence of proportionality in the relationship between two quantities is of significant importance. It is a critical form of thinking that requires the firm grasp of elementary fraction, operation and measurement concepts and lays the groundwork for algebra and other higher levels of mathematics. (Lesh et al. 1988) For most of my students, proportional reasoning will be a new way of thinking within mathematics. Because of this, my unit aims to develop proportional reasoning by centering my instruction on understanding proportional relationships. The National Council of Teachers of Mathematics lays out ten different essential understandings of ratios and proportions for grades sixth through eight. I have developed the essential understandings for my unit from four of these. They are as follows:

Essential Understanding 1. Reasoning with ratios involves attending to and coordinating two or more quantities.

Essential Understanding 2. A ratio is a multiplicative comparison of two quantities, or it is a joining of two quantities in a composed unit.

Essential Understanding 6. A proportional relationship is an equality between two ratios. In a proportional relationship, the ratio of two quantities remains constant as the corresponding values of the quantities change. This is the constant rate of proportionality.

Essential Understanding 7. Proportional reasoning is complex and involves understanding that-

- If one quantity in a ratio is multiplied or divided by a particular factor, then the other quantity must be multiplied or divided by the same factor to maintain the proportional relationship

(Lobato et al. 2010)

When students can apply the above understandings flexibly to solve ratio and proportion problems, they will be exhibiting thorough proportional reasoning. To achieve this ambitious goal I will reflect on my past teaching of the subject and use research and my work with this seminar to develop better strategies and approaches.

Theory Behind Order, Problem Types, and Problem Contexts

“Implicit learning must reflect the desired explicit teaching.” This idea has been central in Roger Howe’s seminar “Problem Solving and the Common Core”. The concept stems from the premise that a large portion of student learning happens due to the order in which content is presented and the variety of problem types and contexts or lack thereof students are exposed to. To expand on this idea I will explain how order, problem type and context can impede the development of proportional reasoning and then outline how my unit will take a different approach.

In reflecting on my own practice, this concept of implicit teaching and learning is something that I often lose site of when designing lessons and analyzing curriculum. A general illustration of students implicitly learning from a lack of context is in the teaching of a triangle. A common mistake primary educators make is when they teach the basic elements of a triangle only using examples that are equilateral with a horizontal base. If this is the only manner in which students first see triangles, they are likely to implicitly learn that triangles are shapes that are limited to three equal side lengths with a horizontal base. Even though this was not explicitly taught, the majority of students will likely take on this narrow view of triangles due to the lack of context. Analogous to this example, if ratios and proportions are not taught in proper order and without a variety of problem types and contexts, narrow and insufficient proportional reasoning will likely develop. This unit will provide a purposeful sequence of concepts and a diversity of problem types that promote the essential understandings of ratios and proportions.

Order

The typical progression of concepts learned in a similar unit is as follows:

1. Students learn different ways to write ratios.
2. Students learn that proportions are two equivalent ratios and can be found by multiplying both quantities by the same value.
3. Students learn how to find unit rates by dividing both quantities by the second quantity.
4. Students then are taught the cross-multiplication algorithm to solve proportion problems with missing values. (Lobato et al. 2010)

This order roughly outlines the progression that my district-mandated curriculum uses. There are several flaws with this order. Students are first introduced to the concept of ratios by identifying basic relationships such as red marbles to blue marbles or boys to girls with the purpose of teaching what is and how to set up a ratio. See two examples of such problems:

Example D: There are 6 blue marbles and 4 red marbles. Write a ratio comparing red to blue marbles and blue to red marbles.

Example E: There are 15 boys and 8 girls in class. What is the ratio of boys to the total number of students in class?

In the above problems aim to address the concept of ratio in isolation, however there is no purposeful context. Meaning, there is no proportional relationship for which these ratios might be the constant rate of proportionality. These problems simply prompt students to order quantities in the form of a ratio without reasoning about the relationship between the quantities. This approach could potentially lead to misconceptions about the fundamental idea of ratio and makes the concept seem arbitrary and abstract.

