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Equations in the Common Core: Algebraic Reasoning and Problem Solving

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Background

Brisbane School District

The Brisbane School District comprises two elementary schools and one middle school totaling 550 students. It is located just south of San Francisco and pulls students from Brisbane and portions of South San Francisco and Daly City. There is a strong district-wide emphasis on the practice of Highly Effective Teaching (HET) that stresses the development of the whole student through rich, cross-curricular themed projects and the practice of Lifeskills.

Lipman Middle School, where I teach 8^{th} grade, is the only middle school in the district, bringing together students from both of the elementary schools. We have a total enrollment of 166 students, about a third of whom would qualify as socio-economically disadvantaged. Lipman recently received a state ranking of 7 out of 10 (10 being the highest score) compared to similar schools statewide. Classes at Lipman are taught in a core structure, with each grade having one Language Arts / Social Studies core class and one Math / Science core class. As a result there is a very small faculty at Lipman, allowing for little collaboration at grade level.

District-wide, the Brisbane Schools have been involved with substantial math professional development leading up to and following the adoption of the CCSSM. In conjunction with a number of neighboring districts and the Silicon Valley Math Initiative, teachers have attended numerous trainings and follow up meetings regarding the creation of common core transition units and best practices for improving student outcomes within the new standards. These practices include the use of MARS tasks, Formative Assessment Lessons (FALS), Problems of the Month (POMs) and number talks. I have participated in all of these trainings over the past two years and have worked to implement these practices in my teaching.

Context

I am a passionate instructor and I want my students to “get” math. As a student, I did well in math because I could memorize algorithms and apply them where necessary. This got me as far as my required math course in college and I had no interest in pursuing math any further. It was when I started teaching math, however, and began to understand the connections and the math behind the algorithms I had previously memorized that I discovered I truly enjoyed math and wanted to better understand it. As a result, it is forefront in my mind when planning curriculum that the goal is not strictly related to math fluency and the use of algorithms but rather an in depth conceptual understanding and application of mathematical principals, patterns and properties.

Equations and Expressions is one of the five domains making up the common core math standards in grades 6-8. In sixth grade students begin to reason with numerical expressions. Progressing into seventh grade, students are expected to continue to develop their understanding of generating equivalent expressions and to apply their knowledge of writing and solving equations to solve real life problems. In eighth grade, students begin to connect linear equations with their graphical representation on the coordinate plane. Additionally, students are expected to give examples of linear equations with one solution, no solution or an infinite number of solutions. Finally, students in eighth grade are expected to apply their knowledge of linear equations to solving pairs of simultaneous linear equations. In order for students to be successful with the eighth grade content, it is key that they have a solid foundation in algebraic reasoning and extensive experience generating and manipulating equations. Without this base, students will have difficulty progressing towards more complex algebraic reasoning.[1]

This unit’s emphasis, therefore, is on the continued development of a solid foundation in algebraic reasoning. In general, this strong understanding is key to a student’s continued success in the academic and real world. Mastery of Algebra is often viewed as a gatekeeper and key predictor of success in higher-level mathematics in high school and beyond. As many studies have shown algebraic reasoning is essential to further mathematical study, “because it pushes students’ understanding of mathematics beyond the result of specific calculations and the procedural applications of formulas”.[2] The development of these skills creates more flexible thinking, preparing the learner for more abstract mathematical concepts.

Mathematical Background

In thinking about how to help my students understand the manipulation and solving of equations, I consider two major categories. First, the essential prerequisite knowledge for basic algebraic thinking, and second, the essential strategies or skills (sometimes referred to as “tricks”) needed to successfully manipulate and solve equations. However, before going into either of these, it is worthwhile to explore some common mistakes I’ve witnessed in student work indicating the need for further emphasis in the development of algebraic reasoning.

Misunderstandings in Algebraic Reasoning

I have encountered a series of common mistakes when students possess a limited sense of algebraic reasoning. The following represent a collection of common errors I have seen in my classroom.

Misunderstanding the Use of the Inverse

Example of Student Thinking	Explanation
2x=8 2x-2=8-2 x=6	This misconception shows that the student does not understand the purpose of using an inverse operation. In this instance the goal would be to transform 2x to x (or equivalently 1x, the multiplicative identity of two time x, by multiplying by ½, the multiplicative inverse of 2). Instead the student subtracted the two, clearly misunderstanding the purpose and possible use of inverse operations. Additionally, this may indicate that the student does not understand the notation of the term and that the 2 and the *x* written side by side indicates multiplication.
x-4=2 x-4-4=2-4 x=-2	Again, this example demonstrates that the student lacks an understanding of the inverse operation. In an attempt to change **-4** into its additive identity**, 0**, the student subtracts four from each side. This will not result in zero, however, as **-4 – 4 = -8**. Of course, this may also represent an error in integer operations.

