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Beyond the Number Line: Coordinate Systems and Vector Arithmetic

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Introduction

If you ask a student of physics what a vector is, you can expect the standard answer: "a quantity with a magnitude and direction." They may even have an innate notion of why we care about direction so much in physics, but a deeper understanding of what vectors are and what they represent is something that takes experience and insight to develop.

Since the mathematical foundation of introductory physics is based on the concept of vectors, doing justice to physics requires using vectors and using vectors requires that students learn some "new math." Here lies the challenge: How can I teach the foundational math in a way that is more intuitive and accessible so that my students' conceptual understanding is not limited by a computational bottleneck?

My students arrive having never really been held truly accountable for considering units. They do not have much experience making measurements. Quantities they have studied are typically only related to countable amounts of objects, areas and volumes. Physics cares about numbers in lots of exciting ways that students have never had to care about them before. Positions change as time elapses, and the way you describe a space dictates how you observe it and how you quantify the observations made.

The purpose of this unit is to provide a language and understanding of number lines as coordinate systems we can construct and place at will in the physical world. Units are intrinsically tied to the coordinate systems we create and measurements are made against the coordinate systems in multiples of the units. More than anything, I want my students to realize that arithmetic operations *change* things, and understanding physics involves understanding of stretches, sides and spins applied to objects in our world.

Demographics

I work at a small magnet high school with an incredibly diverse student population. Despite being a magnet with academic entrance requirements, it is not atypical for students to make it to 11th grade lacking expected math skills. With the exception of having one section of AP Physics 1, our school does not track students for my classes, so I have a full range of abilities in each classroom. In addition, the population of students I teach is indicative of the demographics of Philadelphia as a whole. I have a large number of recent immigrants and English language learners, both from Southeast Asia and Latin America. The wide range of backgrounds and skill levels I have in my room on any given day is always in my mind as I plan.

This is designed to be the second unit for my 11th grade General Physics and AP Physics 1 classes. My school is set up so that physics is the required science course for all juniors. Over two thirds of my 150 students will be concurrently enrolled in Algebra 2. The remaining one third will be in a Precalculus class. Of the students in Algebra 2, a small handful will be simultaneously repeating Geometry. Of the students in Precalculus, twenty or so might also be in AP Statistics. Because of this huge variety, I am targeting students who have passed Algebra 1 and Geometry and allowing room to differentiate my instruction toward students who come in above or below this mark. It is also important to note that this unit does not depend on any specific physics content knowledge, with the exception of some connections to introductory kinematics. The unit, with little modification, could also fit well into a Geometry, Algebra 2 or Precalculus class offered at the high school level that is concerned with grounding mathematics in the physical world, measurement or transformations.

The final demographic consideration is that I teach in the Philadelphia School District, which is the eighth largest school district in the country, serving the entire city of Philadelphia. The financial state of the district is such that schools function with operating budgets so slim that basic needs go unfilled on a daily basis. Realizing this as my own reality and the potential reality for other teachers, the activities and strategies are designed to be free or extremely cheap.

Rationale

Physics takes place in three-dimensional space where many fundamental quantities are represented by vectors, which are mathematical quantities that have a magnitude (or length) and a direction. For the introductory Newtonian formulation of mechanics, vectors provide the mathematical framework. Any displacement, or change in position, can be thought of as a vector pointing from the initial position to the final position. Force is a vector quantity, as are acceleration, velocity and momentum. Further down the line, their rotational counterparts are all also vector quantities. Dealing with all of these quantities requires us to pay particular mind to their directions, but what is the mathematical context that surrounds a high school junior enrolled in a physics course? How can the treatment of vectors be best linked to concepts already present in a student's math background?

My guess is that the best place to make the connection to their previous math education is the long abandoned number line. Students have not dealt with them in years, but the tidy row of numbers that was pasted high on the wall in every elementary classroom forms the very foundation of a one-dimensional coordinate system. The work of Galileo, Newton and Einstein was deeply focused on the idea of the reference frame. The coordinate system that is tied to an observation provides the information needed to translate measurements to other, equally viable, frames. Constructing coordinate systems from nothing and then making measurements within them are the processes that allow the quantification of observations, an essential tool for any student of physics. Without the understanding that physics is done relative to invisible number lines we construct for ourselves to help communicate observations, concepts of measurement and motion do not become intuitive to students.

The first time students are exposed to vectors is typically in a geometry class, which most students in my school take as sophomores. Geometry seems like the obvious place to take an extremely pictorial approach to vectors, but they are usually just only implemented to describe translations in the plane. This is natural and important, but physics requires more. The text we use at our school represents vectors in a component notation, , that is meant to resemble an ordered pair. Students spend a week or so realizing that the vector moves the vertices of a shape *a* units in the *x*-direction and *b* units in the *y*-direction. Then, as far as students are concerned, the word vector is promptly forgotten and not used again.

I want to pick up my treatment of vectors from here. Viewing them as translations in the plane fits with what their implementation as states and changes. Getting them to this point in an efficient manner is the challenge. This will require a reimagining of the front matter of the course related to units and measurement. It seems that most of things are usually done in a very specific, and consequently not transferrable, context. Measurements are only made with a meter stick and a stopwatch in standard units. Coordinate systems are pre-prescribed instead of generated by students. Unit conversion is done in a way that feels like an arbitrary worksheet exercise as opposed to a necessary step in communicating information.

