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Solving Big Problems: Using Estimation to Develop Scientific Number Sense

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Introduction

Numbers and data are so pervasive in media and culture that our students need to be prepared to interpret and interact with them. Most students going through a physics classroom won't become physicists, so what should they get out of the course? The key offerings of a physics education are an intuition about the physical world and a critical eye that can make observations, ask questions and answer them. The foundational skills for this are a strong number sense and the ability to decompose complex problems into smaller ones.

The fact of the matter is students arrive in my classroom without experience in those skills and with habits that act as a direct barrier to their acquisition. Their issues with number sense are hindered by not being able to relate numbers to each other. What would happen if you asked your students, "What's the difference between 10^{17} and 10^{-17} ?" If they're like mine, the best answers you'll get involve a long string of nines. This question succinctly highlights two conceptual deficiencies. The first is that students are not yet fluent in techniques for relating numbers of different size. Math educators would say, they think additively, not multiplicatively. That is, when you say “difference”, they think of subtracting, rather than taking a ratio, i.e., dividing. The second is that they don’t yet understand precision, its implications and the limitations it imposes on the world. This ends up with students writing down all of the digits that show up in their calculator, all of the time. Not all of those digits are created equal, and the place values they hold are far more telling than the digits themselves.

Students also arrive to my class with very limited experience with problem decomposition and perseverance. The way word problems are frequently taught in lower grades leaves students in the habit of taking all of the given information and multiplying it together as a starting point, without considering the narrative of the problem and charting the steps that need to be followed to arrive at whatever is being asked of them. This problem compounds when the given information starts to disappear and they're left to find or produce it on their own. It's part of what makes problem solving a difficult skill for them to translate to the real world. In the work force, tasks aren’t assigned like traditional textbook problems; problems are phrased generally and need to be solved independently.

As a final observation, my students have trouble communicating about their work. They fail to attend to units in a way that preserves the meaning of the quantities involved, losing the computational narrative that runs through the problem and its solution. Without this ongoing understanding of what they're doing and why, students frequently can't articulate the justifications for the steps they're taking and the trajectory of their solution. This unit seeks to increase student understanding of number sense, problem decomposition and communication through the use of a style of estimation questions known as Fermi Problems.

Demographics

This unit is designed to be the introduction for my 11th grade General Physics and AP Physics 1 classes. The majority of students who are taking it are concurrently enrolled in Algebra 2, though a handful of the AP students will be in Precalculus. The unit doesn't depend on any specific physics content knowledge. It would, therefore, be immediately applicable as an introductory unit in a chemistry or environmental science class. The unit, with little to no modification, would also fit well into an Algebra 1, Algebra 2 or Precalculus class offered at the high school level. The complexity of questions and techniques can be adjusted depending on the age or ability level of the students.

I work at a small magnet high school where the students themselves are incredibly diverse. Despite being a magnet, it's not atypical for students to come to me below grade level in reading and/or math. This contrasted with the incredibly high abilities of some of them, means that every classroom contains the full range of abilities. In addition, students come from a variety of backgrounds that reflects the demographics of Philadelphia as a whole. There are a large number of recent immigrants and English language learners, both from Southeast Asia and Latin America. This enormous variety of backgrounds and skill levels is at the front of my mind as I design this unit.

The final demographic consideration is that I teach in the Philadelphia School District, an enormous and diverse urban district that serves the entire city. The financial state of the district is such that schools function with operating budgets so slim that basic needs go unfilled on a daily basis. Realizing this as my own reality and the potential reality for other teachers, the activities and strategies are designed to be free. No special equipment, manipulatives or physical resources are required.

Rationale

Enrico Fermi was an Italian physicist who, among other things, worked on the Manhattan project. As legend goes, during the Trinity test, where the first nuclear bomb was detonated, Fermi took a handful of strips of paper and dropped them into the oncoming blast wave. By pacing off how far backwards the strips were blown, he made an estimate of the energy released in the explosion. Using these seemingly crude methods, he estimated it was equivalent to 10 kilotons of TNT, within a factor of two of the true value.

A little bit below the surface, however, you realize his methods weren’t crude at all. Fermi was meticulous in everything he did, and his estimates were no different. His assumptions that formed the foundation of the estimates were clearly cataloged and defined. He exploited specific symmetries inherent to problems and drew upon his deep physical intuition to decompose the complex into the manageable.

Fermi’s ability to get very good results through a series of estimates is now the stuff of science education legend. That’s why Fermi questions now bear his name! He would give these questions to his students in an attempt to train them to think about problems that seemed beyond calculation by breaking them into pieces that can be reasonably estimated and then combined into an answer. His students drew on observations, experiences and logic to answer questions that, at first, they thought they couldn't even scratch the surface of.

My students love to ask "when will I need this in life" and this unit attempts to give them something they can use every single day. The true value of a physics education comes in the ability to ask questions, make observations and solve problems. Not just math problems or physics problems, but real life problems. Should I rent or buy? What's this noise my car is making? Having the ability to step back and decompose questions into their parts puts answers and understanding within reach. This ability isn't intrinsic to certain persons, either. It can be taught explicitly and developed through experience.

Good number sense is fast becoming just as important in our data-driven society as basic literacy. Numbers are cited constantly in media and literature, but how deeply do our students really understand them? What's the difference between a million dollars and a billion dollars? Numbers so huge may seem out of reach to our kids. How much smaller is a bacterium than a grain of salt? Again, not a hard question, but the small size of both masks the vast comparative difference. Consider this: In our own bodies, bacterial cells outnumber "our" cells – the eukaryotic cells that carry our genes – by about 100 to 1, but their total weight is only a couple of pounds out of our 100 – 200. So while bacterial cells are orders of magnitude more abundant, this implies that they must be many more orders of magnitude smaller than our own cells. Number sense, especially when orders of magnitude are concerned, is a tough skill to develop and internalize. Physics provides the perfect platform for developing an initial number sense and cultivating a fundamental understanding of the powers of ten in students over the course of a year.

Once an understanding of orders of magnitude and relative size is established, the importance of precision becomes paramount. Precision and relative accuracy are concepts that are baked into the nature of science, but students rarely develop real competency in. How precise does an answer need to be? Is writing down more digits better? If the calculator gives me all these digits, I'd better write them down, right? To students, the answers to these questions seem obvious. To science teachers, these student intuitions are obviously wrong.