Misunderstanding the Distributive Rule

Example of Student Thinking	Explanation
2(x+2)=8 2x+2=8	From this example, it is clear that the student does not understand that the parentheses indicate that the 2 multiplies the sum of x and 2, rather than just the x.

Misunderstanding of the Properties of Equality

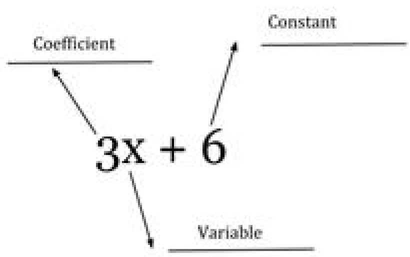
Example of Student Thinking	Explanation
2x+1=10 2x/2+1=10/2 x+1=5	This student attempted to isolate the variable by dividing both sides of the equation by the same value. The student's mistake, however, is that they have only divided the term 2x by 2 as opposed to the entire expression 2x + 1. They have therefore not imposed the same operation on both sides and the equation is no longer true. This additionally demonstrates a broader lack of understanding as to putting together the properties of equality to solve an equation as well as an additional example of misunderstanding the distributive rule.
4x+2+x=10 (4x-4x)+2(x-4x)=10 2+(-3x)=10	This example also shows a misunderstanding of the properties of equality. This student, eager to eliminate the term 4x, subtracts it twice, once from 4x and once from the term x. This fails to account for the fact that both 4x and x are on the left side of the equation and they have therefore subtracted 4x from one side of the equation twice, destroying the equality. This also shows a lack of understanding of the utilization of the commutative and distributive rules and their role in combining like terms.

Essential Knowledge

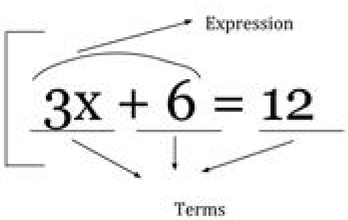
When considering equations, it is first essential that students understand the difference between a variable, a constant, an expression and an equation. Students will also need to know when a number or set of numbers represents a solution to an equation and when it does not.

Parts of an Equation

Students will need to know what makes up an equation. An equation is a statement that two expressions are equal. When one or both of the expressions have a variable the equation is essentially asking for what value(s) of the variable(s) is the equation a true statement? A value that makes the equation true, that is, does give the same value to both expressions, is called a *solution* to the equation. Take for example 3(x+2)=12, this is by definition an equation and x=2 is a solution to this equation.



Within the expression, you may have a variable (a letter used to represent a value), a coefficient (a number multiplied by a variable), and a constant (a number on its own with no variable).



Terms can be either variables, variables with coefficients or constants. Terms are combined by mathematical operations, such as addition or division, to make expressions.

Equivalent Expressions

Different expressions can represent the same quantity. Expressions are considered equivalent when the same number(s) is substituted for their variable(s), both expression yield the same result.

Example: 2(x+4) and 2x+8 are equivalent expressions

We can check if these are equivalent expressions by substituting the same value for x in both equations and determining if the result is the same. Let's try 3.

2 (3+4) = 2(7)=14

2 (3+8) = 6+8=14[2]

The result was the same for both when x = 3. However, in order to be equivalent expressions, each expression must yield the same result for *every* value of x. Therefore, it is not enough to simply substitute in values, we must utilize the Rules of Arithmetic to justify the equivalence. If you look at the expressions again, you can see that the second expression can be obtained from the first using the distributive property, and vice versa. Therefore, the two expressions are equivalent.

2 (x+4) = 2 (x)+ 2 (4) =2x+8

Here is another example,

2x+6+20x-10 = 2x+6-10+20x

Using the Commutative Rule of Addition, the terms 20x and -10 have changed positions, but the expressions are still equal.

Solution or Not

A key practice in algebra is the process of checking your answer to determine if it is in fact a solution or not. It is a practice often overlooked by students and instructors alike as it is tedious and may seem redundant. However, it is very important that students understand the process so that they are always able to check their work. This aligns well with Mathematical Practice 6 in the Common Core Standards “attend to precision”.[3]

The process of checking an answer to an equation is simple. If the answer you determined is correct, you can plug that number into the equation in the place of the variable and the equation should be true.

