Curriculum Units by Fellows of the National Initiative
2016 Volume V: The Number Line in the Common Core

Decimal Expansion: An Address System for All Numbers

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by Jung Min "Jade" Lee

Introduction

A long integer number line extends across my classroom wall. Each student also has a number line of positive and negative numbers taped to his or her desk. Visual representations are an important tool for students to make connections and relate mathematical concepts. Despite having number line posters everywhere in the classroom, it has been one of the tools least used by my students. One of the rare times where students actively utilize the number line is when they add integers. Students have been taught to move to the right when adding a positive number and to the left when adding a negative number. During the course of adding integers on the number line, students often *count* units out loud as they physically move their pencil to mark each moving unit on the number line. When students are asked to find fractions or percent, however, they prefer using area models rather than the number line, but fractions and decimals exist on the number line, too! I would like my students to see each number as representing a length, specifically a distance from zero, and to appreciate the number line in terms of measurement, rather than as a counting tool.

According to "Progressions for the Common Core State Standards in Mathematics," written by the Common Core Standard Writing Team in 2013, decimals are a special sort of measurement on the number line, namely one that you get by starting with unit intervals, and partitioning into the equal subintervals, then partitioning each of these units in equal intervals and so on. Each unit interval between two whole numbers is marked off into tenths, each of which is marked off into 10 hundredths, each of which is marked off into 10 thousandths, and so on, and over. These finer and finer partitions constitute a sort of address system for numbers on the number line.[1] For example, 8.26 is, first, between whole numbers 8 and 9, then in the subinterval between 8.2 and 8.3, then exactly at 8.26, six tenths of the way between 8.2 and 8.3.[2] Teaching decimal expansions in depth will allow students to make connections among different representations and to see how they are interconnected on the number line. Decimals should not be treated as an isolated mathematical concept, but understood as a precise system to locate a number on the number line with its unique address.

School Background

George M. Shirakawa Sr. Elementary School is a K-8 school in the East Side of San Jose, where gang activities, low academic achievement rates, and low socioeconomic backgrounds are known to be some of the neighborhood characteristics. Our two major ethnic groups are from Hispanic (42%) and Asian (43%, predominantly Vietnamese) families and about 40% of those students are categorized as English Language Learners (ELL). We also serve Filipino (5%), Caucasian (4%), African American (2%), and mixed race (4%) populations in our student body.[3] Approximately 70% of the student population are eligible for free or reduced lunch. Against those odds, Shirakawa has earned the title of being a California Distinguished School in 2008. In 2015, we celebrated our success as a California Honor Roll School, recognized by Educational Results Partnership (ERP) and the Campaign for Business and Education Excellence (CBEE), for demonstrating consistently high levels of student academic achievement and reducing achievement gaps among student populations. According to the 2015-2016 SBAC preliminary results, 70% of 8th grade students have met or exceeded English Language Arts Common Core Standards and 74% have met or exceeded Math Common Core Standards.

Since we are a large school with nine grade levels, different grades have voluntarily built their own small learning communities (SLC). For example, 7th and 8th grade teachers regularly meet to discuss curriculum lesson plans, student discipline issues, and coordinate student events only for middle school students. Although I am the only teacher who is in charge of 8th grade math curriculum, I collaborate with 6th and 7th grade math teachers in an effort to interpret Common Core Standards and to improve student achievement. Our small professional learning communities are largely responsible for Shirakawa's success over the years. Shirakawa continues to work tirelessly to provide a safe learning environment, support every student and teacher, and foster a love of learning through our A, B, C's: Academic excellence for ALL students, Building culture and community, and Collaboration.

Rationale

Many of my 8th grade students come to my classroom with gaps in content knowledge, especially in understanding basic mathematical concepts and operations. Since Shirakawa is a K-8 school, all three 8th grade classes have homeroom teachers and students are not regrouped by math performing levels. The academic gap seems even larger when there are 32 students with heterogeneous mathematical backgrounds. A lot of 8th grade students have difficulty developing a conceptual understanding of decimals and decimal operations and often treat them as an isolated mathematical concept. Students should understand that mathematics is just a collection of rules, but it is logical and makes sense. Most of the mathematical mistakes my students produce are due to memorizing steps and rules when solving problems. One dominant misconception my students have when solving decimal problems is about dealing with the decimal point. I conducted an informal survey online asking students what the most confusing part is when working on decimal operations. Here are some of the most common responses I have received: [4]