Fermi became known not just for his accomplishments as a physicist, but also for asking and answering big questions in a way that required a set of justified assumptions in order to arrive at an answer. Students can learn to take a question that, on the surface, seems impenetrable and break it into smaller pieces (or constituent quantities). These pieces can be related to each other by their units and arranged in an order that produces the desired result. Then students have to make a judgment and decide if each quantity should be looked up or estimated. When they estimate, students have to realize that they've thrown any hope of a precise answer out the window. Since different people may estimate differently, there needs to be a rationale behind the choice of numbers. Students have to be able to communicate the assumptions they've made to arrive at a solution, and to justify them. To answer Fermi Problems requires students to not only develop skills that aren't actively taught in other classes, but also shake free of most of what they've learned about solving problems. There isn't going to be an exact right answer and that causes students a great deal of stress, but the payoff is a robustness in problem solving and thought that isn't developed at any other point in secondary education.

These skills are all highly transferable. Estimation and number sense can partner to help students be more critical of statistics and more data literate. The understanding of quantities as they relate to the real world can contextualize them in a way that by-the-book pure math classes often fail to do. Routinely deconstructing problems can help students to be more independent problem solvers, by developing their ability to take the first steps toward finding their own answers to questions. Finally, all of these skills are moot if the results can't be effectively communicated and supported. This unit seeks to set students up with foundational skills that will lead to increased success in the physics classroom and beyond.

From the scenarios presented in the problems in this unit, students should get a better sense of what sorts of questions they can answer with the math they already know. The surprising result is that they have access to way more computational power than they'd ever expect with just unit sense, order of magnitude sense and some multiplication. Students are also going to need to develop a systematic approach to problems where the components they need to solve it aren't immediately provided. Hard problems are separable into easier problems. Students can apply an iterative process of problem solving steps until they develop an intuition about answering questions.

When students invest time in thinking about the problems, they should obtain a sense of how situation and purpose inform an appropriate level of precision in the numbers they use. Significant figures are more than just a set of rules that artificially constrict the answers in a science classroom. As students look numbers up, measure quantities and estimate, they will be challenged to develop an understanding of where the idea of significant figures actually comes from and a sense of how to use them in a meaningful and intuitive way. Students will also need to examine the quantities they need and sort them into categories based on what qualifies as “encyclopedia knowledge” and can be looked up, what needs to be estimated, and what needs to be further broken down and calculated. As students deal with these different numbers of different orders of magnitude, they'll also develop a sense of how they relate to each other.

Finally, from solving the problems, students will learn how quantities of different orders of magnitude interact with each other. Operationally, addition and subtraction will behave extremely differently than multiplication and division. How do numbers get bigger or smaller when combined? When do numbers have little to no effect when combined? Students will also need to understand why estimates are acceptable in some situations and not others. For some reason, students are irrationally tied to all of the digits that pop up in a calculator window, even when infinite precision is completely inappropriate. The unit will explore how to break them of this habit and also get students discussion the rationale and process used in solving problems, instead of just supplying a functionally meaningless string of digits.

Background

Fermi problems are a class of estimation problems that rely on taking a set of justified assumptions and extrapolating them into estimates in order to answer a “big” question. The classic example asks students to estimate how many piano tuners are in Chicago. This can be answered by making some assumptions about how many households are in Chicago, what fraction of households have pianos, how often they need to be tuned and how much work an individual piano tuner can do. If the assumptions you make are carefully documented, you can arrive at a surprisingly accurate result, even with no explicitly given information. Even if individual estimates aren't great, the accuracy comes from the hope that some estimates will be high, some estimates will be low and on average, the effects will balance each other.

This style of question is indicative of the types of questions students naturally generate on their own or encounter in higher education and the workforce. There’s a research component, an intuition component and the idea that no one is going to hand you the “given information” and check your result against an answer key. Using Fermi problems, we can develop independent problem solving skills in students that will empower them with the ability to ask and answer questions for themselves. The ideal outcome, then, becomes students who not only revel in the challenge of solving new problems, but also seek to ask new questions of the world around them. These questions can then be posed, pondered and traded with other like-minded problem solvers.

One of my favorite published collections of this type of problem appears in Weinstein and Adams’s book *Guesstimation*. Their problem selection is interesting and engaging. They address some of the “historic” Fermi problems and they provide a good framework for approaching them. Each question comes with a full solution, as well. The approach they outline, however, is somewhat casual and doesn’t quite hit all of the skills I want my students to gain, so below I’ll break down what I want them to get out of this unit in a more systematic fashion.

Problem Decomposition

Fermi Problems are presented as questions. They ask for a specific answer, but typically don't provide (or even hint at) the path to get there. Students, then, should be able to trace back from a request for information to the necessary pieces that comprise the final quantity. I'll refer to these pieces as *constituent quantities* for the remainder of this unit. Students then combine these pieces to assemble the final calculation that gives them their answer.

Identifying constituent quantities is the bulk of the battle for my kids. They're making a shift from being provided with information that can be arranged and assembled to form an answer to a world where a request is made and the appropriate information to arrive at the solution is not specified. They are making the decisions about what information is important to their calculation and what information is not, because the importance of a number can't be inferred from the fact that it was or was not provided to them in the problem itself. A task I’ve found helpful is to ask my students make a list of information that might be important. There's no requirement to use all (or any) of the quantities that end up getting listed, but having that list is a good starting place for the steps that follow.

To illustrate this better, let's use an example: If you laid all the French fries you ate in a year end-to-end, how many city blocks would they stretch? There isn't much information. There isn't even a definite solution, but a good approximation can be made. The place to start is by listing out what pieces of information might be important to the problem. The immediate constituent quantities for this calculation might be the length of one French fry, the length of a city block and the number of French fries you eat in a year. The number of fries you eat in a year could be further broken down into the number of fries in a box, the number of boxes consumed per week and the number of weeks in a year.

Once these quantities are identified, they need to be numerically defined. Some of them are quantities that have known values or can be looked up, like the length of a city block or the number of weeks in a year. Other things that fall into this category are physical constants or easily Google'd measured quantities. Quantities that can't be looked up readily are marked for estimation. This is a departure from the method proposed by Weinstein and Adam, where every quantity should be estimated[1] , but students always have cell phones with them, so it doesn't make practical sense to deprive them of easily accessible information. Plus, the accepted values lend themselves to discussions about how good their estimates would have been.