Another and even more harmful impact of this typical sequence is how early the cross-multiply and divide method is presented. When students are given this algorithm too early in the development of concepts around proportional reasoning, it turns off the thinking switch when solving problems. I have experienced this in my own classroom when presenting this strategy to solve proportion problems. Shortly after the procedure is given, it is the only method used. Kids find the algorithm to be quick, comfortable, and fool-proof. Very little thinking happens thereafter. “Proportional reasoning involves much more than setting two ratios equal and solving for a missing term." (Lesh et al. 1988) This procedure is efficient and one that students should learn to utilize eventually. However, it should be avoided until students develop a conceptual understanding of proportions.

The sequence of my unit will have two principal objectives: (1) unlike traditional progressions, the sequence will keep the idea of a proportional relationship at the fore by developing the concepts of ratio in *conjunction* with the concepts of proportion, not in isolation, and (2) the sequence will introduce skills and strategies that make the concepts of ratios and proportionality more accessible before teaching any procedural algorithms.

Context of Problems

When first developing proportional reasoning, students need to work with problems that help students visualize and grasp the desired outcomes. If students are given problems with scenarios in which they cannot easily model for themselves, they are more likely to look for procedural solutions and abandon the task of reasoning. They should be exposed to simple, familiar contexts and extend to more complex and/or unfamiliar ones as proportional reasoning develops. (Cramer and Post 1988)

Lessons in my unit will first consist of problems that are accessible and of high interest to my students. The contexts will be tangible so that students can use their own experiences to deepen their understanding of concepts. Examples of these contexts are "chocolateness" of chocolate chip cookies, strength of fruit juice, cost per item, length of time it takes to run around a track, and crowdedness of a room. These scenarios are easy to visualize and will provide an entry point for all students in whole-group discussion. In addition, they demand proportional reasoning and give purpose to comparing the relationship of different quantities.

Problem Types

The variety and order in which different problem types are presented to students plays a significant role in the development of proportional reasoning. Students need to be exposed to a range of problem types so that they develop reasoning that can be generalized to all situations involving ratios. As is true with all areas of math, if a student's problem solving is limited to a subset of problem types and if the key issues are never presented or discussed, they will likely develop a narrow and less flexible understanding of the area's concepts. The curriculum that I have used in the past has taught specific proportional reasoning strategies in isolation and dealt only with specific problem types. For example, problems that require students to find a missing value are traditionally used to teach the cross-multiply and divide algorithm. If students do not see a substantial variety of problems, they tend to stop analyzing and reasoning through problems. Seeing the same problem type repeatedly elicits the same strategy repeatedly, and this leads to a fixed and inflexible use of reasoning when presented with a problem type that could require a different strategy or approach. To avoid this, my unit will consist of mixed problem sets that contain a combination of many types of problems so that students have space to apply their different forms of reasoning under varying conditions.

There are three different problem types that require the use of proportional reasoning. They are: (1) missing value, (2) numerical comparison, and (3) qualitative prediction and comparison. (Post, 1988) The following sections provide a description and example of each type.

Missing Value

In the various curriculums I analyzed, missing value problems constituted a large majority. A problem of this type gives one complete ratio with two given quantities and another incomplete but proportional ratio with only one given value. Consider the following problem:

> 36 chocolate chips are used to make 6 chocolate chip cookies. If 12 cookies are made with same number of chips per cookie, how many chips are needed?

Here the complete ratio of 36 chips to 6 cookies is given with the task of finding the missing number of chips for 12 cookies in a proportional relationship. The degree of difficulty changes with different numerical contexts. The problem is generally easier if the relationship between the ratios or within the ratio is integral, as in the previously mentioned cookie problem. When the relationship between or within is fractional, the problems of course become more difficult. Because this unit aims to introduce and develop proportional reasoning for the first time, whole number relationships will be used to start with. Once students are comfortable with this idea of proportion, students will be exposed to proportional relationships with larger whole numbers, unit fractions, and general fractions with gradually increasing numerators and denominators. Using this strategy will allow different abilities to be challenged at an appropriate level throughout the unit.