To do this differently, I want to look back toward elementary explorations of the number line, motion along it, and concepts of measurement. The purpose of dredging up these old sense-making tasks is to exhibit the unspoken connection between numbers in math class and quantities in physics. The learning my students do is often very isolated and they are frequently unable to connect concepts they have learned in math to things we do in physics unless the relationship is built up for them.

Once they have got this mathematical language to describe quantities, it is time to bring transformations into the picture. Composition of translations is vector addition. For students the connection is evident because they move quantities, but preserve scale. Dilation from the origin is multiplication by scalars. Students can see this because they stretch the vector and change the unit length, but preserve orientation (or at least restrict it to the original direction or the exact opposite direction). Seeing how quantities relate and change is at the heart of the study of physics and it is my hope that this dynamic representation of arithmetic will help students better visualize and understand physics.

Background

This unit is divided into three sections, with each section building upon the previous ones. Conventions of set theory and transformations will be used to demonstrate concepts generally, with direct applications to physics given at the end of each section.

Coordinate Systems and Measurement

For the first section, we seek to establish the number line as the foundation of a one-dimensional coordinate system, make measurements within that coordinate system and then geometrically interpret arithmetic operations as measurements along the line.

Coordinate Systems and Their Construction

To do physics in one dimension, we need to start with an appropriate one-dimensional coordinate system. To construct this, we will begin by considering a straight line, which we will call *L*. To understand the structures of the line and operations on the line, we must consider the important geometric features of *L*.

The first of the two foundational features is order. Points that lie on the line are ordered, and this ordering it *total,* meaning that any two points are comparable. For example, if you have two different points *p* and *q*, one of them is greater than the other.

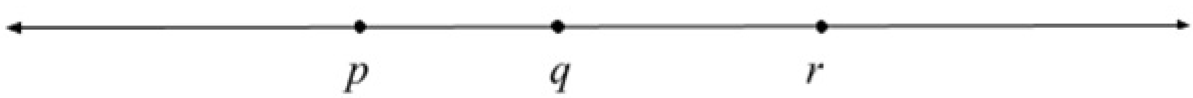
$$q > p$$

This ordering is also transitive. If another point *r* is greater than *q*:

$$r > q$$

Then its relationship to *p* is implied:

$$r > p$$



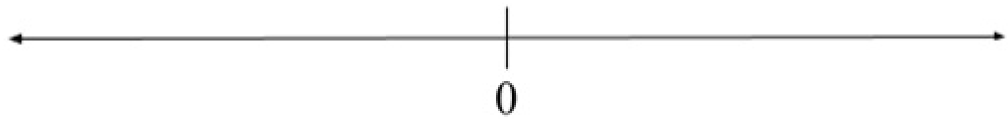
As a consequence, this also establishes the idea of orientation, or direction, along the line. This orientation corresponds to the sign of numbers, such that numbers greater than 0 are positive and numbers less than 0 are negative. Successively more positive points extend out in the positive direction while successively less positive points extend out in the other, negative, direction.

Order alone is not enough for us to build a viable coordinate system however, because the line itself could to be stretched and squeezed locally while still preserving the order of points. The second feature we need for the line is the inherent notion of distance. Labeled points on the line have a physical separation that can be compared to (and combined with) other separations. On the line, if *r* is between *p* and *q* (that is, $p < r < q$, or the reverse), and *D(p,q)* is the distance between *p* and *q,* then:

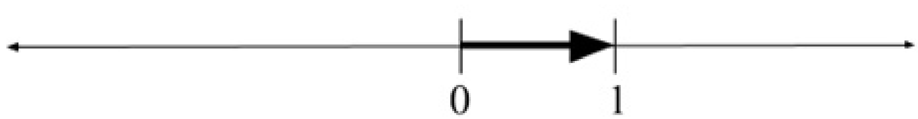
$D(p,r) + D(r,q) = D(p,q)$

It is the combination of order and distance, along with origin and unit that allows us to coordinatize the line. To do so, we begin from our straight line, *L*:

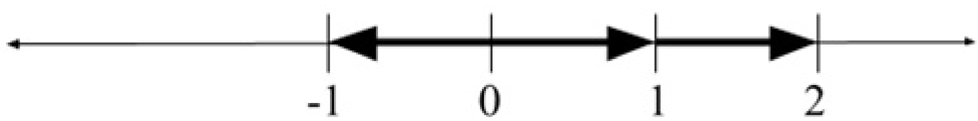
To construct a coordinate system on *L*, that is, to turn *L* into a number line, we only need to carry out two steps. The first is to choose an origin. This point represents the zero of our coordinate system.



The second is to choose a unit distance, or the distance between 0 and 1.



Given these two choices, all other points on the line can be labeled. If you tick off an additional unit distance beyond one, you arrive at the point 2. As you continue adding unit distances, the numbers will be more and more positive. If you travel a unit distance in the opposite direction from zero, you arrive at the point -1.



In general, the number labeling any point on the line is the ratio of the distance of that point from 0 to the unit distance, with the sign of the number indicating direction. If that sign is positive, it is in the same direction from 0 as 1. If that sign is negative, it is in the opposite direction.

Every point on the line can be labeled, not just the whole number multiples of the unit distance. All points in between whole numbers can be labeled using rational numbers, decimal expansions or other processes. Regardless of how a point is labeled, the absolute value of the number that labels a point is the ratio of the distance of that point from 0 to the unit distance and the sign of the number indicates the direction from the origin.