The next step is to attend to units. What is the unit on each of these quantities? This becomes an ideal time to dig into the meaning of "per," the standard linguistic signifier of a rate or ratio. What is a mile per hour or a meter per second? How is that similar to some of the units that would pop out of our example question, like boxes per year, fries per box, weeks per year, meter per fry and meter per block? Every quantity has a unit and students should develop the habit of always supplying the unit with computations.

In practice, those first three "steps" happen simultaneously. You can identify the constituent quantity, supply the unit and mark it to be looked up or estimated all at once. Once the list of possible constituent quantities is assembled, they need to be formed into a calculation that results in the desired quantity. Chemistry teachers will recognize this as a process that's functionally identical to stoichiometry, but that prior knowledge isn’t required. The idea is to assemble the constituent quantities in such a way that the extraneous units "cancel," leaving only the final unit and thus the final answer. This somewhat formalizes and somewhat diverges from the (admittedly easily accessible) process outlined by Weinstein and Adam[2] , but was selected to specifically ground this process in other commonly accepted science practices.

Let's apply this to our example. To estimate the total length, in blocks, of all the French fries I eat in a year, I'm going to need to know the following quantities, with their unit in parenthesis: the number of fries I eat in a year (fries/year), the length of one fry (m/fry) and the length of a city block (m/block). I'll further break down the number of fries per year into the number of fries in an order (fries/order), the number of orders I eat in a month (orders/month) and the number of months in a year (months/year). Since I'm looking for a result in blocks/year, the units could be arranged as such, paying attention to the fact that the reciprocal of the length of a block is used:

$$\left(\frac{\text{m}}{\text{fry}}\right)\left(\frac{\text{fries}}{\text{order}}\right)\left(\frac{\text{orders}}{\text{month}}\right)\left(\frac{\text{months}}{\text{year}}\right)\left(\frac{\text{blocks}}{\text{m}}\right)=\left(\frac{\text{blocks}}{\text{year}}\right)$$

Estimation

Estimation, as a skill, is founded in equal parts on number sense and self confidence. Students need to narrow the order of magnitude, establish a range of reasonable values, pick a central value to represent that range and then be confident enough in the assumptions they've made for their estimation that they feel comfortable leaning on it in a calculation. Making this work for students requires that they develop advanced number sense with special emphasis on orders of magnitude and precision.

Orders of magnitude, or powers of ten, is a term used to describe the dominant place value in a quantity and by how many decimal places two quantities differ. We operationally know that adding a zero to the end of a number makes it ten times bigger, but it's often hard to imagine what that might look like. Learning and developing estimation techniques can help bring us closer to a more intuitive understanding of orders of magnitude, because they are the fastest way to pinpoint quantities on a spectrum of values.

To apply to our previous example, let's try to find the order of magnitude for each of the quantities we'll have to estimate and see how that begins to shape our answer. These extremely rough estimates will be refined later, but just trying to pin the result down within an order of magnitude is a strong way to begin estimating. A French fry is probably a single digit number of centimeters, so we have a number on the order of 0.01 m, or 10^{-2} m/fry. An order of fries doesn't have hundreds of fries in it. And a box doesn't have a single digit number of fries; it has tens of fries. So we're looking at 10^{1} fries/order. I don't eat fries more than ten times a month, so we have 10^{0} orders/month. Months per year is tricky. We know there are 12 months in a year, but we’re not looking to be that precise yet. Twelve is on the order of 10^1 and the unit is months/year. In Philadelphia, a city block is a little longer than a football field, so there are on the order of 10^2 m/block, which translates to 10^{-2} blocks/m. Let's put that together and narrow down our answer.

$$\left(10^{-2}\frac{\text{m}}{\text{fry}}\right)\left(10^{1}\frac{\text{fries}}{\text{order}}\right)\left(10^{0}\frac{\text{orders}}{\text{month}}\right)\left(10^{1}\frac{\text{months}}{\text{year}}\right)\left(10^{-2}\frac{\text{blocks}}{\text{m}}\right)=\left(10^{-2}\frac{\text{blocks}}{\text{year}}\right)$$

Once students are comfortable with making these order of magnitude estimates for individual quantities, they need to be reflected upon and refined. The initial result of 10^{-2} blocks per year seems small. Does it make sense? When I look back, one French fry is probably much closer to ten centimeters than one. There are definitely more than ten fries in a box, but not 100. I eat fries more than once a month. There's 20% more than ten months in a year, but there are slightly less than 10^{-2} blocks/meter. All combined, this reinforces the initial notion that the order of magnitude estimate was too small. Now that we have an understanding of our computation, we're in a position to refine it.

Before that happens, now is the time to address matters of precision. Science classes push the use of significant figures, and rightfully so. The stumbling block is that students view them as a set of rules as opposed to the natural extension of the relationship between measurement and estimation that they are. Digits are significant when you're sure of their quality and typically you're entitled to estimate one digit. Let's say I need to measure a piece of wood. I can use a meter stick, that has mm markings or I can use my thumb, which is about 2 cm wide. Both methods should give me the same order of magnitude, say tens of centimeters or hundreds of centimeters, and probably even the same leading digit. After that, they'll likely diverge, because the level of allowed precision is intrinsic to the measuring device used – this is the purpose of the scientific practice of using significant figures. The meter stick allows me to be sure of millimeters. My thumb requires me to estimate the number of centimeters. Looking at significant figures then becomes a way to quickly evaluate the quality of a measurement or calculation, but also a way to keep computation honest. A calculation is only as good as the worst measurement, so if one value could be off by up to 50% in either direction, your result could be off by up to 50% in either direction. Better measurements yield higher precision and that precision is manifested in more significant figures.

How is an estimate made? A good technique to use is to start by setting reasonable boundaries and then pick a number in between. Students will gravitate toward taking the average, or mean, but that's a construct more suited to situations involving addition and subtraction, rather than for the order of magnitude estimates and multiplicative constructions inherent to this type of problem. A more appropriate tool is the *geometric mean*, which we can think of as a multiplicative average. First let's look at the formula for arithmetic mean (AM) between two numbers, *a* and *b*:

$$AM = \frac{a+b}{2}$$

The geometric mean (GM) between *a* and *b* is found by "escalating" that calculation. Instead of adding *a* and *b*, they are multiplied. Instead of multiplying by one half, their product is raised to the one half power, which is the square root:

$$GM = \sqrt{ab}$$

For an arithmetic mean, if you add or subtract the same value from the central point, you get to the bounds; it represents the additive midpoint between the two numbers. For the geometric mean, if you multiply or divide the central point by the same value, you arrive at the bounds. This gives you a multiplicative midpoint, analogous to the function of the arithmetic mean. Another way to look at this is if you add the two original numbers, you get the same answer as when you add the arithmetic means to itself; and when you multiply the two original numbers, you get the same answer as when you multiply the geometric mean by itself.