Numerical Comparison

Numerical comparison problems require the evaluation of two given ratios. Consider the following problem:

Mixing flavored powder with water makes Kool Aid juice. Grandma made Kool Aid by mixing 3 cups of water with 9 tablespoons of powder. Grandpa made Kool Aid by mixing 6 cups of water with 12 tablespoons of powder. Which Kool Aid will taste stronger?

In order to reasonably answer the question, students must compare the two given ratios, 3 cups to 9 tablespoons and 6 cups to 12 tablespoons. This problem type requires students to apply proportional reasoning and determine a greater or lesser value in relation to the question. Although these problems do not explicitly ask the solver to express a proportional relationship, one must reason with the implicit ones to answer the question. Here, a proportional relationship was implied in each of the recipes. Grandma would use 3 tablespoons of powder for every cup of water and Grandpa soul use 2 tablespoons to every cups of water.

Qualitative Prediction

My analysis revealed that qualitative prediction is the most unrepresented problem type in middle school curriculum. It consists of a scenario that requires a comparison that is not dependent on numerical data. Consider the following problem:

Two friends each hammered a line of equally spaced nails into different boards using the full length of each board. Bill hammered more nails than Greg. Bill’s board was shorter than Greg’s. On which board are the nails hammered closer? (Cramer and Post 1988)

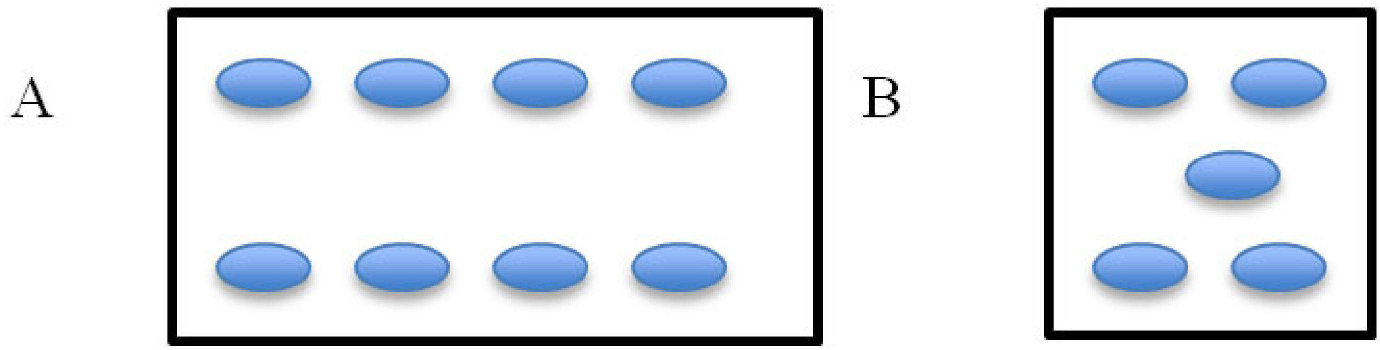
The problem requires the comparison of the board length and number of nails. The absence of numbers forces the solver to focus on the relationship of two different ratios in order to answer the question. The implicit proportional relationship that must be considered is the length of a row of evenly spaced nails to the number of nails. This type of problem is valuable because it demands proportional reasoning and provides no opportunity to apply a procedure or algorithm. Although I have found that this type of problem is not typically assessed, it will be a valuable exercise in developing a complete understanding of proportion.

Unit Learning Outcomes

Attending to and Coordinating Two Quantities- Essential Understanding #1

Students will first develop the concept of ratio by solving a set of comparing and missing value problems. Ultimately students need to see ratios as describing the relationship between two quantities. "Before children are able to reason with ratios, they typically reason with a single quantity." (Lobato et al. 2010) Students must first reach this understanding of attending to and coordinating *two* quantities before they can reason with ratios. Consider the following problem that will be used to develop this concept.