2x+10=20

2x=10

x=5

I can check this solution, by plugging 5 in to the original equation in the place of x. If both sides of the equation remain true, my solution is correct.

2(5)+10=20

10+10=20

20=20

Since my equation is still true because twenty does equal twenty, I know I have found a solution.

It will also be important that students are able to determine what is a reasonable answer. I will use the above equation to demonstrate. In the equation we are taking a number *x* and multiplying it by a factor of two and adding ten, the result of this addition is twenty. Students should have a basic understanding then that the variable, *x,* is not going to be smaller than one. If it were a number less than one then the product of two and the variable would not be nearly large enough to combine with ten to get twenty. Conversely, students should be able to determine that an answer like 100 is also unreasonable as it is much too large. Students should be able to use clues from the equation and the context to mentally check that their answer is reasonable.

Properties of Equality

While this should be review for my students, it is worth the instructional time to make sure that students' understanding of the concept is strong because it is the foundation of algebraic reasoning. For example consider the following equation:

$x-2=10$

This equation is unchanged (read: the two sides of the equation are still equal) if the same number is added to both sides of the equation. Since subtraction is just addition of the additive inverse, or negative, the same rules apply for subtraction. We'll call this "Equals added to equals make equals".[4] Thus, if we add 10 to both sides, we may conclude that

$x-2+10=10+10$

$x+8=20$

This same property of equality exists for multiplication and division. Thus, again start with the equation:

$x-2=10$

The fact that these two expressions are equal will not be changed if *both* sides of the equation are multiplied by the same number. Again, this is true for division as well since division is multiplication by the multiplicative inverse, or reciprocal. We'll call this principle "Equals multiplied by equals make equals".[5]

$3(x+2)=3(10)$

These properties are essential knowledge for my students. A firm understanding of these principles will allow them to manipulate any equation to either solve or make it easier to solve. It will also eliminate many common mistakes students make when solving equations.

Inverse Operations

In addition to the properties of equality, a strong understanding of inverse operations and the equivalence of subtraction with adding the additive inverse, and division with multiplying the multiplicative inverse, will be key to ensuring continued success with algebra. In order to understand inverse operations, students must first grasp the concept of the identity element for addition, and the identity element for multiplication. When using the inverse operation the purpose is to transform the term into its identity value so as to remove its impact on the equation. The key Rules of Arithmetic that relate to this situation are the Identity Rules:

$a+0=a$

$a\text{x}1=a$

These equations demonstrate the basic facts that the sum of any number *a* and zero is *a,* the variable is unchanged. To refer to this, we call 0 the *additive* identity. Similarly, the product of *a* and 1 is *a*, the variable is also unchanged. Thus, we call 1 the multiplicative identity. This is important, because if we can transform terms into either their additive or multiplicative identities then we can essentially remove or eliminate them from the equation because they will have no impact on the result. To do this with addition, we use the *additive* *inverse* or negative of an element. To do it for multiplication, we use *multiplicative inverse*, or reciprocal.

$x+(-x)=0$

$7x*1/7x=1$

The following are the Rules of Arithmetic governing inverses. The additive inverse of a number, when added to that number, will result in 0, or the additive identity

$a + -a = 0$.

Likewise, the product of a number and its multiplicative inverse will be 1.

$a \times 1/a=1$.

Employing the additive or multiplicative inverse is often referred to as "doing the opposite".

Order of Operations

This is essential knowledge to prevent common algebraic mistakes that will change the value of the result. PEMDAS standing for Parentheses, Exponents, Multiplication, Division, Addition, Subtraction is a frequently used acronym to introduce students to order of operations. This, however, can mislead students into believing multiplication supersedes division when in reality they should be considered at the same time from left to right. The same is true of addition and subtraction. A more appropriate acronym is GEMS (Grouping Symbols, Exponents, Multiplication and Division, Subtraction and Addition) because it emphasizes that multiplication and division should be considered at the same time, as should subtraction and addition.

Infinitely Many Solutions, No Solutions

Students will also need to understand when they have found one solution to an equation, no solutions or infinitely many solutions (8.EE.C.7.A). Situations in which the equation has one solution should be relatively obvious to students as this the result that they are typically used to seeing (all of the aforementioned examples have one solution). Students will naturally be less familiar with equations that will result in no solutions or an infinite number of solutions. I will utilize the Hands on Equation model (described in full below) to demonstrate instances of infinitely many or no solutions to give them a physical representation of each scenario.