It’s also interesting to note that in all cases, except for where a=b, the arithmetic mean is always larger than the geometric mean. In the case where a=b, the arithmetic mean and the geometric mean are equal, and AM = GM = a = b.

Let's compare the arithmetic and geometric means between 1 and 100. From above, the arithmetic mean is computed by adding the bounds and dividing by two:

(1+100)/2 = 50.5

If 49.5 is subtracted from or added to the arithmetic mean of 50.5, the bounds of 1 and 100 are recovered. Thus, 50.5 represents the number midway between 1 and 100.

Contrast this with the geometric mean. From above, the geometric mean is computed by multiplying the two bounds (instead of adding) and then taking the square root (instead of dividing by two). Therefore:

$$\sqrt{1 \times 100} = 10$$

If the geometric mean is divided by ten, you recover the lower bound. If it is multiplied by ten, the upper bound is recovered. Similar to above, 10 is “midway” between 1 and 100, but in a multiplicative, not additive, sense.

When performing estimates, feel entitled to one digit and an order of magnitude. Anything more precise than that is a lie. Getting my students to internalize this idea and not report results with much higher precision will be a struggle, because they view every digit that pops up in a calculator as equally important, a misconception that we debunked in the rationale for this unit.

One example in the trap of false precision comes in estimating the radius of the Earth in miles and then converting that to kilometers. The average radius of the Earth is around 4000 miles, a number that has to come with a fair amount of qualification, seeing as the Earth is bulgy around the equator, not a sphere. The Earth also has tall mountains and vast undersea trenches, and therefore can’t be considered a smooth object. The conversion factor between miles and kilometers is known to very high precision: there are 1.609344 kilometers in one mile. So let’s find the average radius of the Earth in km.

$$(4000\ \text{mi}) \times \left(1.609344\,\frac{\text{km}}{\text{mi}}\right) = 6437.367\ \text{km}$$

Wait, what just happened? Just because we converted the units doesn’t mean we made a better measurement! Even though the conversion factor is extremely precise, we’re still only entitled to the significant digits we started with. 4000 mi is good to two significant figures (maybe even better, despite the roundness of the number), so we can round our conversion to a more reasonable 6400 km. Students have a natural inclination to view unit conversions as perfect operations, but still remind them to attend to allowable precision.

When students are unsatisfied with their single significant digit and an order of magnitude, we need to remind them that precision isn't the point of every calculation. They're going to be setting themselves up to verify a more complex future calculation, ballpark a new number to put an idea into perspective or judge whether or not a figure someone's quoted seems reasonable. For these purposes, a single significant digit is perfect. Stress to your students that they're trying to understand the world better, not build a bridge. No one is in danger of dying if they're off by a factor of two, but they can learn a great deal if they're within a factor of ten.

The other desired effect of getting students away from reporting meaningless digits is an increased understanding of place value and relative accuracy. How often have you run into a situation where you tell students that an answer is “around 2000,” and someone shoots their hand straight up into the air? I can already predict what they’re going to say. Probably, “I got 2031.8, is that okay?” Normally I’d respond with, “Calculate the percent difference and tell me how you feel about that,” but I’d like to illustrate the point with an example.

Think about the square root of two. I'd probably write down 1.4 or 1.41 off the top of my head, depending on the situation and what I was trying to do. Ask a student, however, and you'll probably see them scribble down 1.414213562... until they run out of digits on their calculator display. They don't have a sense of how meaningful each digit is. Not all ones are created equal! Look back at that "student response." The leading 1 represents over 70% of the actual quantity being expressed. That second one, just two decimal places away, represents less than 1%. And that third one, three more decimal places away, is *less than 0.001%*.

Another good exercise can come from modeling the Earth as a perfect sphere and calculating the surface area. Looking up the mean Earth radius gives a value of 6371 km (which doesn’t even make sense to ±1 km, for the reasoning above), but let's use successively more precise values for the radius and perform the calculation, given that the surface area of a sphere is $4\pi r^2$:

$4\pi(6 \times 10^3 \text{ km})^2 = 5 \times 10^8 \text{ km}^2$

$4\pi(6.4 \times 10^3 \text{ km})^2 = 5.1 \times 10^8 \text{ km}^2$

$4\pi(6.37 \times 10^3 \text{ km})^2 = 5.10 \times 10^8 \text{ km}^2$

$4\pi(6.371 \times 10^3 \text{ km})^2 = 5.101 \times 10^8 \text{ km}^2$

The difference between the least precise result and the second least precise result is 2%. The difference between the two most precise results is 0.02% and the most precise result doesn't impart any additional insight, especially when you consider all of the complicating factors that were brushed over to arrive at these answers in the first place. When students do this process for other calculations and compare the percent difference, they should start to realize almost all of the value in the calculation is in the first digit. The second can modify the answer somewhat but past that, the differences are inconsequential. It’s also worth noting that when you round a number between 10 and 20 to the nearest 10, the percentage error may be much larger than when rounding a number with a leading digit greater than one.

With this understanding of reasonable and relative precision and the tool of geometric mean, let’s refine the estimates we made for our French fry problem. A typical French fry is between 6 and 10 cm. There are somewhere between 30 and 100 French fries in a typical order. I eat French fries somewhere between two and 8 times per month and a city block is between 100 and 200 meters. 12 months in a year is a known quantity, so we’ll use that. Let’s compute the geometric means and keep only one significant digit:

$$\sqrt{0.06 \times 0.10} = 8 \times 10^{-2} \, \frac{\text{m}}{\text{fry}}$$

$$\sqrt{30 \times 100} = 5 \times 10^{1} \, \frac{\text{fries}}{\text{order}}$$

$$\sqrt{2 \times 8} = 4 \, \frac{\text{orders}}{\text{month}}$$

$$\sqrt{100 \times 200} = 1 \times 10^{2} \, \frac{\text{m}}{\text{block}}$$

The calculation now becomes:

$$\left(8 \times 10^{-2} \, \frac{\text{m}}{\text{fry}}\right)\left(5 \times 10^{1} \, \frac{\text{fries}}{\text{order}}\right)\left(4 \times 10^{0} \, \frac{\text{orders}}{\text{month}}\right)\left(12 \, \frac{\text{months}}{\text{year}}\right)\left(1 \times 10^{-2} \, \frac{\text{blocks}}{\text{m}}\right) = \left(2 \times 10^{0} \, \frac{\text{blocks}}{\text{year}}\right)$$

Two blocks per year is, as suspected, significantly larger than what we had before and much closer to my gut feeling. It's also somewhat depressing and makes me seriously question my dietary choices. But, the calculation did its job. It provided perspective and insight, both into the problem solving process and the scenario we considered.