Let the two rectangles represent different sized cow pens and the ovals represent equally sized cows. Pen A has an area of 8 square yards. Pen B has as area of 4 square yards. Which cow pen is more crowded?



The point that will be drawn from this example is that the relationship between two different quantities, in this case number of cows and area, which can be expressed, for example, in number of square meters, needs to be reasoned with in order to answer the question. A first intuition might be to count the number of cows in order to determine crowdedness. In this line of thinking, a student is only attending to one quantity. This logic can be easily proven wrong by drawing a grid over the rectangles showing that pen A has 1 square yard for every cow whereas pen B only has 4 square yards for 5 cows making it more crowded. Another way to approach this problem is by doubling B making the area the same as pen A but the number of cows greater by 2. This too visually shows that pen B is more crowded than A. After students have grasped the concept of attending to different quantities and their relationship, they will be explicitly taught the different ways of notating ratios.

Although the focus of the cow pen problem is on prompting students to begin using ratios to coordinate and compare two different quantities, it also implicitly calls students to consider proportional relationships. For example students might arrive at the conclusion that pen A has 4 cows per 4 square meters, thus arriving at a proportional ratio. This link should be explicitly shown to students in discussing ratio problems. Even though the unit starts with a focus on ratio, ideas of a proportional relationship will be addressed in conjunction. When a solid foundation of ratio and its utility is established, students will move on to explicitly grapple with proportions.

Proportion - Essential Understandings #2 and #7

Two ratios that are in proportion are also called equivalent. This concept will be initially developed by applying it to familiar contexts such as measurement conversions and price per item. Consider the following problem:

> John had 3 yardsticks and he noticed that there were 9 feet in 3 yards. He used the yardsticks to measure the length of the classroom. The classroom measured 12 yards long. What is the length of the classroom in feet?

This problem provides an excellent entry point to understand proportions in two different ways. One approach to this problem is to look across the ratios and see that 12 yards is 4 times as many as 3. So to complete this proportion one can multiply 9 feet by 4 to get 36 feet. In addition, students should ultimately see the relationship within both ratios realizing that 9 is 3 times 3 so 12 can be multiplied by 3 to reach the same answer of 36. This, I will later explain, is the constant rate of proportionality (namely 3 feet per yard), a more advanced concept.

(9 ft × 4)/(3 yd × 4) = (36 ft)/(12 yd)

(9 ft)/(3 yd × 3) = (36 ft)/(12 yd ×3)

The first example demonstrates the understanding that two corresponding quantities in one ratio will relate to any two other corresponding quantities by the same factor. In this case the quantity 9 feet relates to the quantity 36 feet by a factor of 4, as does 3 yards to 12 yards. This is the fundamental property of proportional relationships, and should be focused on first. The second example demonstrates that there is a constant relationship *within* a ratio and that is the same for all other proportional ratios. In this case the constant is 3. So for any number of yards, the number of feet will be 3 times as many. In this example it is obvious, due to its being a simple whole number, but when the constant rate of proportionality is larger or fractional, this concept becomes more difficult for students to grasp. Both examples are important uses of multiplicative comparison and both will be developed and taught explicitly via these types of problems. With this knowledge students can evaluate ratios for proportionality and generate equivalent ratios. In order to develop flexible proportional reasoning students need these understanding before learning any algorithm.

Taught alongside multiplicative reasoning should be the idea of a unit rate. A unit rate is a version of a ratio that expresses the relationship between a quantity and 1 unit of the other quantity. Unit rates are only defined when the units of each quantity are specified. Often there is an obvious or convenient unit to choose for the denominator, but the fact that a choice has been made is important and should be addressed. Once the units of each quantity have been chosen, the unit rate is a well-defined number. Common examples of unit rates seen in an everyday context are miles per 1 hour or cost per 1 item. Consider the following problem:

If 12 sodas cost $18, how much would 20 sodas cost?