Units, Measurement and Precision

If the process of selecting an origin and unit interval seems arbitrary, that is because it is arbitrary. It does not matter where we choose our origin, and it does not matter what the unit distance is, it only matters that the unit distance is established and agreed on in some way. In physics, we have a set of units that we typically like to perform our measurements in, such as meters, kilograms and seconds. That does not mean those are the only viable units, they have just been selected for the sake of convenience and consistency. As long as the unit is specified, measurements can be made using any coordinate system.

The Measurement Principle relates positions on the number line to distances. Formally, the Measurement Principle states: the number labeling a point on the number line tells you the distance of that point from the origin, as a multiple of the unit distance. For example, the point 4 is four times the unit distance from the origin. The Measurement Principle assures that whole numbers are evenly spaced along the number line and dictates where the numbers in between them are placed. Even though the line has order, it is the Measurement Principle that allows us to assign each number to a physical location.

Once constructed, the number line allows us to assign a length to any line-like object. To do so, place the object alongside the number line, with one end lined up with 0. Then the number labeling the point where the other end of the object falls is its length. For example, if the unit distance is 1 cm and a pencil stretches from the origin to the mark labeled 12, that means the pencil has a length of 12 times the unit distance, or 12 cm. While that seems obvious for a ruler, it gives us a clear rule to use to coordinatize the line and an explicit meaning to the numbers that label it.

It is important to have students consider physical limits on precision. When considering a measurement principle, students will naturally connect the number line to a ruler. Even though the number line allows us to label all points using whole numbers, fractions, decimal expansions and rational numbers, a physical tool like a meter stick does not afford the same luxury. If you restrict points on the line to whole number multiples of the unit (or particular labeled fractions), the construction of a physical measuring device puts a hard limit on the precision available when making measurements with it. Specifically, the size of the unit length is going will be a limiting factor. If you take a ruler with a unit length of 1 cm, the measurements that can be made will either be a whole number of centimeters “plus a little more” or “a little less than” a whole number of centimeters. Realistically, that slop at the end of the measurement can be up to half a centimeter. If you refine the unit interval to one millimeter, the maximum size of the “little more” or “little less” is now a tenth of what it was before. Scientists express this as an estimated digit at the end of a measurement.

Measurement Interpretation of Arithmetic

Lengths naturally lend themselves to being combined and compared. This even happens before numbers are defined on the line, because combination and comparison are inherently nonnumeric operations. If you have two pieces of rod, you can put them end-to-end and see the total length. If you would like to see the difference in length between two rods, you can lay them side-by-side and measure the difference. All of this might feels very elementary to students as they do it, but is actually quite subtle and forms the basis for further understanding of the numeric and symbolic representations of arithmetic.

For addition, the picture is easy. Addition is often seen as an operation that combines amounts. Lengths are very simply combined when there is a number line to measure them against. The sum of the lengths is the length of the sum, that is to say the total length of any collection of objects is the sum of the individual lengths. How do we interpret 5 + 7 in this picture? Start from the origin and lay a rod of length five to the right. From the end of that rod, lay an additional rod of length seven. The final sum can be found by measuring the distance from where the rods start to where the rods end and reading off the measurement of the total: 12 units.

Subtraction is slightly more complex. It is frequently viewed as a comparison operation, and it can fill that role. Especially in elementary grades, subtraction can also “take away.” It can also do that. To perform the operation 7 - 5, students may want to take their seven unit rod and their five unit rod and lay them next to each other and see that the difference in size is two units. Along the number line, they would lay the seven unit rod with the left end on the origin and the other end extending to the right. They would then take the five unit rod and line up the right side of it with the right side of the seven unit rod and look at where the left end falls on the number line: two units.

For my juniors, I want them to have the idea that these are merely interpretations of subtraction in a context, not the nature of subtraction in and of itself. Subtraction is, operationally, the addition of the negative. Since positive and negative are defined as specifying direction, we have to care now about how we orient the lengths we combine. To compute 7 - 5, we first consider the addition 7 + (-5). Begin with the left end of the seven unit rod at the origin, allowing the length to extend to the right because it is positive. The five unit rod begins where the seven unit rod ends and the length of it extends to the left, because it is negative. The distance is measured from where the seven unit rod starts to where the five unit rod ends and the result is read off as a position on the number line. Just as above, the result is two units.

Multiplication can be thought of with a similar representation, where 5 x 7 would involve laying five copies of a rod of length seven end-to-end starting from the origin and extending to the right. The result is found by measuring the total length. Division can be similarly represented as the partitioning of a rod, so that 24 ÷ 4 would be breaking a rod of length 24 into four equal pieces and measuring the length of one section. For this unit, however, these representations are significantly less useful than the purposed representation described below of multiplication as a dilation along the line, despite the obvious applications of the measurement interpretation of addition and subtraction.

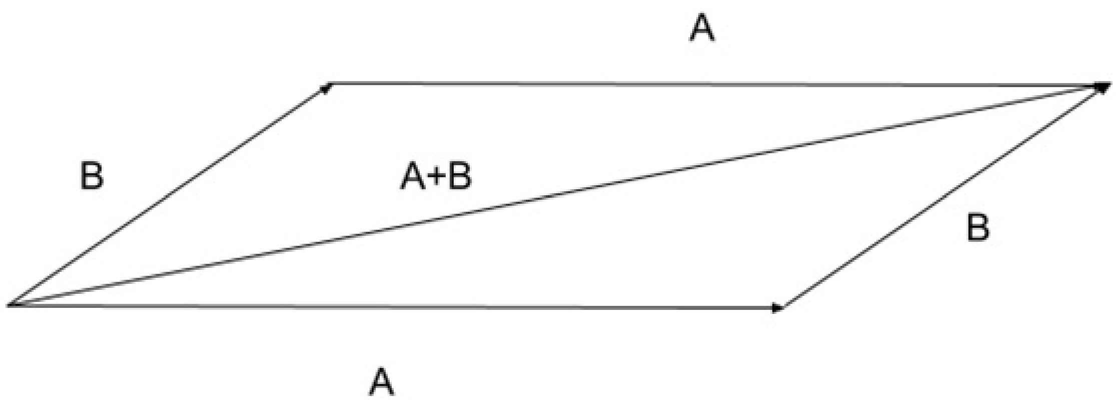
Vectors and Translations

After students have mastered measurement and the establishment of coordinate systems, it is time to explicitly teach the difference between scalar and vector quantities as well as the representation of arithmetic operations as translations along the line.