Essential Strategies

The following represent the strategies that will be essential to my students' success with equations. Each strategy will be discussed as a class, included in students' interactive notebooks and also coupled with a visual representation using manipulatives from the Hands on Equations collection. Additionally, there will be an extensive use of word problems to not only give students an understanding of how to translate words into algebraic expressions and equations but to also give them an opportunity to continue practicing these essential skills.

Distributive Property

Students will need to be able to use the distributive property to simplify more complex equations.

3(x+2)=30

The expression x+2 can not be simplified because x and 2 are not like terms. Therefore to simplify the above equation, another method must be employed, the distributive property. Many students will have this method memorized (“the double rainbow, right?” students ask), but I am more interested in them truly understanding *why* it works. This can be demonstrated with a relatively simple example proof.

5(3+2)

Since this expression does not have a variable it can be easily simplified and evaluated without the Distributive Rule. We will use that knowledge to check our work with the proof. Students know that multiplication is simply repeated addition so there are then five groups of the sum of three and two.

5(3+2)= (3+2)+ (3+2)+ (3+2)+ (3+2)+ (3+2)

=25

Using the associative property and commutative property of addition many, many times, the numbers can be arranged with the twos preceding the threes.

5(3+2)= 3+3+3+3+3+2+2+2+2+2

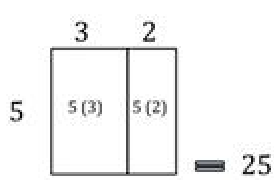
=25

From the above students should now be able to see that two is added five times, and three is added five times. Therefore:

5(3+2)= (5x3) + (5x2)

=25

There is also the geometric representation to consider demonstrating for students, in which the values become side lengths and the product is the area. The shape shown below is a composite of two rectangles. There are two ways to find the area of this shape, breaking it into two smaller rectangles and adding the areas, or adding the side lengths together and finding the area of the larger shape. This mirrors the process of the distributive method above, and offers further proof as to why this method can be used.



5(3)+5(2) = 5(3+2)

In this model, the idea that five will be multiplied to both 3 and 2 and that the resulting products will be added together is clearly illustrated, this should be convincing for middle school math students, who have quite a bit of experience with the area of quadrilaterals. This model will also be useful as students move further in algebra and begin to multiply binomials and polynomials.

Sample Problem: Amanda wanted to bring some cookies to a party. She and her son purchased a package of ten cookies and then made *c* additional cookies from scratch. Becca made twice as many cookies as Amanda and her son brought to the party. Represent the number of cookies Becca made as an expression in terms of *c*.

Explanation: If Amanda had purchased 10 cookies and then made c amount of cookies we can represent the total amount of cookies she has as c + 10. Since Becca had double this amount multiply the original expression by 2, 2(c +10). Using the distributive, the expression is then expanded to 2c + 20.

2 (c+10) =

2 (c) + 2 (10) =

2c+20

Combining Like Terms

Many students will have experience with this essential skill, but are often lacking a deep understanding of *why* it is mathematically possible. It is imperative to the simplification of more complicated equations and will be a major underlying component of much of the 8^{th} grade content. Building of students' knowledge of the distributive property can aide their understanding of like terms and why then can be added together. In fact, combining like terms is simply an application of the distributive property.

Like terms are monomials that contain the same variables raised to the same power. They can be combined to form a single term. Take the following as an example.

5x+3x

Using the distributive property we can rewrite this expression.

5x+3x=x(5+3)

5x+3x=x(8)

5x+3x=8x

It's worth noting that the above example also utilizes the commutative rule that states that a x b = b x a, or in the above example x (8) is equal to 8x. The factors being multiplied have not changed, their order has simply been reversed.

In this way students can see why we can combine terms that are like, and why you can not combine terms that are not.

5x+3x^2=x(5+3x)

The two terms in the above expression have x as a common factor. When x is removed however, and are not like terms and 5 + 3x cannot be simplified further.

Sample Problem: Hilary made *c* cookies on Tuesday. She doubled the original recipe on Wednesday. On Thursday, she tripled the original recipe. Write a completely simplified expression representing the number of cookies Hilary baked in terms of *c*.

Explanation: The amount of cookies Hilary made is represented by c. Double that amount would be 2c and triple that amount would be 3c. To determine the total amount of cookies we add the original amount c, to double and triple that amount (c + 2c + 3c). Using the distributive we can combine like terms.

c+2c+3c = c(1+2+3)

=c(6)

=6c

Clear Equations of Fractions

Rational coefficients, and fractions in general, sincerely frighten the middle school math student. “I hate fractions…I don’t do fractions” is a common thread in my classroom. Having a coefficient that is a fraction can complicate the manipulating of an equation but we can tap into students’ previous knowledge of the properties of equalities to make this process easier for ourselves.