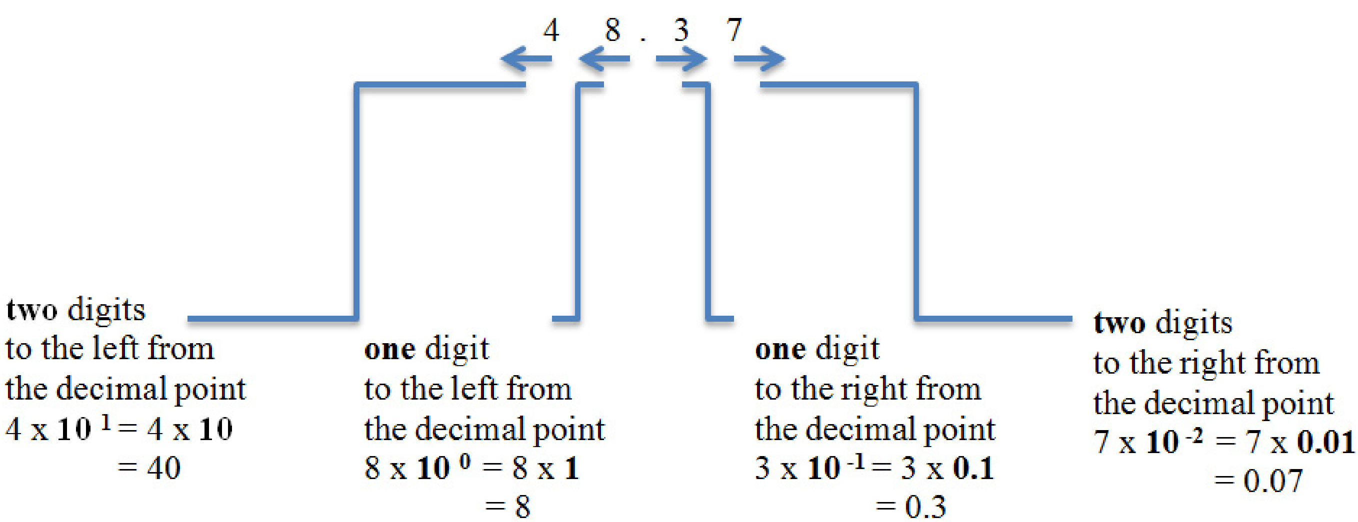
When I first started learning about decimals, I didn't know how to line up the decimals for addition and subtraction. And for multiplication, I didn't know where to put the decimal points when I got the product. – Student A

The most confusing thing for me when I started learning about decimal operations was when "carrying over" in addition and subtraction over the decimal point. For example, 1.37 + 15.8 =? – Student B

Multiplication and division were worse for me because of how the decimal point behaved. All of a sudden, instead of remainders, you have to move a decimal point and add zeroes everywhere and the re-put the decimal point. – Student C

It feels like someone up made up the rules as we went along when it came to where the decimal was placed *after* the calculations. Sometimes it goes in the first spot and sometimes it depends on how many decimal places there were. – Student D

According to my students' comments, the origin of trouble comes from not paying close attention to the decimal point. Decimal points give concrete values to each digit of the number; as the decimal point gets moved one digit to the left, the place value becomes ten times multiplied and as it gets moved one digit to the right, the place value becomes one-tenth of the previous digit's value. This relation stays exactly the same in every scale in all numbers. For example:



Many complain when they do not reach the correct answer adding and subtracting the decimals and would say, "I line up the numbers on the right as I was taught before," but the 'lining up the numbers on the right' rule only applies to whole number operations. When performing decimal computations, numbers must be lined up by the decimal to add digits that are multiplying the same base-ten unit. This rule, however, often is not made clear to most students. Students' lack of understanding of decimal place values results in (1) not counting the correct number of decimal places, (2) not aligning the decimal point when adding or subtracting decimals, (3) not understanding each decimal place is divided into tenths (the magnitude of base-ten scales), and (4) not being able to explain "how" and "why" when operating with decimal points. The number line can be a useful tool in showing the interrelationships of decimal numbers by identifying each decimal point and demonstrating decimal expansions to illustrate how decimals fill in each number line segment.