Vectors

In physics, we define vectors to be quantities that have a *magnitude* (or amount), a unit and a *direction.* In one dimension, simply specifying whether a number is positive or negative is tantamount to providing a direction, so the distinction between “number” and “vector quantity” might not carry a lot of weight for students yet, but many of the major physics quantities, like displacement, velocity, acceleration, force and momentum are all vectors. Vectors are frequently introduced visually and the standard picture we use to represent a vector is an arrow. The length of the arrow is the magnitude and the arrow points in the direction of the quantity that it represents. When referring to vectors symbolically, we typically put an arrow above the symbol for the quantity to specify that it is a vector and the direction is something that matters. For example, $\vec{A}$, $\vec{B}$, $\vec{C}$ and $\vec{x}$ would all be symbolic representations of vector quantities.

Vectors are similar to lengths, in that they are things that naturally want to be added together. This is extremely easy and intuitive to do so using measurement representations of arithmetic. Before, lengths were laid end to end in order to measure the sum. Similarly, vectors are aligned “tip to tail,” meaning that the tip of the first vector touches the tail of the second. It is important to preserve the direction of each individual vector during this process. When performed correctly, the tail of the first vector starts at the origin of your number line and the tip of the final vector in any additive series will live at the sum. Draw a new vector, known as the *resultant,* from the origin to the tip of the final vector in the additive series. When subtracting two vectors, the direction of the second vector is reversed and the addition process is followed. Remember that subtraction is the addition of the negative and realize that reversing the direction of a vector creates its negative. In one dimension, students should be able to grasp this relatively quickly, since it is relatable to the measurement interpretation of addition, only rods have been replaced with arrows. The figure below demonstrates what is often called the “parallelogram rule for addition.” Seeing that the order in which we combine vectors *A* and *B* does not matter, we demonstrate the commutativity of vector addition.



Transformational Interpretation of Arithmetic

Where this unit begins to diverge from conventional presentations in a physics classroom is in the interpretation of arithmetic operations as transformations along the number line. The usual geometric picture of vector addition implicitly assumes that we can translate a vector without changing its length or direction, which allows us to create the standard visual associated with adding vectors. Translating one of the vectors by the other creates the sum. Since vectors in an introductory physics course represent quantities that are associated with motion, the connection is meant to appear naturally. Let us consider first the result of translating an arbitrary point *x* by *a*, written $T_a(x)$. What T_a will do is shift all points on the number line by *a* units. If *a* is positive, it will shift to the right, and if *a* is negative, it will shift to the left.

$T_a(x) = x + a$

This illustrates how the translation T_a, takes any point *x*and moves it to the new position $x + a$. Since all points on the number line are shifted by the same amount, we see that translation preserves the distance between two points. Transformations that preserve distance are known as *isometries*. Isometries also preserve *signed distance,* or a distance that also specifies a positive or negative direction. On the line, this is the same as vector difference. To demonstrate this, let us apply T_a and observe the signed distance between the points *p* and *q* before and after. Before transformation, we find the signed distance, D̀, through subtraction.

$\vec{D}(p ,q) = q - p$

Apply the transformation to both points.

$T_a(p) = p + a$

$T_a(q) = q + a$

Now we will calculate the new signed distance to see that agrees with the previous result.

$\vec{D}(p + a, q + a) = (q + a) - p + a)$

Apply the associative rule:

$D^{\rightarrow}(p + a, q + a) = q + (a - (p + a))$

By the sum rule for inverses:

$D^{\rightarrow}(p + a, q + a) = q + (a + ((-p) + (-a)))$

By the commutative rule:

$D^{\rightarrow}(p + a, q + a) = q + (a + ((-a) + (-p)))$

By the associative rule:

$D^{\rightarrow}(p + a, q + a) = q + ((a + (-a)) + (-p))$

Apply the definition of inverse:

$D^{\rightarrow}(p + a, q + a) = q + (0 + (-p))$

Apply the identity rule:

$D^{\rightarrow}(p + a, q + a) = q + (-p)$

Finally use the definition of subtraction:

$D^{\rightarrow}(p + a, q + a) = q - p$

What does a translation do, then? It adds a fixed quantity to each and every point on the number line, mapping it to a new point. Because distance and scale are preserved, translations are *isometries*: they do not alter the units of a quantity. With vectors, a displacement vector takes the position of an object and maps it to a new position. A displacement behaves like a transformation, in the sense that each represents a change. The difference is that displacements are a change in position of a particular point, while transformations are a change in the entire line: every point is being displaced in the same way at the same time.

The second type of transformation to consider is dilation. A dilation, d_m, will take every point on the line and multiply it by the constant factor m. Symbolically, this is written as

$d_m(x) = mx$

Dilation is not an isometry. To demonstrate why this is the case, let us again look at the distance between points p and q before and after the dilation.