1/5(x+2)=10

Many a student will see this problem on the homework and not attempt it, recalling former difficulties with fractions. But if we recall the multiplicative property of equality, we know that we can multiply both sides of the equation by the same number and maintain the equality of the two expressions. If we multiply both sides of the equation by 5, we get

5[1/5 (x+2)] = 5 (10)

or equivalently, since

5[1/5 (x+2)] = 5(1/5) (x + 2) = 1 (x + 2) = x+2

x+2 = 50

This problem is now more easily solved. Admittedly this is a very simple example of equations with rational coefficients, but with a firm understanding of the properties of equality and the possibilities for manipulating equations, students will be able to use those properties to simplify the process for themselves.

Sample Problem: Pat had *p* pounds of cookie dough in her freezer. At the grocery store she purchased 3 additional pounds. Her daughter asked for $1/4^{th}$ of the total cookie dough Pat had in her fridge. If Pat gave her daughter 3 pounds of cookie dough, how much did she have originally?

Explanation: If Pat originally had p pounds, then 3 more pounds would be p + 3 pounds, so $1/4^{th}$ of that would be (1/4) (p +3). Since that total is 3 pounds the result is the following equation

1/4(p+3)=3

4[1/4(p+3)] = 4(3)

p+4=12

$p=9$

Using Properties of Equality to Solve Equations

Ultimately, the goal is for students to put all of this together and utilize the strategies to solve one- , two- and multi-step equations. At this stage, the use of word problems will also be integrated more heavily into the process. Students will have had experience in previous lessons translating words to algebraic expressions and they will continue that process here with complete equations.

In order to pull together all the above-mentioned essential knowledge and strategies for students to be successful with equations, I will employ a hands on approach to understanding equations conveniently known as “Hands on Equations”. This is discussed further in the Teaching Strategies section below.

Sample Problems

Can be set up with visual model

Monica made some cookies and then bought a package of 12 cookies. Monica’s sister made double the amount that Monica made herself. Together they now have 36 cookies, how many did Monica make to begin with?

Less suitable to visual model

Monica had some cookies. She gave half to her sister and then ate five, and still had four left over. How many cookies did she have originally?

Teaching Strategies

Interactive Student Notebooks

Interactive Student Notebooks (ISNs) are an important part of my math curriculum and are utilized for both note-taking during direct instruction and individual student practice. The right side of the student created notebook is occupied by teacher-determined content, such as a foldable graphic organizer of the rules of exponents. The left side of the journal is reserved for student created content such as reflection on in-class activities or answers to practice problems.

The ISNs are an excellent tool for organization, guiding students to follow various protocols and guidelines as they build the content in their journal throughout the year. In addition to organization, the journals address students’ multiple intelligences. For example, creating or inserting graphs, tables and other images will be beneficial for visual learners. Finally, ISNs can also be used to assess student learning. Teachers can monitor student progress on the left side of the journal while students can self-assess and reflect as they look through the content of the right side of the journal. Overall, the journals allow students the ability to “internalize and personalize the content provided” leading to more ownership over the material covered.[6]

Number Talks

A number talk is a quick 10-15 minute exercise that asks students to practice mental computation around various skills. It is meant to be repeated on a regular basis with set routines and is not intended to necessarily line up concurrently with the ongoing curriculum. In number talks, “students are asked to communicate their thinking when presenting and justifying solutions to problems they solve mentally”. The process is intended to create more flexible, efficient and accurate mental problem solving. In my classroom, I have utilized this structure to touch on various review concepts or skills such as equivalent expressions or order of operations without needing to deviate entirely from the curriculum as scheduled.[7]