Using the unit rate approach, students can find the cost of 1 soda by dividing 18 by 12 to find the unit rate of $1.50 per 1 soda and then multiply this value by 20 to find the cost of 20 sodas. Ratios within a proportional relationship will always have the same unit rate. This is because the unit rate is the constant rate of proportionality, expressed in the given units. Similar to the multiplicative approach, finding the unit rate can help evaluate proportional relationships and find equivalent ratios. Research shows that students used the unit rate strategy more frequently due to its more procedural and intuitive nature. To make sure that students do not narrowly rely on this strategy it is important to show its relation to multiplicative comparison. (Cruz 2013)

Constant Rate of Proportionality- Essential Understanding 6

When comparing multiplicatively within a ratio and when finding the unit rate, the constant rate of proportionality is defined. However this concept will be taught last; it is not enough to simply understand that the unit rate is the constant rate of proportionality. It needs to be understood as a relationship. Consider the following statement letting the variable k represent the constant: if quantities a and b are in proportional relationship, then for any pair of corresponding values,

In any ratio $a/b = k$ so $a = kb$

The constant explicitly shows the relationship between the two quantities in the ratio, *a:b.* This relationship is that for every *b* there are *k* times as many *a*. To reach this understanding it is useful to ask students questions such as: How many times bigger/smaller is *a* compared to *b*? For every increase to *b* what is happening to *a*? Comprehending this idea is an important landmark in proportional reasoning. Ultimately students will be tasked with creating an equation such as the one above to express a proportional relationship. Because of its importance in their future work in grades seven and eight with linear equations and functions, a conceptual understanding of the meaning of a constant rate of proportionality is critical.

A Note on Algorithms

This is an introduction to ratios and proportional reasoning. Students will be developing this sense over the course of the next two years. Teaching of standard algorithms could impede the future development of proportional reasoning. Although my standard curriculum introduces the cross multiply and divide algorithm half way through the unit, I will hold off on teaching this strategy until I am confident in my students' understanding of what a proportional relationship is.

Teaching Strategies

In the following sections I explain three important strategies that I will use to develop proportional reasoning. These approaches will be taught in the order they are explained in this section, but will not be used in isolation. I will explicitly prompt my students to use all strategies throughout the unit so that they are solving problems with a more robust toolkit.

Ratio Tables

Ratio Tables will be explicitly taught and referred to as a strategy in understanding and solving problems. This form of modeling helps represent relationships between two variables by placing the given data of a scenario into an organized table. See the following example.

Mr. Bingea ran 6 laps around the track in 12 minutes. Ms. Barret ran 12 laps around the track in 18 minutes. Compare their speeds.

Mr. Bingea

Laps	**6**	3	1	2	20	10
Minutes	**12**	6	2	4	40	20

Ms. Barret

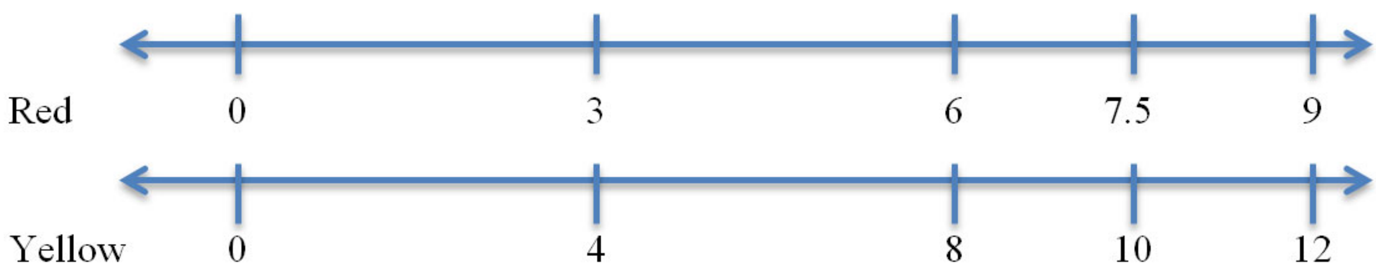
Laps	**12**	6	2	20	10
Minutes	**18**	9	3	30	15

The bolded numbers represent the given data from the problem. With this ratio table, proportional reasoning is seen if you follow the numbers from left to right. The quantities are multiplied by obvious factors, finding similar values for both relationships, making it easier to compare. Giving students this form allows them to see the proportional relationships in multiple iterations reinforcing the notion of a constant rate of proportionality. In addition to using this structure to solve and understand problems, students will also be presented tasks where they are given tables with complete or missing data sets and prompted to draw conclusions.