$d_m(p) = mp$

$d_m(q) = mq$

Calculate the distance between the new points.

$D(mp,mq) = mq - mp = m(q - p)$

Notice now that the new distance between them is *m* times the old distance. Dilation stretches all intervals on the line by a factor of *m*. Because it does not preserve distance, dilation results in a change of units. The distance between zero and one is no longer the same as it was. It is *scaled* by a factor of *m*.

This is precisely what occurs during the multiplication of a vector quantity by a scalar quantity. Scalars are quantities that have numbers and units, but no direction. Multiplying a vector by a scalar transforms it to a new number line with different units, distorting its length in the process. Dilation of the line is the same thing as a rescaling of the line.

Applications to One Dimensional Kinematics

The purpose of considering addition of vectors and multiplication by scalars as transformations of the number line is to add an additional level of visual and spatial understanding to students' understanding of the way quantities combine in physics. To illustrate the idea, first consider displacement.

$D\vec{x} = \vec{x} - \vec{x}_0$

$\vec{x} = \vec{x}_0 + D\vec{x}$

An object's displacement is equal to its change in position, or, stated differently; an object's final position is equal to its initial position plus its displacement. When we look at the physical operation performed by displacement, we can view it as a translation of the object by$D\vec{x}$. Adding the displacement vector to an object's position translates the object's position by$D\vec{x}$ and leaves the unit untouched. To use the notation of transformations, we can write:

$\vec{x} = TD_{\vec{x}}(\vec{x}_0)$

Now consider the displacement undergone by an object moving with a constant velocity over some time interval.

$D\vec{x} = \vec{v}t$

While we say vectors want to be added, for this to work they must refer to the same unit. Positions and displacements, typically, are measured in meters while velocities, typically are measured in meters per second. These are incompatible! Velocities can never be added to displacements. What multiplication by the scalar quantity *t* does, however, is dilate the velocity vector onto a number line where the fundamental unit is the same as that of displacement, allowing it to be treated as a translation of the position vector.

$D\vec{x} = d_t \vec{v}$

$$\vec{x} = T_{\delta_t \vec{v}}(\vec{x}_0)$$

To develop this idea this even further, consider the position equation for a particle undergoing constant acceleration.

Students might be tempted to draw a picture of the particle in the problem, label the velocity and acceleration vectors and then somehow combine them. If they instead view this as two successive translations, first by $\vec{v}t$

and then by 1/2 $\vec{a}t^2$, they can see that while both terms of the equation carry separate physical meaning, but they still translate the position of the particle. Then each piece of the translation can be described by a vector. Even though each piece is based around a velocity or acceleration, they have been dilated by an appropriate factor of time, producing units of a displacement vector. Attending closely to this issue of units separates vectors with incompatible units and reminds students to only add vectors that are expressed in the same unit. Following that line of logic, the standard kinematics equation could then be restated in the notation of transformations as:

The notation is obviously extremely cumbersome, but highlights an essential insight about vectors. Viewing the vectors as transformations emphasizes the impact of physical quantities on each other and provides an intuitive framework by which they can interact.

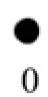
Two Dimensional Extensions

We have seen how the number line effectively constitutes a coordinate system on the line, provides the framework for measurement and grounds our visual representations of vector addition. The language of transformations provides another way to consider vector addition and scalar multiplication that works with our understanding of the number line, not against it. Fortunately for the world (and somewhat unfortunately for our students) physics is applicable in more than one dimension and these concepts can be extended to provide an intuitive two-dimensional treatment of vectors.

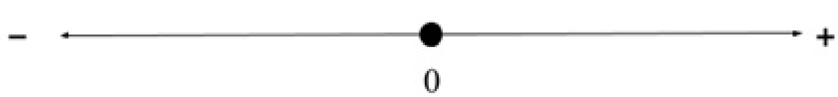
Construction of Two Dimensional Coordinate Systems

When we want to examine motion in the plane, a one-dimensional coordinate system as previously constructed will no longer suffice. That does not mean we ought to disregard our prior work. Indeed as gradually realized during The Middle Ages and finally crystallized by Descartes, we can describe the plane using two number lines.

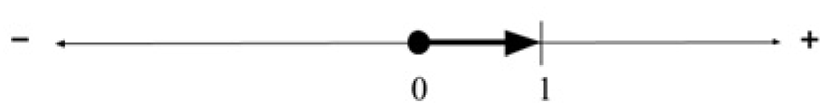
To construct the plane, we begin by choosing a point for the origin.