Kate Kinsella Precision Partnering

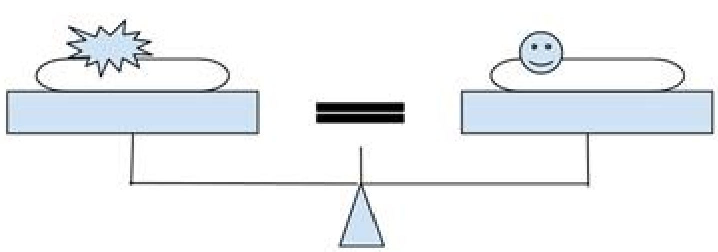
Dr. Kate Kinsella, a professor at San Francisco State University, specializes in ELD teaching strategies and has developed a protocol for effective partnered conversation in any classroom. Titled “Precision Partnering” this practice centers around four actions for students to employ in any partner conversation stated simply as “Look” (at your partner), “Lean” (in towards your partner), “Lower” (your voice) and “Listen” (carefully to your partner). These actions along with varied strategies for eliciting student responses and holding students accountable for each conversation serves to help create effective and productive partnered conversations. I use these strategies regularly in the classroom to ensure that students are on task and engaging with their classmates during number talks or other class discussions.[8]

Manipulatives

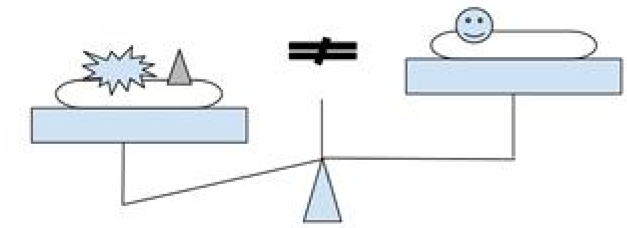
A major component of this unit is a hands on and manipulative heavy approach to teaching equations. Although it is essential that students develop the capacity to solve equations symbolically it has been found that “many students can benefit from working with physical problems that can be symbolized mathematically”.[9] By combining the two strategies, students will develop a more intrinsic understanding of the essential knowledge needed for increasingly difficult algebraic reasoning.

Pan Balance

I will employ a visual model with a balance scale to best physically demonstrate the concept of the properties of equality to students. For example, to demonstrate the addition / subtraction property of equality I will put two unknown but equal weights on a balance scale, students will see the scale is balanced and not tilted to one side or the other.



Students will quickly see that if an additional weight is added to one side, the balance will become off center, teetering to one side.

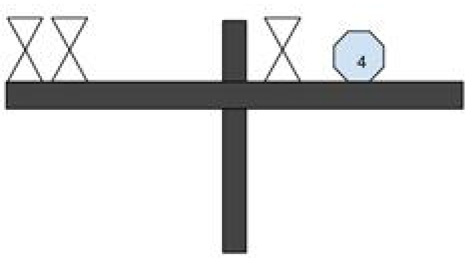


If that same weight is added to the other side, however, the balance will again be even. The same visual representation can be employed for the multiplication and division properties of equality by doubling or tripling the weight in one pan, for example, and doing the same in the other.

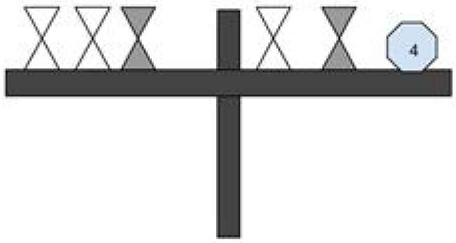
Hands on Equations

One manipulative that I will employ is called "Hands on Equations." This approach uses a scale (similar in theory to the previous pan balance scale activity) and symbolic representations of variables and negatives. Quite a bit of research supports this approach. One such study found that "the isomorphism between the object itself and the mathematical notions implied allows students to form a mental image of the operations that they have to apply. They are able to reactivate this self-evident image at any moment. "[10] While this method will be most useful and relevant for simple and more straight forward problems, the process will solidify students' foundational understanding of the content, allowing them to develop mastery of more complex problems. This tool will additionally be useful in scaffolding the material. Students who absorb the content quickly can move on to a strictly symbolic representation of the problems assigned, whereas students still struggling with the concepts can continue to use the physical, hands on model.

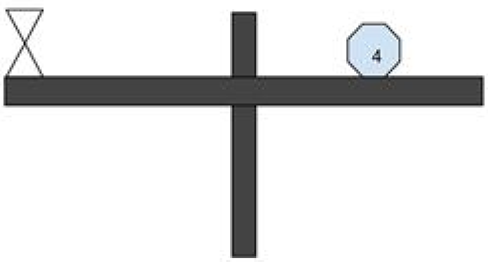
I will use this tool extensively when discussing the essential knowledge and strategies for developing algebraic reasoning and equations. To start, it will be integral in discussing inverse operations. Through the model, students utilize inverse operations to simplify equations. For example with the equation 2x = x + 4 the scale would be set up as follows with two pawns on one side and one pawn on and a four counter on the other side:



The first step students will take to solve this is to move all the pawns (variables) to the left side. To do this, students add a dark colored pawn to each side. The white pawn represents a positive x, while the dark pawn represents a negative x. Adding the dark pawn to both sides represents a "zeroing out" of the white pawn. Symbolically this would be equivalent to **x + -x = 0**.