Our current curriculum, *CPM Core Connections*, encourages students to be responsible for their own learning through working together in small heterogeneous groups of four. My role as a teacher in this unit is to provide support and guidance in student team discussions and present an opportunity to think and examine how decimals are recorded on the number line system. Students will (1) demonstrate their ideas, (2) listen to what group members have to say, (3) see how the same problem can be solved in different ways by seeing their teammates' work, and (4) ask questions to resolve mathematical conflicts together. This curriculum unit will assist me to meet goals of the Common Core Standards and to design a more student-centered learning environment to encourage students to take an active role in cooperative learning.

Objectives

By the end of this unit, students will be able to understand and acknowledge that the number line is a measurement tool rather than a counting tool. As the number line shows the relationship of number placements in different base-ten scales, students will describe and demonstrate decimal expansions on the number line. The metric system (for linear measurement) will be presented as an example of base-ten scales and students will be able to use the number line with millimeter, centimeter, decimeter, and meter markings to measure line segments to the nearest tenth of a centimeter, a tenth of a decimeter, or a tenth of a meter. Presenting the metric system in scientific notation and relating it to the number line in a set of consecutive powers of ten; 1000, 100, 10, and 1 or 1, 0.1, 0.01, and 0.001 will be used as a form of formative assessment of the unit study.

I expect this unit to take around two to three weeks or ten to fifteen class periods. Although most of the class activities are designed to take only one class period, I would like to take enough time to make sure each lesson is fully understood. This unit study will be the first math lesson of the year before opening the textbook! Since the concept of decimal expansion includes Common Core Standards from 4th through 7th grades, students are already familiar with a lot of prior knowledge, and hopefully will be ready to be active participants in this unit study.

This curriculum unit consists of the following topics:

1. Placing whole numbers in base-ten on the number line

This lesson is to introduce the curriculum unit to students by presenting number line segments that are in different base-ten scales. Students will be able to make connections among different numbers and how they are related in different base-ten scaled number line segments.

2. Decimal expansions

After students are introduced to different scales of the base-ten system, they will explore the decimal system. Decimals are in a base-ten number system that provides a more precise collection of numerical expressions than whole numbers. Students will compare the magnitude of different decimal numbers by expanding each digit and place value of a given decimal number on a number line segment.

3. The metric system

Students will apply their knowledge of decimal expansions to the metric system, a decimal system of base-ten applied to measurements, by using a centimeter to represent 1, 10 for a decimeter, and 100 for a meter.

4. Scientific notation

In this section, students will make connections and express length measurement in the metric system in terms of scientific notation, which also is fundamentally based on the decimal system with a base-ten scale.

5. Repeating decimals

This exercise is for students to develop a convincing argument establishing that every real number with a repeating decimal eventually finds its own address on the number line and therefore it is a rational number. Students in small groups will find decimals with a periodically repeating expansion and investigate systematic rules associated with different patterns when converting fractions to decimals.

The decimal expansion method will repeatedly be used throughout the entire academic year, and especially when we study the irrational numbers towards the end of the 8th grade. Students will be able to use the Pythagorean theorem to locate some irrational numbers, such as Ö2, Ö3, and Ö5 geometrically on the number line and investigate the precision of the placements by showing decimal expansions.

Teaching Strategies and Classroom Activities

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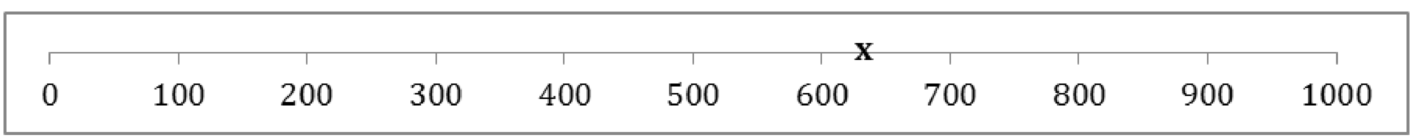
1. Placing whole numbers in base-ten on the number line

Number lines are a linear representation of all numbers in the order of size.

Example 1. Estimating whole number placement in different base-ten scales.

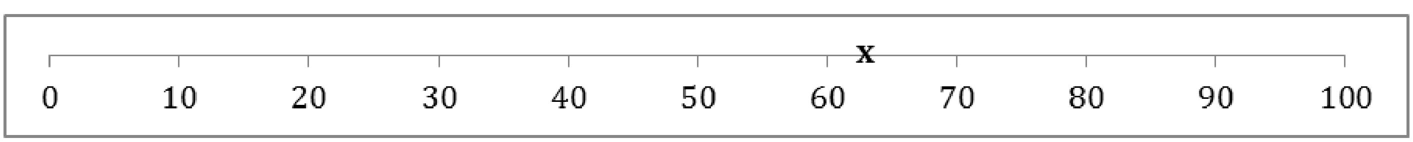
Place 638 on the number line.