Justification

The final piece of the puzzle is being able to justify and explain what's been done. If you'll notice, during the estimation phase I made some assertions that you may or may not have agreed with. The most controversial is probably about the size of city blocks. The justification I'd use for my estimate is that since I'm from Philadelphia, I'm talking about Philadelphia city blocks, which are different than New York or Chicago city blocks. I also eat French fries probably once per week. By clarifying the assumptions that led me to the estimates that I made, I've showed two things: First, that my end result should be reasonable if you agree with the assumptions I made in the estimations. Second, that I've considered the numbers that I'm writing down instead of just pulling them out of thin air.

Under the Common Core, the justification skills that go into backing up an estimate are easily assessed under the Argumentative Writing rubric. Students should be providing evidence for what they do and considering alternate positions. Obviously there won't be a sustained narrative surrounding the solution to a Fermi problem, but students should be able to explain their rationale and calculations to other students in either a paragraph or short presentation. Requiring students to explain their work with more depth than just listing the steps they followed will force them to be more meticulous about how they consider their work at all steps of the process. It also helps me assess their work and provide specific feedback by making clear the logic and rationale behind each step.

Teaching Strategies

Using an example problem has allowed me to break the process down for clarity of exposition, but this is not the way I will present it to my students. Some suggestions are provided for illustration, but they are not exhaustive. The classroom activities are presented as skill building activities first. The skills will first be developed in isolation and then built into multi step problem solving through a process of careful scaffolding. Here I'll assemble some strategies that guide my selection of classroom activities to build order of magnitude sense, justified estimation, dimensional analysis and problem decomposition. In the next section, I'll outline specific activities I'll use with each of these strategies. Refer to the appendix for additional problems that are aligned with each topic.

Orders of Magnitude and Estimation

I plan on teaching estimation by teaching two different techniques. Students will start doing one-step estimations, or working with quantities they can estimate directly. The first, order of magnitude estimation, requires pinning down a result within a factor of ten. I will be sure to be clear with my students about what they're trying to do, which is pick a power of ten that is closest to the true value of the quantity they're trying to estimate. Think about a bag of Swedish Fish (my favorite candy!). What order of magnitude goes with the number of fish in a small personal-sized bag? There are tens of fish in there, but not 100, so the estimate would be *on the order of* 10^1 *fish.* How about the thickness of a penny in meters? It seems to be around a millimeter or two, so it would be *on the order of* 10^{-3} *meters.* I will stress to them that they're just trying to find the power of ten that would go with the real amount.

After some practice with nailing down orders of magnitude, tougher estimates can get much better through a technique known as *bounding*. The idea is to specify reasonable upper and lower bounds for the quantity you want to estimate and then pick a number somewhere in between. To present the idea to my students, I'll have them pick a number that they know is too small *(lower bound)* and a number they know is too big *(upper bound)*. Then, looking at these boundaries, I want to ask them to try and "squeeze" them down. As an example, let's consider the circumference of the Earth in miles. You could be extremely conservative and say it's more than one mile and less than one million miles, and you'd be right, but we can move those bounds in by quite a bit. The distance from New York to Los Angeles is about 3000 mi, so that'll be our new lower bound. For some reason I always remember that the distance from the Earth to the Moon is around a quarter of a million miles, so that can be our new upper bound. This gives us a range less than one-fourth our original, so we're making progress! With some clever thinking about time zones, we could narrow that down even more, but let's say one of my students stalls out there. How might I help them pick a number somewhere in between?

Assuming they did an order of magnitude estimate (and know the distance between NYC and LA), they probably guessed that it's on the order of 10^4 mi for a whole trip around the Earth. When picking a number "in the middle" of two other numbers, my students will automatically default to taking the average, or *arithmetic mean*, which yields a number on the order of 10^5 in this case. I want to make them feel unsettled about this, because it doesn't fit with what they know, so it must be reevaluated! For estimates like this and situations where they'll be multiplying a number of estimates together, finding a multiplicative average, or *geometric mean*, is preferable. They can of this as the multiplicative analog of the arithmetic mean. I don't want to make this concept more intimidating than it needs to be for my students. They remember that the arithmetic mean between two numbers involves adding the numbers and then dividing by two. For the geometric mean, adding the two numbers becomes multiplication. Division by two is replaced by the square root.

In general, I want them to know that the geometric mean between lower bound *a* and upper bound *b* is:

$$GM = \sqrt{ab}$$

So for our example:

$$\sqrt{(3000\text{ mi}) \times (250,000\text{ mi})} = 27,386\text{ mi}$$

Then, I want to demonstrate to them that multiplying (not adding) the lower bound by a certain number gets us to the geometric mean. Multiplying the mean by that number gets us to the upper bound.

30 x 9.13 ≈ 27,386

27,386 x 9.13 ≈ 250,000

And in general:

$$a \cdot \sqrt{\frac{b}{a}} = \sqrt{ab}$$

$$\sqrt{ab} \cdot \sqrt{\frac{b}{a}} = b$$

In keeping with our earlier talks about precision, our result from using the geometric mean to estimate the circumference of the Earth is far more precise than we have any business being, so let’s round that result to one decimal place. This gives a result of 3 x 10^4 mi, consistent with the order of magnitude estimate and within 20% of the frequently used value of 25,000 mi for Earth’s equatorial circumference.