Since the white pawn has been "zeroed out" it can now be removed from the right side of the equation and the solution can be clearly seen.



We can see now that each pawn is worth 4. Through exercises similar to this one, students will become familiar with the process of using inverse operations to eliminate terms from the equation. This model is additionally helpful for reinforcing the property of equality as it shows students that the equation maintains equality when the dark pawn is added to *both* sides.

Examples of equations that result in either no soluyion or infinitely many solutions, can also be demonstrated with the Hands on Equations balance scale to give students a physical representation of the equations. The equation 2x+3=2x+6**,** for example will reduce down to 3=6, students will easily be able to see on the balance scale that the 3 cube is not equal to the 6 cube, and therefore no value of x could make that equation true. Similarly, an equation with infinitely many solutions could also be represented on the balance scale. 2x+3=2x+3, for example, would reduce down to 3=3. From this result a discussion would follow about what values for x would make this equation true, upon trying a few potential solutions student should discover that any number substituted for x would yield the same result. This, in turn, should lead them to the conclusion that there are infinitely many solutions for this equation.

Bar Models

The *model method* was developed in Singapore in response to students' reliance on key words and other superficial clues to solve word problems. The model method, often referred to as bar models, "puts the focus back on the relationships and actions presented in the problem, and helps students choose both the operations and sequence of steps that are needed to solve a problem".[11] In an algebraic context, the bar models can be used to help students visualize a problem in a similar way to the Hands on Equations balance scale. Two bars are set equal to each other, divided into various parts that represent different values in the equation. Through a process of elimination, a physical representation of the properties of equality, students can visualize the process of solving an equation. Consider the following problem as an example:

Jimmy bought two bags of assorted candies each with the same number of candies in each. He also bought two individual candies. If Jimmy had a total of 24 candies, how many were in each bag?

X X 2

24

Although this is not drawn to scale, it is clear that the two from the first bar and two from the second bar can be eliminated because the two bars will remain equal if we remove the same amount from each.

X X

22

We can now see that the two x's are worth 6 total. Or one x is worth three, x = 3.

X X

11 11

This model can help students create a visual representation of the information presented in a problem. The bar model can also be very useful with more complex algebraic problems, take the following as an example.

Al and Bob lift weights to gain for football. Al weighs 195 pounds. Bob weighs 205 pounds. Al wants to gain three pounds a week, Bob wants to gain two pounds a week. How long will it be until they weigh the same?

Set up the known information in two bars. Let w represent the number of weeks gaining weight.

Al

195 w w w

Bob

195 10 w w

What the bar model above makes apparent is that even though Al starts off weighing less, he is gaining more per week. By setting up the bar model with the w's being equivalent, it also becomes apparent that w and 10 are equivalent. It will therefore take 10 weeks for their weight to be the same.

Appendix

Problem Sets

Translating words into algebraic expressions

1. Take a number and add 8.
2. Take a number and reduce it by 5.
3. Take a number and multiply by 3.

4. Take a number and multiply it by one half.

1. Add 8 to twice a number.
2. 5 fewer than half of a number.
3. 3 times one less than a number.
4. One half of three times a number.

Translating words into one and two-step equations

1. 8 more than a number is 15.
2. 5 fewer than a number is 20.
3. 3 times a number is 9.
4. One half of a number is 100.

1. 8 more than twice a number is 28.
2. 5 fewer than half a number is 20.
3. Half of 3 fewer than a number is 10.
4. 3 times one less than a number is 24.
5. One half of three times a number is 15.

Solving Multi-Step Equations with Real World Context

Monica made some cookies on Monday using a recipe her mother had given her. On Tuesday, she doubled the recipe and on Wednesday she tripled the recipe. After three days, Monica had 60 cookies, how many cookies did the original recipe make?

Let x be the number of cookies in the original recipe,

$x + 2x + 3x = 60$

Monica had some cookies, she gave half to her sister and then ate five and still had four left over. How many cookies did she have originally?

Let x be the number of cookies Monica had originally,

$x – 1/2x – 5 = 4$

Monica made some cookies, her sister made half as many. They combined their cookies and ate ten of them, if they had 8 left over, how many did Monica make?