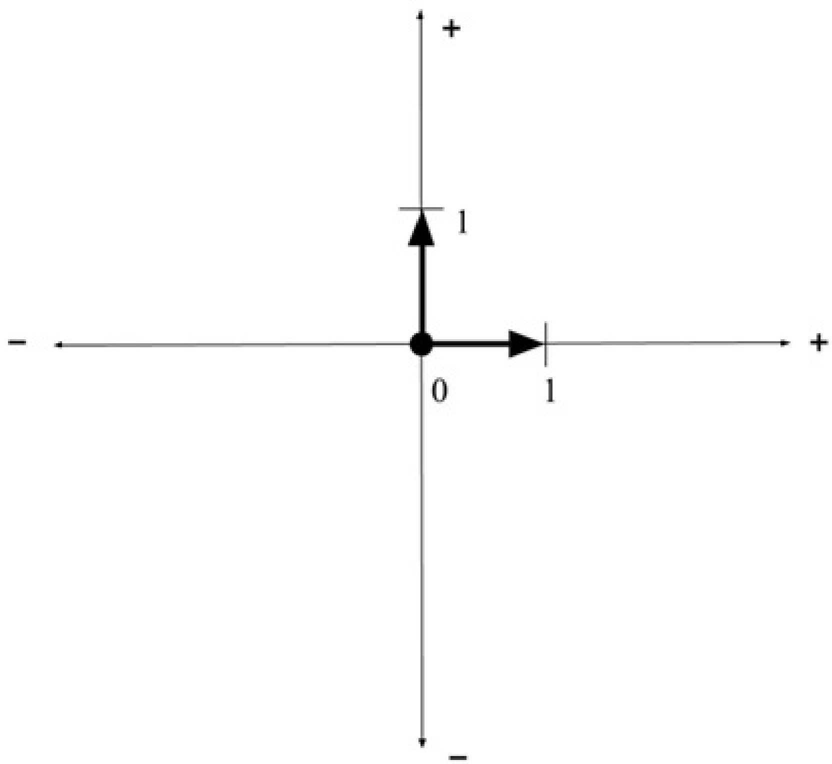
Next, we choose an oriented line going through the point for the x-axis.



Next, we select a unit distance on the x-axis to set the scale.



Next we construct a second line, perpendicular to the first one at its origin. On this, the y-axis, we orient another unit distance that is the same length as the first. This sets both axes to the same scale while simultaneously orienting the y-axis.



Each line, or axis, retains all of the properties of the original line, such as order and directionality and distance, but we can now describe the position of a point anywhere in the plane, even if it does not lie precisely on one of our axes. The usual convention is to refer to these axes as *x* and *y*, with *x* being associated with the horizontal direction and *y* being associated with the vertical direction, though other orientations can be equally valid.

The most common way we locate a point in the plane is using Cartesian coordinates. This is a pair of numbers, *(x,y)*, such that when perpendicular lines are drawn from the specified point to each axis, these lines intersect the axes at the values *x* and *y.*

A less common way of describing position in the plane is that of polar coordinates, where a number line is "tacked" at the origin but free to rotate and points are specified based on their distance from the origin and the orientation of the line used to measure them. These coordinates are reported in the form *(r,Θ)*, with *r* being their distance from the origin and *Θ* being an angle that describes the orientation of the line of measurement. For standard coordinate systems, the angle *Θ = 0* describes a line of measurement that points directly along the *x* axis and the positive direction for *Θ* is defined to be counterclockwise from there. Polar representations better display magnitude and direction but Cartesian representations support vector addition. The big issue is that polar coordinates are wedded to the point around which they are defined, and do not support simple descriptions of translation the way Cartesian representations of components do. To switch between representations, one only needs to know a small amount of trigonometry to derive the following relations:

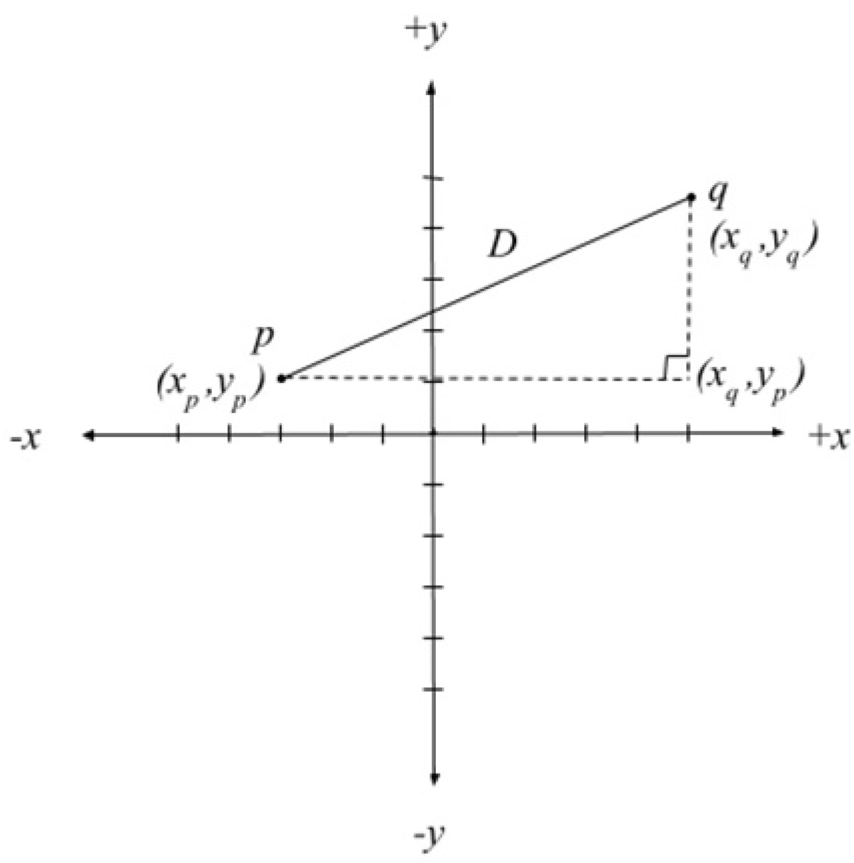
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$\begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \tan^{-1}\dfrac{y}{x} \end{cases}$$

For students, this can all be derived using only the sine, cosine and tangent relationships as well as the Pythagorean Theorem. In practice, this typically warrants an early crash course in trigonometry for the benefit of those who have not taken a course in it yet and periodic refreshers throughout the year.