Learning to be a good estimator comes with practice, because estimates are usually better when you have a known quantity in mind to compare them to. Both of these are reasons why all the warm-ups I’ll do during this unit will be quick estimation tasks. Future estimates can build on past estimates, or students can even perform estimates on the way to the final estimate. I require them to specify the upper and lower bounds and the rationale for their result. When they’re done, they will calculate a percent difference, as this can often put numbers that “feel” far apart into better perspective. I’m very visual with data for my classes, so I’ll make histograms for the class and bin their estimates based on percent difference. That way, the students can chart how much their skills are improving over time. Plus I get to teach them how to use histograms, so it’s a double win.

Dimensional Analysis and Unit Conversion

My class benefits from every student having taken chemistry the year prior, because the unit conversion process that I teach looks an awful lot like stoichiometry. The basic idea of the process is that if we place equivalent quantities in different units in the numerator and denominator of a fraction, the value of the entire fraction is one. Then, when a quantity is multiplied by one, its value doesn’t change. This allows the *units* of a quantity to change while the total *amount* expressed remains the same. As a one step unit conversion, I might ask my class to express 450 feet in meters. One meter is close to 3.28 feet, so we set up a *conversion factor,* in the form of a fraction, and complete the calculation:

$$(450\text{ ft})\times\left(\frac{1\text{ m}}{3.28\text{ ft}}\right)\approx 137\text{ m}$$

What students should see is that the unit of feet appears in both the numerator and denominator, so it "cancels," leaving a unit of meters. This is a one step conversion, because it requires one conversion factor. As a two-step conversion, I might ask: Not counting the end zones, how many rods is a football field? Football fields are expressed in yards, one rod is 16.5 feet, so they'll also need to convert yards to feet so everything lines up:

$$(100\text{ yds})\times\left(\frac{3\text{ ft}}{1\text{ yd}}\right)\times\left(\frac{1\text{ rod}}{16.5\text{ ft}}\right)\approx 18.2\text{ rods}$$

I want them to see that yards appears in the numerator of the first quantity and the denominator of the first conversion factor. Feet appears in the numerator of the first conversion factor and the denominator of the second. Finally, rods, the remaining unit appears in the numerator of the final conversion factor, making it the unit of the result. I'll point out to students that units that they don't want to appear in their answer need to show up in both the numerator and denominator of the original quantity or conversion factors. The power of this technique is that it works just as well with exotic as with mundane units and will serve as the framework for the problem decomposition and solving techniques. Additionally, units never cease to be important. In a physics class, I always think it's a great idea to have my students perform this process, known as *dimensional analysis,* on the equations they're using to make sure they're getting the answers they want in the units they require.

Another way I want to tie unit conversion back to precision is to have my students evaluate approximate conversions in order to simplify them. Conversion factors are well known and easily looked up, but to my students they feel very carved in stone. I want to ask them what percentage error is involved in saying one foot is equal to 30 cm? Or that one mi/h is equal to 1.5 ft./s? Or that one year is $\pi \times 10^7$ s? Then, in what sorts of situations is the approximate conversion good enough? These sorts of questions will fit well into an introductory homework assignment or as a few warm-ups when they first come into my class. The end goal is to help my students get a better feel for how different units relate to each other on an intuitive level, so they can get develop an increased fluency in approximately equating different quantities and expressing their estimates.

Problem Decomposition

When addressing problem decomposition, I steadily keep in mind that I am building students toward the end goal of performing multistep estimates on their own. I have students identify what they're looking for and the unit. Then, I like to have them make a list of any constituent quantities that might help them solve the problem, along with the appropriate units. At this point, it doesn't matter if they list quantities they won't actually need. My students will be started with "one step" decompositions, or questions that split once, cleanly into two parts. As an example, let's examine a one step question: What is the weight of a bathtub full of water in kilograms?

The question specifies units, so we're looking for an answer in kilograms per bathtub. We'll probably need the density of water (kg/L) and the number of liters in a bathtub. (L/bathtub). Students might want other information, like the number of cups in a gallon or the temperature of the water, but when the calculation is arranged, it should become apparent that the units of these other quantities don't fit. Let's construct the skeleton of the calculation that gets us to our result:

$$\left(\frac{\text{kg}}{\text{bathtub}}\right)=\left(\frac{\text{L}}{\text{bathtub}}\right)\times\left(\frac{\text{kg}}{\text{L}}\right)$$

Notice that kg was in the numerator of our answer, so the constituent quantity with kilograms was arranged so that kg was in the numerator. The same attention was paid to "bathtub," which needed to appear in the denominator. We picked our constituent quantities well, so that liters appeared in both numerator and denominator, but if it didn't, we could have added more constituent quantities or conversion factors to get the other units to cancel each other out. From here, the computation is trivial, because we can just put in values for the capacity of the bathtub and the density of water. An average bathtub holds around 150 L of water, so the calculation becomes:

$$\left(150\,\frac{\text{L}}{\text{bathtub}}\right)\times\left(1\,\frac{\text{kg}}{\text{L}}\right)=\left(150\,\frac{\text{kg}}{\text{bathtub}}\right)$$

That's over 300 lbs. of just water in a bathtub!

Let's now apply this to the multi step question of how many miles a person walks in a lifetime. The answer should be in miles per lifetime. The constituent quantities might include how long a person lives (years per lifetime), how much time a person spends walking in a normal day (hours per day) and how fast a person walks (miles per hour). Notice that this doesn't quite work out:

$$\left(\frac{\text{miles}}{\text{lifetime}}\right)\neq\left(\frac{\text{years}}{\text{lifetime}}\right)\times\left(\frac{\text{hours}}{\text{day}}\right)\times\left(\frac{\text{miles}}{\text{hour}}\right)$$

Lifetime and miles appear where they should and hours appears in both the numerator and the denominator, but years and day are left. What we can use is a conversion factor, with days in the numerator and years in the denominator, to make sure the units agree.

$$\left(\frac{\text{miles}}{\text{lifetime}}\right)=\left(\frac{\text{years}}{\text{lifetime}}\right)\times\left(\frac{\text{days}}{\text{year}}\right)\times\left(\frac{\text{hours}}{\text{day}}\right)\times\left(\frac{\text{miles}}{\text{hour}}\right)$$

It's also important to note that this isn't the only way to get to a valid answer. Someone in my class might argue that it makes more sense to estimate how many miles you walk in a day directly or that a person's walking speed is nowhere near a steady pace. Beyond that, the results will vary pretty substantially from person to person. The beauty of this type of problem is that different approaches and results are expected, as long as the thought process is justified.