(1 unit = 0.5cm. Total length of the number line segment = 5m)\



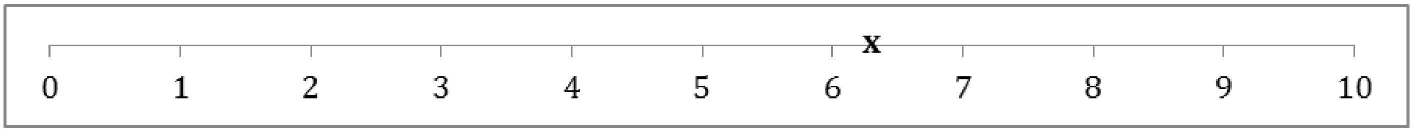
Place 63.8 on the number line.

(1 unit = 5cm. Total length of the number line segment= 5m)



Place 6.38 on the number line.

(1 unit = 50cm. Total length of the number line segment = 5m)



Students will be able to see how the relationship of the places of 638, 63.8, and 6.38 are the same when each number line segment is scaled 1/10 of the previous one.

Classroom Activity 1 (Introducing the unit – one to two class periods)

To start off the unit in an interesting way, I plan to invite my students to create our own "human number line." The human number line is to not only introduce the unit in an engaging way, but also present the number line visually as a measuring tool. This activity will be conducted outside the classroom on the basketball blacktop.

I will prepare a 5-meter yarn with ten equal spaces marked on the yarn. Students will pick a piece of paper from a box to find out what their job assignment is. The box will have job assignments written on small pieces of paper. Each assignment has a direction based on the number line of Example 1. For example, twelve students will randomly pick 0, 100, 200, 300, 400, 500, 600, 700, 800, 900, 1000, and 638 to create the first number line graph of Example 1. Students who pick "0" and "1000" will be responsible for holding the yarn number line straight. The other students will walk around to find their number positions and the student who picks 638 will use educated guess to locate his or her position on the human number line. As I lead discussion questions about the number line, the rest of the class will share their observations. Some example questions are as follows:

- What do you notice about this number line segment?
- What are the minimum and maximum values of this number line segment?
- Why do you think I marked the intervals equally?
- How did he or she find the position of 638?
- How do we measure the distance of 638 on this number line segment?
- What does each interval represent on this number line segment?

When presenting the first number line of Example 1 is done, students who participated in making the number line will return their job assignment paper to the box. Then another group of twelve students will pick their job assignments from another box that has numbers from 0 to 100 in intervals of 10 and 63.8 as the second number line of Example 1 displays. I will do another round of questionnaires that are similar to the ones I asked in the previous round. I will also add some new questions:

> "What are some of the similarities and differences of this number line segment compared to the number line segment we just made before this one?"
>
> "What would you say about the place of 63.8 compared to 638 of the previous number line segment?"

We will repeat the same routine to create the third number line segment of Example 1 and discuss the same questions. When students come back to the classroom, they will be given five minutes to write their math journals to reflect what they have learned from this activity. If we have enough time, students will share what they have written in their journal with their group members.

Classroom Activity 2 (one to two class periods)

After students finish writing their journals, I will distribute a piece of 5-meter long paper to each group. Each paper will have a horizontal line with ten equally spaced intervals. I will ask each team to randomly pick its assignment from a box. Each assignment sheet will have the same basic instructions, but with different scales and numbers. Some of the example assignments will be as follows:

Assignment A

Place 745 on the number line segment. Each marked interval on your paper has

length 100 units. Your minimum unit is 1 and each unit equals 5mm or 0.5cm.

Assignment B

Place 74.5 on the number line segment. Each marked interval on your paper has

length 10 units. Your minimum unit is 1 and each unit equals 5cm.

Assignment C

Place 7.45 on the number line segment. Each marked interval on your paper has

length 1 unit. Your minimum unit is 1 and each unit equals 50cm.

Assignment D

Place 481 on the number line segment. Each marked interval on your paper has

length 100 units. Your minimum unit is 1 and each unit equals 5mm or 0.5cm.