Measurement and Vector Notation in Two Dimensions

Measuring the distance between two points in a plane can be accomplished by using the Pythagorean theorem. Starting with two arbitrary points *p* and *q*, we label the coordinates of these points (x_p,y_q) and (x_q,y_p) respectively. Placing them on the plane allows us to visualize the distance between the points, *D*, as the hypotenuse of a right triangle formed with *p* and *q* as the acute vertices.



To find *D* we write the relationship between the side lengths of this triangle using the Pythagorean theorem:

$$(x_q - x_p)^2 + (y_q - y_p)^2 = D^2$$

$$D = \sqrt{(x_q - x_p)^2 + (y_q - y_p)^2}$$

Solving for *D* gives us a generalized two-dimensional distance formula. Moving forward, we express the distance between two points *p* and *q* as *D(p,q)* such that:

$$D(p,q) = \sqrt{(x_q - x_p)^2 + (y_q - y_p)^2}$$

As in one dimension, the distance between two points *p* and *q* is the magnitude of the vector drawn from *p* and *q*. Unlike in one dimension, merely specifying a positive or negative number is no longer sufficient to fully define that vector in two dimensions. The mention of polar descriptions of points comes from the fact that, especially in applied physics, quantities are most often reported as an overall measurement in a particular direction. The entire notion of measurement in the polar system arises from the ability to align your number line in the direction you want to measure, report how long it is and then specify how you were holding your ruler when you made the measurement. This is intuitive and easy until you need to combine or transform measurements. The danger is that lengths can be combined in any orientation, but the treatment of that is more complex if they are not pointing in the same direction. To get around this, physics typically deals with the *components* of vectors, which are the perpendicular pieces that lie along a fixed pair of axes and combine to form a vector.

There are three common notations used at this level, each with benefits and drawbacks. The most common, and simplest, is known as *ordered set notation.* Consider a vector that points from the origin to a point *(x,y)* in the plane. This vector would also have some length, *r,* and some direction, *Θ* as found from the relations above. In ordered set notation, we can report this vector as

Instructional Strategies

Debate and Discussion

Science and learning are not activities that occur in isolation. Students should have an opportunity and space to challenge ideas and work them out with the teacher and each other. Because of this, activities are designed to occur in front of others in a collaborative space, but it is important to develop a culture of respectful questioning, discussion and debate. One tool that I have found is the TIPERS[1] series of books. These provide thought provoking questions that warrant a great deal of discussion among students, helping to draw out misconceptions and set up a place where students are able to learn from each other.

Measuring

Physics is all about making observations and learning from them. Because of this, students need to be able to make measurements and understand the measurement process. Coordinate systems are an essential part of this process, so this unit seeks to have students actively measuring with a variety of devices in a variety of units in order to understand how to measure properly as well as why we use the systems we do. Becoming proficient at measuring and observing forms the basis of good lab work for the rest of the course and beyond. This unit hopes to bring students up to speed and give them the proper skills to do well in physics.

Modeling

The real major focus of this unit is modeling. The entire purpose is to take something that is frequently treated in a purely symbolic and algebraic fashion and make it explicitly geometric. Physics describes the real world and the behavior of objects in it, so the understanding of concepts must be directly transferable to the real world. Making physical coordinate systems and physicals representations of mathematical ideas, as well as focusing on diagramming and drawing relationships is meant to help students internalize and understand vectors arithmetic, as opposed to just perform steps and report a result.

Classroom Activities

These three activities are by no means exhaustive, but should provide a firm basis for execution of the unit.

The Arbitrary Ruler

As an introductory activity to understanding measurement, I want to give my students blank rulers of various lengths. I will cut these with a band saw from scrap 2x4's I have lying around, but even old rulers with the print sanded off or pieces of (well sanded!) pallet cut to various lengths should work well. Each of these will be representative of a unit length. Students will then take a measurement of marked distance in the classroom in terms of their unit. From these, they should compute conversion factors to convert from their unit length to that of another student. These conversion factors they calculate can later be identified as dilation factors that transform from their unit length to another.

Once students have taken a measurement with their blank ruler, it will be time for them to subdivide their ruler in some way. They can mark their ruler in smaller intervals based upon a new unit interval, like the width of their thumb or the length of a locker key, in order to construct a more precise measuring instrument. Students should then find the dilation factor that transforms their new unit length into the old one and measure something smaller. Students will pair up and using their new conversion factors, each will predict what the other will measure for a specific object, like a table length. Students can then calculate the percent difference between their predictions and the actual measurements.

What students should get out of an activity like this is that the choice of unit distance does not impact how measurements are made, but does impact how simple they are to perform and how precise they are. Students should see that smaller unit distances produce more precise measurements. Communicating about the units they use is also important, because it allows other people to take their measurements and use them. Finally, I want my students to arrive at the idea that having standardized units of measurements is a smart idea because it simplifies communicating about data.

Transforming the Clothesline Number Line

The next major activity in my unit will begin with students placing numbers onto a clothesline number line. I want them to establish an origin and a unit distance and then use that to demarcate points along the line in both directions with clothespins and note cards. Once students form this large number line, I will have them measure objects along it and perform addition and subtraction by combining and comparing the lengths of those objects. From here, the objects will become vectors that are to be added and subtracted visually. This physical model will appear identical to the way one dimensional vector addition is frequently drawn – as arrows along a number line. Students will perform the computations and then draw the pictures to solidify what they are doing.