To implement this in my class, I will start with an example or two that we handle together. I'll ask students what they might need to know in order to answer the question and compile the list on the board, being sure to pause and ask for units. Then I'll reference the unit conversion procedure with students for guidance on how to arrange the units to set up the calculation skeleton. After each calculation is set up, students will estimate (or look up) the values of each piece and complete the calculation as a group. Starting them with the one-step examples, I'll work them up to more complex questions that add more quantities. My main focus here is making sure they are attending to all of the skills and necessary justification that they'd developed along the way.

Classroom Activities

The following activities are modular frameworks and discussion strategies that I plan to use with each of the topics outlined above.

Estimate and Measure

Especially at the beginning, I want to have my students estimate things that can be measured. If they're trying to estimate the length of a hallway, the groups that finish early can grab a few meter sticks and go take the measurement. If they're trying to estimate the population of a city, I'll have them look it up. Even if it's the number of jellybeans in a jar, have someone count on the spot so they can quantitatively evaluate the quality of their estimate by calculating the percent error:

$$\%\ \text{error} = \left| \frac{\text{actual value - estimated value}}{\text{actual value}} \right|$$

When they calculate the percentage error, they can get a feel for how close their estimate was and then begin to discuss with other groups how their assumptions and justifications differed. When they reflect on and adjust their estimates, they should become better estimators. The instant feedback of estimating a measurable (or verifiable) quantity should also ground the activity in reality. Estimation is a skill that requires practice to develop, so I want to be sure my students have ample opportunity to not only practice, but also to self-check their results.

Think-Pair-Share

One of the first student engagement techniques I learned was the Think-Pair-Share, where students complete independent work, share it with a partner and then share it with a larger group. For this unit, it works exceptionally well. I'll have students complete whatever estimate, decomposition or problem they need to and then pair with another student to compare results and revise their work. I'm big on setting up norms for discussions and group activities like this, so I'll post and review guidelines in advance. The difference here is that they don't have to agree, as long as they are respectful of each other's work. Since there can be more than one correct path to a solution, they should be presenting and working through each of their thought processes instead of just agreeing on one right answer. Once they've developed their ideas with a partner, I'll have them join a larger group and "pitch" their ideas. Then they'll develop all the solution paths that are proposed. After I've allowed some time for this, have the entire class come together and share their results, while students identify similarities and differences in their own approach. The goal is to have my students pursue their intuition with the support of other students who may not have looked at the problem from the same angle while also having those who are comfortable with the techniques support those who are less comfortable. Having them pair first also gives students multiple chances to talk through their ideas, which can make their self-editing process easier.

Facilitating Discussion/Pondering

Holding whole class or large group discussions in science and math classes can be challenging, because students are really tied to the idea of right and wrong answers. Especially with estimation tasks, results are more subjective and rationales are open for debate. In my physics classes, I frequently use qualitative questions, referred to as *ponder questions*, and public debates to help students better understand material. The debate format must be taught at the beginning of the year, but serves to help draw students into the discussion by separating out just their ideas for consideration.

First, the questions are *pondered* (hence the name) a few nights in advance, so that all students come to class with something to contribute or discuss. One or two students are selected for each question to lead and moderate the discussion. They begin by reading the question and then clarifying what it is asking for in their own words. They then present only an answer, with no explanation. Other students are then allowed to contribute their answers, without explanation, as the moderator records them on the board. Once the answers are collected, they are numbered and referred to as "solution one" or "theory two" or "explanation four." Then, one by one, someone who agrees with each answer goes through their solution, rationale and justification, attempting to convince the rest of the class they are correct. Other students may raise their hands to ask questions and the discussion continues until the class reaches consensus on the validity, or lack thereof, for each solution.

As the teacher, my first job is to teach the format and make sure students adhere to it. Second, I want to make it clear to the students that the ideas are what is being discussed, debated and debunked, not each other. Allowing anyone to argue for or against an idea presented by anyone else (and numbering them) helps to depersonalize the discussion. Third, I want the discussion to be student generated and student centered. Frequently this will mean large periods of waiting. Just remind students they are being evaluated on their participation and contribution to the discussion. It can feel really time consuming, but ends up being one of my favorite class activities. My students get really into it once their comfort level grows, as well. For this unit, I'll use this discussion format to go over homework problem sets and also to facilitate class solutions to challenging questions.

Poster Talks

Another of my favorite activities is having groups of no more than three students work together to solve a problem and make an informal poster to visualize their approach. I do this frequently in my class, so I like to have a large roll of butcher paper and bold colored markers for this very purpose. I bought my butcher paper cutter and roll online and found that despite the larger initial buy-in, the butcher paper tends to be around five times cheaper per square foot than chart paper pads. In our 48 minute class periods, my students have enough time to come in, do a warm up, be assigned groups, select questions and finish their posters. I store these overnight for presentations on the following day. With my own kids, things that go home frequently don't come back, so if they need more time to work it must be done in school during free periods. For me, an interesting and efficient way to assign questions is by passing out a set of three cards with a different question on each to a group and having them select one to solve. The remainders can be reshuffled and dealt to later class periods. For four class periods with ten groups in each, I'll prepare 60 different question cards. That way every group gets three questions to pick from and every group also gets a unique question.

In general, I evaluate my students' presentations using a Common Core-aligned argumentative writing rubric, with a focus on clarity, organization and justification of ideas. In Pennsylvania we use the Keystone Argumentative Writing Rubric, but whatever is locally mandated should be fine. Be sure to communicate expectations up front and model good presentation skills. I find this works best when there is an understanding that the talks are short and informal, but not casual. Personal rules I have regarding presentations include severe penalties for reading off of a poster or slide and additional deductions for rambling to fill time. Clarity and efficiency when communicating ideas are what I hope to develop in them. I find the most enjoyment and learning comes from a short and informative presentation followed by a question and answer period where students are prompted to expand upon and justify their methods and results. Other students should be encouraged to ask respectful questions. Since every group had a unique question, I like to display them in the hallway outside my class to inspire discussion across class periods.

Appendix

Example Problems

Key to this unit is the selection of problems, of which I've specified four classes below. Estimation problems and decomposition problems each have two levels that are differentiated by the instructions supplied and what the students are asked to do.

One-Step Estimation Problems

Estimate the order of magnitude of the requested quantity. Justify your estimate in a sentence or two describing any observations, assumptions or prior knowledge you used.