Let x be the number of cookies Monica made,

$x + 1/2x -10 = 8$

Monica had some cookies; her sister had three times as many. If they combined their cookies and then gave half of the total to a neighbor, how many cookies did Monica make if they gave their neighbor ten cookies?

Let x be the number of cookies Monica made,

$(x + 3x) / 2 = 10$

Monica had some cookies. She gave seven to her sister. Monica then gave half of the remaining cookies to a neighbor, and she still had five cookies. How many cookies did she have originally?

Let x be the number of cookies Monica had,

$(x-7) / 2 = 5$

Monica made some cookies and then bought a package of 12 cookies. Monica's sister made double the amount that Monica made herself. Together they now have 36 cookies, how many did Monica have to begin with?

Let x be the number of cookies Monica made originally,

$x + 12+ 2x = 36$

Monica made some pies to sell at the bake sale. The school cafeteria contributed four pies to the sale. Each pie was then cut into five pieces and sold. There were a total of 60 pieces to sell. How many pies did Monica make?

Let x be the number of pies Monica made for the bake sale,

$5 (x + 4) = 60$

Monica needed to make 100 cookies for her school bake sale. She saved $1/5^{th}$ of each batch for her own family. If she had to make five batches to have enough for the bake sale, how many cookies were in each batch?

Let x be the number of cookies in each batch,

$100 = 5 (x – 1/5x)$

Monica's cookie recipe called for chocolate chips. She bought a total of 36 ounces of chocolate chips. In her first batch of cookies she used a set number of chocolate chips and then to experiment she used twice as many chocolate chips as she did in the first batch for her second batch. To continue the experiment she used half as many chocolate chips in her first batch in her third batch. If she used all 36 ounces in the three batches, how many chocolate chips were in the first batch?

Let x be the number of ounces of chocolate chips in the first batch.

$x + 2x + 1/2x = 36$

Monica made a total of 98 cookies of three different flavors- chocolate chip, oatmeal raisin and peanut butter. The number of oatmeal raisin was twice the number of peanut butter. The number of chocolate chip was three times more than the number of oatmeal. How many of each type did Monica make?

Let x be the number of peanut butter cookies,

$X + 2x + 3(2x) = 98$

The PTO set up a buffet fundraiser at school at a total cost of $15 per person, with an extra cost for dessert. If a family of 5 all ate the buffet and dessert for a total of $90, what is the total cost of dessert?

Let x be the cost of dessert,

$5\ (15 + x) = 90$

Ms. Nowakowski filled her gas tank completely before leave for LA. After she had driven 300 miles she discovered she still had a quarter tank left. How far can Ms. Nowakowski travel on a full tank?

Let x be the miles traveled on a full tank,

$(3/4)x = 300$

Ms. Nowakowski commuted to school and back from her house, two days this week. She commuted the other three days from her Mom's house to school and then back. At the end of the week, she had driven 140 miles. If Ms. Nowakowski's mom lives 10 miles closer to school than Ms. Nowakowski, how far is her daily commute?

Let x be the number of miles on Ms. Nowakowski's daily commute,

$2x + 3(x-10) = 140$

Ellie got her test back from Ms. Nowakowski, she saw that Jonah's score was ten points higher than her score and that Lev's score was 7 points lower. If together they averaged a score of 89, what was Ellie's score?

Let x be Ellie's test score,

$[X + (x + 10) + (x – 7)] / 3 = 89$

Ms. Nowakowski's class has some boys and some girls. There are 5 more girls than there are boys. If there are 25 students in the class, how many are boys?

Let x be the number of boys in class,

$x + (x +5) = 25$

There are a total of 20 students in Ms. Waldo's elective, 6^{th} , 7^{th} and 8^{th} graders. If the number of 7^{th} graders is two more than the number of 6^{th} graders and the number of 8^{th} graders is one more than the number of 7^{th} graders, how many students of each grade are there?

Let x be the number of 6^{th} *graders,*

$x + (x + 2) + (x + 2 + 1) = 20$

Ms. Nowakowski decided to rent a car on her vacation. She can either rent a car for $35 day plus $0.40 a mile or for $20 a day plus $0.55 per mile. How many miles would Ms. Nowakowski have to drive for the two rental cars to cost the same amount?

Let x be the number of miles driven,

$35 + 0.40x = 20 + 0.55x$

Becca wants to paint a ceramic planter at a local potter shop. The total price of the planter is the cost of the planter plus the hourly painting rate of $6 an hour. How many hours did Becca paint if the planter cost $9 and she paid $33 for the total bill?

Let x be the number of hours Becca painted,

$33 = 9 + 6x$

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