Assignment E

Place 48.1 on the number line segment. Each marked interval on your paper has

length 10 units. Your minimum unit is 1 and each unit equals 5cm.

Assignment F

Place 4.81 on the number line segment. Each marked interval on your paper has

length 1 unit. Your minimum unit is 1 and each unit equals 50cm.

We will then display the same number group number line segments (i.e., 745, 74.5, and 7.45) vertically on the hallway wall to see how the relationship of the places of 745, 74.5, and 7.45 are the same when each number line segment is scaled 1/10 of the previous one.

2. Decimal expansions

Students need to acknowledge why understanding the nature of decimal expansion, especially in comparing the magnitude of different decimal numbers, is important. Decimals are in a base-ten number system that provides a more precise collection of numerical expressions than whole numbers. I have always been fascinated by how quickly students grasp the concept of decimals in our money system (dollars and cents), but when the decimal system is presented in a course of study, students rather seem to have a hard time processing its nature. Typical students often have difficulty linking conceptual understanding of decimals to the rules of manipulating symbols and solving decimal problems (Hiebert & Wearne, 1986).[5] I have witnessed in my class that students often feel challenged just by having a decimal point in their math problems because they understand the decimal point gives entirely different place values to each digit in numbers. For example, students would not feel uncomfortable to solve 137 + 458. When facing 1.37 + 45.8, however, students feel they need to apply procedural steps such as "lining up the decimal point," as a memorized rule, rather than implementing the nature of decimal digits in magnitudes of ten. Some students would ignore the decimal point and treat the digits as whole numbers. Without explicit connections to link the decimal system to the number line system, students often perceive the decimal system to be a new symbol system representing new concepts rather than an extension of an already partially mastered system of numeration (Hiebert & Wearne, 1986).[6] Decimal expansion lessons will be beneficial to those who have only applied the rules of whole number operations in rational number computations without developing a full understanding of the magnitude relation of decimals. It would help to discuss about the expanded form for decimal fractions, such as 1.37 = 1 + 3 tenths + 7 hundredths and 45.8 = 4 tens + 5 ones + 8 tenths. By expanding the decimals fractions, it may be clear to students that they should add the pieces of the same order of magnitude with regrouping whenever one gets more than ten of a given base-ten unit. Then it may be clear why 1.37 + 45.8 = 4 tens and 6 ones and 11 tenths and 7 hundredths, or 47.17. I hope my students finally see the decimal calculation works by adding the ones to ones, the tenths to the tenths, the hundredths to the hundredths, etc. after the decimal expansion lesson.

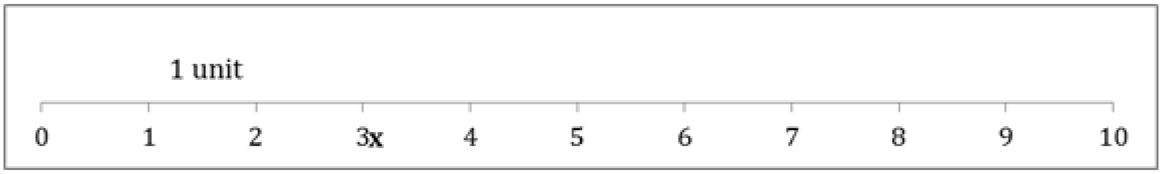
The expanded form of a decimal refers to the value of a number written as a sum. For example, the expanded form of the decimal 0.125 is $1/10 + 2/20^2 + 5/10^3$ or one-tenth + two-hundredths + five-thousandths, which is closely related to the notion of expanded form used at the elementary level.[7] Students will show how decimals fill in the gaps between two integers on the number line and show the decimal place value of tenths. They will expand the line between two of these decimals in order to show even more decimal numbers between them. As students continue to break intervals into ten smaller units, they will review different decimal place values located on the number line. It is also important to emphasize that decimal numbers on the number line represent their distance from zero.