Estimate the requested quantity by setting upper and lower bounds and finding the geometric mean. Justify your estimate by describing any observations, assumptions or prior knowledge you used.

1. What is the height of the school in meters?
2. How many skittles come in a bag?
3. How much does the teacher weigh?
4. How many jellybeans are in a jar?
5. How high above the street is a traffic light?
6. How many seconds have you been alive?
7. How fast do you walk? One mile per hour is about 0.5 m/s and about 1.5 ft/s.
8. How much does a stack of pancakes weigh?
9. If you woke up late, what's the fastest you could get ready and get to school?
10. How much water could this (oddly shaped real-life) bowl hold?

Unit Conversion Problems

Express the requested quantity in terms of the specified unit. Show the conversion calculation and justify your answer in a sentence or two.

One Step:

1. How many minutes is 10^3 seconds?
2. How many centimeters are in a kilometer?
3. How many feet are in 60 miles?
4. A man weighed 180 lbs. What is his weight in kilograms?
5. A large coffee is 20 oz. How many liters is it?
6. Six nanometers is how many millimeters?

7. If an average double spaced essay has 250 words per page, how many pages is a 5000 word double spaced essay?
8. Everest is 8.8km tall. How many feet is that?
9. How many meters tall is the 973 ft. tall Comcast Center in Philadelphia?
10. A typical housefly can live up to 30 days. How many hours is that?

Multi-Step:

1. Human hair grows at a rate of about 6 inches per year. How fast is that in meters per second? How many meters per second are all the hairs on your head growing in combined length?
2. An NBA court is 94 ft. long and a standard basketball is 25 cm in diameter. How many basketballs long is a basketball court?
3. How many days is 10^6 seconds? How many years is 10^9 seconds?
4. The speed limit on US roads is 65 mi/hr. What is that in meters per second?
5. The longest game in Major League Baseball history was 8 hours 25 minutes. How many seconds long was it?
6. The Suzuki Hayabusa was once the fastest production motorcycle, traveling at 194 miles per hour. What is this speed in meters per second?
7. A "stone" is a unit of measure equal to 14 lbs. If a half-ton pickup can transport a half-ton of cargo, how many stones can it transport?
8. Abraham Simpson drives a car that gets 40 rods to the hogshead. If a hogshead is 63 gallons and a rod is 16.5 feet, how many miles per gallon did his car get?
9. A man is 5'10". How many meters tall is he?
10. If a car tire is 60cm in diameter, how many times does it rotate every mile?
11. Compare the population densities of India, the United States, China and Canada.

Decomposition Problems

Specify the unit on the requested quantity. Then, decompose each quantity into its constituent quantities and specify their units.

Estimate the answer, showing the constituent quantities used to arrive at your estimate. Justify your answer.

One Step:

1. What is the orbital speed of the Earth as it travels around the sun?
2. What is the volume of the Earth?
3. If you turned it all into one-dollar bills and stacked it, how tall would a billion dollars be?
4. How many people are eating dinner right now?
5. How long would it take to drive to the moon?
6. How many hairs are on your head?
7. What percentage of Earth's total mass is people?[3]
8. How many functioning ballpoint pens are in the school right now?
9. How much would it cost to survive on ramen noodles for one year?[4]
10. How many blades of grass are on a football field?

Multi Step:

1. How many football fields could you cover with all the pizza the school cafeteria serves in a year?
2. What's the weight of the atmosphere?
3. How massive is a mole of tennis balls? How does this compare to the mass of the Earth?
4. How many cubic meters of trash does Philadelphia generate in a year?
5. During a big California earthquake, two million books fell off the shelves at a university library. How many students would need to be hired to reshelf all of the books in three weeks?[5]
6. Picture a yellow school bus filled with high school students. By what factor do the weight of just the students and the weight of just the bus differ?
7. How much would the ocean's surface rise if the Antarctic ice sheet melted?
8. How much land area would be needed for solar panels to provide the United States with all of its power needs?
9. If you were to convert Wikipedia to a physical encyclopedia, how thick would it be?
10. How long will it take until the surface of the Earth is entirely covered in gravestones?
11. If everyone in the world went swimming in Lake Michigan, how much would the water level rise?
12. Compare the daily increase in the human population with the total population of various animals – e.g., lions, tigers, rhinoceroses, elephants and bison.

Content Standards

The Next Generation Science Standards call for a shift from "skills" to "practices" that are germane to science and engineering and should be engaged in by students across all grade bands[6] . Eight practices are outlined in the NGSS, of which this unit aids development in four. By solving these problems, students engage in the first practice of asking questions by not only asking themselves what they need to know, but also by asking what the answers might mean. Students will also use the fifth practice, using mathematics and computational thinking, increasing their fluency in the types of calculations that are common in science and engineering. By having students justify, report on and discuss their calculations, they will be engaging in the seventh and eighth practices of engaging in argument from evidence and obtaining, evaluating and communicating information.

For a mathematics classroom implementing the Common Core Standards, this unit addresses two High School Standards explicitly. Under the High School standards for Number and Quantity, the Quantities heading[7] specifies that students reason quantitatively and use units to solve problems. The very approach outlined has students using units as a way to understand problems and guide the solution of multi-step problems. The Common Core also asks that students be able to define appropriate quantities as well as choose a level of accuracy appropriate to limitations on measurement when reporting quantities. All of these are integral to the approach students will take when solving these problems.

Modeling gets its own heading in the High School Common Core standards[8] , and their description of the skill and its application embodies what students are being asked to do in this unit almost exactly. The Common Core emphasizes choices, assumptions and approximations, which I've set out to focus on and develop explicitly within the problems for this unit. Modeling goes beyond what this unit asks students to do, however, to draw in graphical and statistical tools. It's my hope that solving these problems will provide students with a solid foundation that will allow them to incorporate and understand these other tools with more depth at a later date.

The final notes on standards come from the English and Language Arts. Under the Common Core, there is an upgraded emphasis on argumentative writing[9] . Teachers of all subject areas are using these standards to improve student communication and writing of non-fiction and technical texts. Having students write justifications for their reasoning sounds small, but forces them to document their thought process for others in a way that they more than likely don't frequently practice. The Common Core also contains a set of Speaking and Listening Standards[10] that ask for participation in discussions and giving presentations in a way that I find extremely productive and helpful to developing and cementing student understanding of their work.

Annotated Bibliography

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Problem Collections

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Notes

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