Double Number Line

A double number line is an important tool that provides a visual representation of a ratio and further demonstrates the constant rate of proportionality. See the following example:

Zack's orange paint is made by mixing 3 cups of red for every 4 cups yellow paint. How many cups of red would be used if 10 cups of yellow were used?

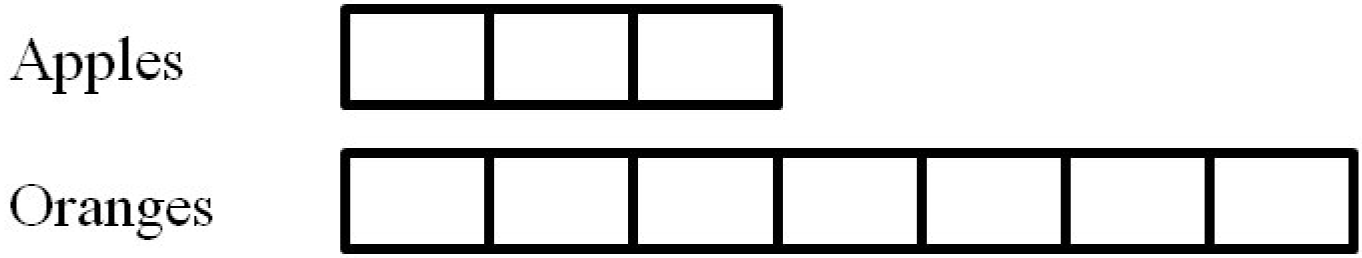


This model vividly teaches the attention to and coordination of different quantities. Here the ratio of 3 to 4 is meaningfully represented in the relation between the length scales on the two lines. The answer to the problem is achieved by reasoning that 10 is half way between 8 and 12, therefore the solution lies half way between 6 and 9 which is 7.5. This example is advanced and should not be used to introduce the double number line. Instead examples where there is an integral unit rate should be used to first introduce ratios this way. It is also prerequisite that students have a firm grasp of representing value on a number line. Students must be comfortable with length measurement, and the need to specify a unit length on a given number line. If not, this skill should be reviewed before introducing the double number line.

Tape Model

Another way of visually reasoning with ratios and proportions is by means of a tape model. Consider the following example:

In the school cafeteria, 3 apples are eaten for every 7 oranges eaten. If 80 pieces of fruit have been eaten, how many of each fruit can we predict have been eaten?



In the tape model, 3 boxes represent 3 apples and 7 boxes represent 7 oranges. The given ratio is clearly seen with a total of 10 boxes for the 10 pieces of fruit. Because the ratio will not change in a proportional relationship we can ask ourselves: What does each box represent if there are 80 pieces of fruit? Because there are 10 boxes of equal value, it is apparent that each box represents 8 pieces of fruit. So we can predict that there would have been 24 apples and 56 oranges eaten. The visual nature and focus on the constant rate of proportion makes this model a valuable tool in this unit. However the order of introducing this model must be considered. This strategy will be introduced after the previous strategies and when students have a sufficient understanding of the multiplicative relationship of ratios in a proportional relationship. They are excellent models to represent and solve missing value problems but have the potential to be used by students without a conceptual understanding. If this is the case, students tend to use this strategy as an algorithm without understanding the reasoning behind the model.