The second part is going to be having students model the same addition and subtraction they just did as translations by making a second, identical, number line below the first. This second number line is to be made of bungee cord, not rope, and put under some tension. Be careful with this, however! You can store an extremely large amount of energy in a bungee cord! Students will shift the second number line and use the translation of all of the points on that line to visualize addition. Subtraction will then be viewed as a translation in the negative direction. Students can repeat the exercises from the previous activity or perform multiple sequential translations until they start to see how vector addition and translation by a constant are related to each other.

The third part is to bring dilation into the picture. Since the second number line is made of bungee, increasing or decreasing the tension in the cord can dilate it! Again, be careful that the bungee is not put under too much tension, because that could cause a dangerous condition. Students can use this to model scalar multiplication as a dilation. An interesting way to set this up might be to have a loop of bungee with pulleys at the end. One pulley can remain fixed while the other can slide and be fixed in place with a setscrew. This sort of set up will allow for students to combine dilation with translation, simultaneously modeling scalar multiplication and vector addition.

Students should take away a few different visual representations of addition and subtraction, as well as a concrete visualization of multiplication as a dilation of the line. The purpose is to get them comfortable with the process and visualization of arithmetic as vector operations and transformations.

Two Dimensional Transformations

For this activity, I want my students to see how translations and dilations look in two dimensions. I like to use big sheets of butcher paper because it is inexpensive and a great way to have students quickly generate posters that can be hung up and discussed. Each group of three will be given a sheet of butcher paper, a card and some markers. The card will have a collection of points that the students must first plot and connect to make some sort of shape. Also on the card will be some translations that students need to carry out that are specified by vectors. One will be purely one-dimensional translation. One will be two-dimensional translation. The third prompt will be a composition of two translations. There will also be some scalar dilations and translation-dilation compositions that students will need to plot and draw with their original figure. It will be important for them to label the vector that describes the translation of each point on their posters.

The next task will involve them taking their own drawings and describing the transformations that provide specific instructions on a card. For example, students may draw a card that asks them to describe a transformation that moves the center of their figure to the point (5,5) and triples the size. The group would then describe the dilation and translation that produce that end result. The pictures will be hung up along with the prompt and peer-reviewed during a gallery walk type activity.

The final activity will involve assigning each group a random locker and a random starting point on the floor where my classroom is located. Starting points will be labeled with letters on the ground in painter's tape. Each team is tasked with describing a vector that points from their starting point to their assigned locker. This vector must be specified in terms of magnitude and direction on the back of the starting point note card. The cards will be shuffled and dealt out to different groups who must then use the vector and starting point on their card to recover the assigned locker. It helps that my school's floor has tile that approximates a grid laid into it.

The major student takeaways should be a visualization of how vectors are represented in the plane, how they combine, how they dilate and how this all fits together in a physical sense. The activities are designed to be big and engaging, because can be a challenging topic, especially for students who have limited exposure to trigonometry.

References

"FLP Vol. I Table of Contents." FLP Vol. I Table of Contents. Accessed August 05, 2016. http://www.feynmanlectures.caltech.edu/I_toc.html. This is a free-to-read online version of *The Feynman Lectures on Physics*. There is a chapter with a particularly nice treatment of vectors, though it might not be appropriate to give directly to students.

Hieggelke, Curtis J., David P. Maloney, and Stephen E. Kanim. *Newtonian Tasks Inspired by Physics Education Research: NTIPERs*. Boston: Addison-Wesley, 2012. This book is a collection of conceptual questions and ranking tasks that are excellent for starting class discussions and debates on physics concepts. This particular volume is a large collection of great questions on mechanics. There is another volume for electricity and magnetism topics.

"Set-Builder Notation." Set-Builder Notation. Accessed August 05, 2016. http://www.mathsisfun.com/sets/set-builder-notation.html. This site is a short and simple web explanation of the mechanics of set builder notation.

"Vector Addition 2.02." Vector Addition 2.02. Accessed August 05, 2016. https://phet.colorado.edu/sims/vector-addition/vector-addition_en.html. This is an excellent interactive web app built by the University of Colorado's physics education research team to illustrate addition of vectors in two dimensions. There are a lot of visualization options available within the app and it is free to use.

Implementing District Standards

This unit is designed to implement ideas from both the Common Core and the Next Generation Science Standards.

Common Core

The Common Core Standards for Mathematics have two distinct parts: content standards and standards for mathematical practice. Under high school standards for Vector and Matrix Quantities (N-VM), students are expected to Represent and Model with Vector Quantities and Perform Operations on Vectors. Both of these are explicitly the focus of this unit. Under the standards for mathematical practice, students are expected to construct viable arguments and critique the reasoning of others, use appropriate tools strategically and attend to precision. The notions of precision, modeling and measurement are baked directly into the content and classroom activities. Constructing arguments and critiquing the reasoning of others is an instructional strategy used to facilitate deeper thinking and discussion about the content.

Next Generation Science Standards

Despite being a foundational tool for the study of physics, specific NGSS standards do not exist for vectors. This unit is, however, aligned to the eight practices of science and engineering outlined in their framework. These main practices students will be engaged in are: developing and using models, using mathematics and computational thinking, constructing explanations and engaging in argument from evidence. The focus of the approach and activities in this unit is in taking physical and pictorial representations of vectors and using the models as a vehicle to relate the deep mathematical underpinnings and facilitate problem solving in students. Since understanding vectors involves working through misconceptions, students will always be focused on argument and discussion, as well as on communicating their ideas and understanding.

Endnotes

1. Hieggelke, Curtis J., David P. Maloney, and Stephen E. Kanim. *Newtonian Tasks Inspired by Physics Education Research: NTIPERs.*