Activities

Structured Problem Solving

The general form of lessons I will use in this unit will be centered on structured problem solving. This is different when compared to a typical gradual release lesson format where students are taught a strategy explicitly, practice the strategy with heavy teacher guidance, and then eventually apply it through independent practice. In a problem-based format, students are presented a problem to be first worked on independently. Here students will apply their own mathematical knowledge in an effort to develop strategies to reach a solution. Once students have had time to work independently, their different strategies and solutions are shared and discussed in groups or as a class. During this time students will be able to see multiple approaches, discuss misconceptions, and come to new conclusions about the material. The discussion must be carefully facilitated in order to reach the desired outcomes for the lesson. If all student ideas have been exhausted and the strategies or understandings the lesson set out to achieve have still not been reached, explicit teaching will then take place. Finally the students are given a set of additional problems to apply and practice what they learned from the problem solving and the discussion. The key idea behind a structured problem solving approach is that students are first given the opportunity to reason and construct their own concepts and strategies, which leads to a deeper understanding of the content covered in a given lesson. In the pursuit of nurturing proportional reasoning, a new way of thinking for 6th graders, this approach will be fundamental.

Structured problem solving is laid out in Akihiko Takahashi’s paper titled, “Characteristics of Japanese Mathematics Lessons”. Takahashi stresses that in addition to the attention devoted to extensive discussion, the selection of problems and activities needs to be carefully considered as well. (Takahashi 2006) For the purpose of developing proportional reasoning, the progression of problems in this unit is designed to bring out concepts and strategies that cohesively build on each other. In general, each lesson will present a new problem scenario for students to solve. This scenario will be engaging and include the use of visuals, multi-media, and props to ensure all students access and become invested in the problem. After the structured problem-solving process has taken place, students will be given a similar scenario if not the same but with different numbers. These exercises will give students the opportunity to process, apply, and generalize the previously discussed strategies and understandings.

The problems that I will use to center my lessons around require a greater explanation and use of images. For access to these, visit the website www.mrbingea.blogspot.com or email me at aaron.bingea@gmail.com.

Number Talks

A number talk is a five to ten minute routine that I will use to start most class periods. The purpose of this activity is to review prerequisite or fundamental concepts to the day's lesson in an efficient and meaningful way. As stated earlier, proportional reasoning requires the mastery of many skills learned prior to sixth grade, and it is likely that my students will have large gaps in these areas that could hinder their learning about proportional relationships. I will use number talks to gradually remediate and develop these more basic understandings throughout the entire year. The routine follows a simple format that is centered on the purpose of bringing all students into the thinking and discussion around the lesson's concepts.

A number talk begins by all students putting away materials to ready themselves for the prompt that will only require the use of mental math. The problem is presented on the board for all students. They then solve the problem mentally and put their thumbs up when they have reached a solution. Next, volunteers share their strategies as the teacher models each student's line of thinking on the board. Finally the class discusses the accuracy and reasoning of each strategy. By the end of this discussion students can see multiple, valid approaches to solving a relatively simple problem. Mastery of any one strategy is never the goal for a single number talk; instead it is a daily practice for students to gradually acquire a wide range of skills with basic operations and number sense. For a more thorough description of number talks refer to the book written by Cathy Humphreys and Ruth Parker, "Making Number Talks Matter". It details the procedure as well as how the teacher should facilitate student thinking and discussion. The book also gives example prompts and different strategies to help students develop. (Humphreys and Parker 2015)

The concept of focus in each number talk will be directly related to the lesson's learning objectives. For example, when teaching the concept of equivalent ratios in a proportional relationship, the number talk will cover the concept of equivalent fractions. This sets up the opportunity to make connections to these foundational concepts in later class discussions. The following table shows several problems that will be used in number talks to prime lessons in this unit and the linked foundational concepts.

Foundational Concept	Problems for Number Talks
Concept of multiplication and division	14×25=60 ÷ 15
Multiplicative Comparison	128 is how many times 32? 5 is ½ of what number? 5 is ¼ of what number?
Comparing Fractions	Which is larger 5/8 or 21/32? Which is equivalent to 3/7 ? 21/49 or 12/21

Each of these problems requires a skill or concept covered in grades three through five. Students will most likely know or be partially familiar one way to solve each problem. Through this routine students will be tasked with generating multiple strategies to solving these problems, making students more flexible with numbers and operations and ultimately being able to better access concepts within proportional reasoning.

Appendix 1- Standards

This unit has been written in accordance to the Common Core State Standards, which have been adopted by Chicago Public Schools. Specifically this unit will cover a set of standards within the strand of ratios and proportional relationships.

6.RP.A.1 Understand the concept of a ratio and use ratio language to describe a ratio relationship between two quantities.

6.RP.A.2 Understand the concept of a unit rate a/b associated with a ratio a:b with b ≠ 0, and use rate language in the context of a ratio relationship.

6.RP.A.3 Use ratio and rate reasoning to solve real-world and mathematical problems, e.g., by reasoning about tables of equivalent ratios, tape diagrams, double number line diagrams or equations.

6.RP.A.3.a Make tables of equivalent ratios relating quantities with whole-number measurements, find missing values in the tables, and plot the pairs of values on the coordinate plane. Use tables to compare ratios.

6.RP.A.3.b Solve unit rate problems including those involving unit pricing and constant speed.

6.RP.A.3.d Use ratio reasoning to convert measurement units; manipulate and transform units appropriately when multiplying or dividing quantities

Appendix 2- Sample Problems

Below is a set of sample 6th grade problems for ratios and proportions. These are problems that will be used for practice and assessment throughout the unit. They are not problems that will be used for the basis of structured problem solving-based lessons.

The ratio of the number of boys to the number of girls at a school is 4:5. If there are 120 boys, how many students are there all together?

Sam’s two new aquariums each hold exactly 200 gallons of water. One aquarium will hold small fish and the other will hold large fish. He will buy 5 small fish for every 10 gallons of water in the aquarium. He will buy 8 large fish for every 40 gallons of water in the aquarium. What is the total number of fish Sam will have? What will be the ratio of Sam’s small fish to large fish?

A total of 300 trees will be planted in a park. There will be 2 pine trees planted for every 3 maple trees planted. How many of each type of tree will be planted in the park?

A food company that produces peanut butter decides to try out a new version of its peanut butter that is extra crunchy, using twice the number of peanut chunks as normal. The company hosts a sampling of its new product at grocery stores and finds that 5 out of every 9 customers prefer the new extra crunchy version. If the company is planning to produce 90,000 containers of crunchy peanut butter, how many of these containers should be the new extra crunchy variety, and how many of these containers should be the regular crunchy peanut butter?

The producer of the news station posted an article about the high school's football championship ceremony on a new website. The website had 500 views after four hours. Create a table to show how many views the website would have had after the first, second, and third hours after posting, if the website receives views at the same rate. How many views would the website receive after 5 hours? (It should be noted here that "same rate" means: in any two equal time periods, there are the same number of views.)

Hank bought 5 meters of ribbon for $10. How much does the ribbon cost per centimeter?

A runner ran 20 miles in 150 minutes. If she runs at that speed, how long would it take her to run 6 miles? How far could she run in 15 minutes? How fast is she running in miles per hour? What is her pace in minutes per mile?

Sally drives 66 miles in 3 hours and Molly drives 72 miles in 4 hours. What is the difference between their average speeds, in miles per hour?

A stand is selling 8 mangos for $10 at a farmers' market. The grocery store is selling 12 mangos for $15. Which has a cheaper deal on mangos, the farmers' market or the grocery store?

A line measures 30 centimeters long. A second line measures 500 millimeters long. Which line is longer?

Who walks at a faster rate: someone who walks 60 feet in 10 seconds or someone who walks 42 feet in 6 seconds?

During the winter of 2012–2013, Buffalo, New York received 22 inches of snow in 12 hours. Oswego, New York received 31 inches of snow over a 15-hour period. Which city had a heavier snowfall rate?

One math student, John, can solve 6 math problems in 20 minutes while another student, Juaquine, can solve the same 6 math problems at a rate of 1 problem per 4 minutes. Who works faster?

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