The goal of this exercise is for students to recognize the general nature of the *decimal expansion as an address system* and that each decimal place is related to neighboring places by a factor of ten; the address system gets ten times more precise as each decimal place is added on the right. Students will define decimal place values, decimal digits, and equivalent decimals. Number lines in Example 2 describes how decimal numbers fill in the locations on the number line between whole numbers.[8]

Example 2.[9] Plotting 3.14159… (π = 3.14159265359…)

Step 1) Whole numbers on a number line

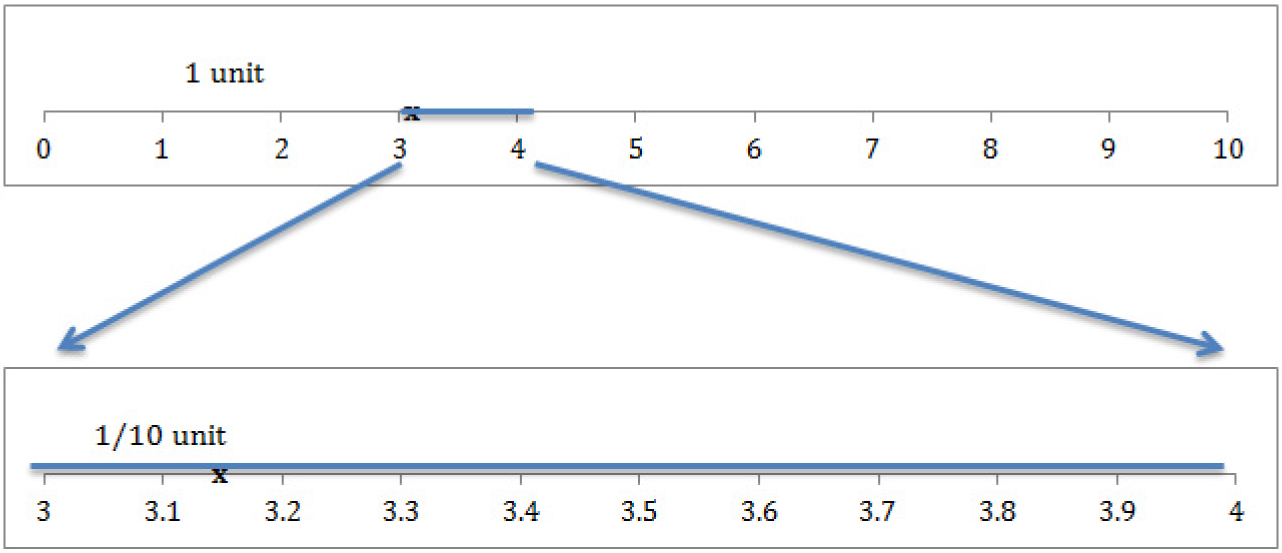
Integers are placed on a number line and each consecutive number is one unit apart.



Step 2) Each unit is divided into tenths.

Real number π satisfies $3 \leq \pi < 4$

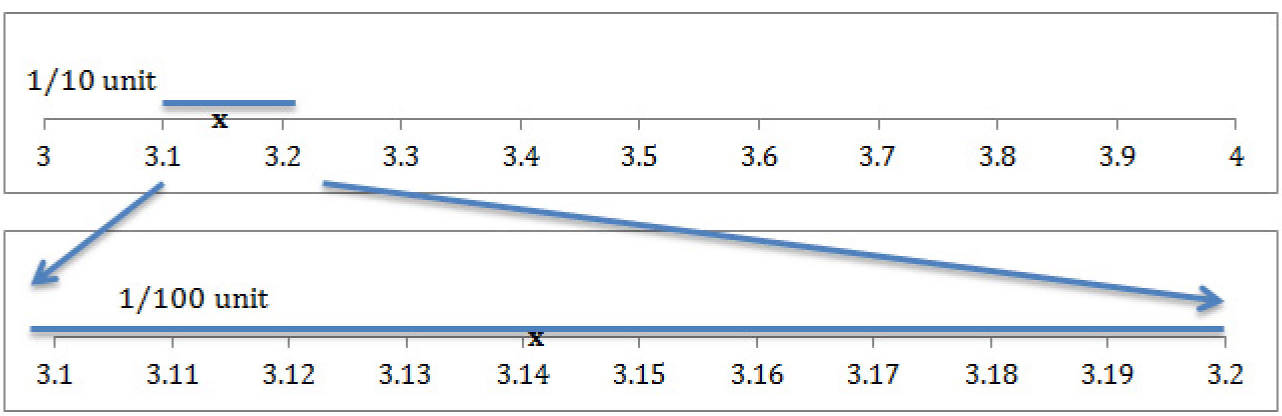
The real number π is placed in the tenths place with the ten smaller intervals, each 1/10 of one unit long.



Step 3) Each 1/10 unit is partitioned into ten one-hundredths.

Real number π satisfies $3.1 \leq \pi < 3.2$

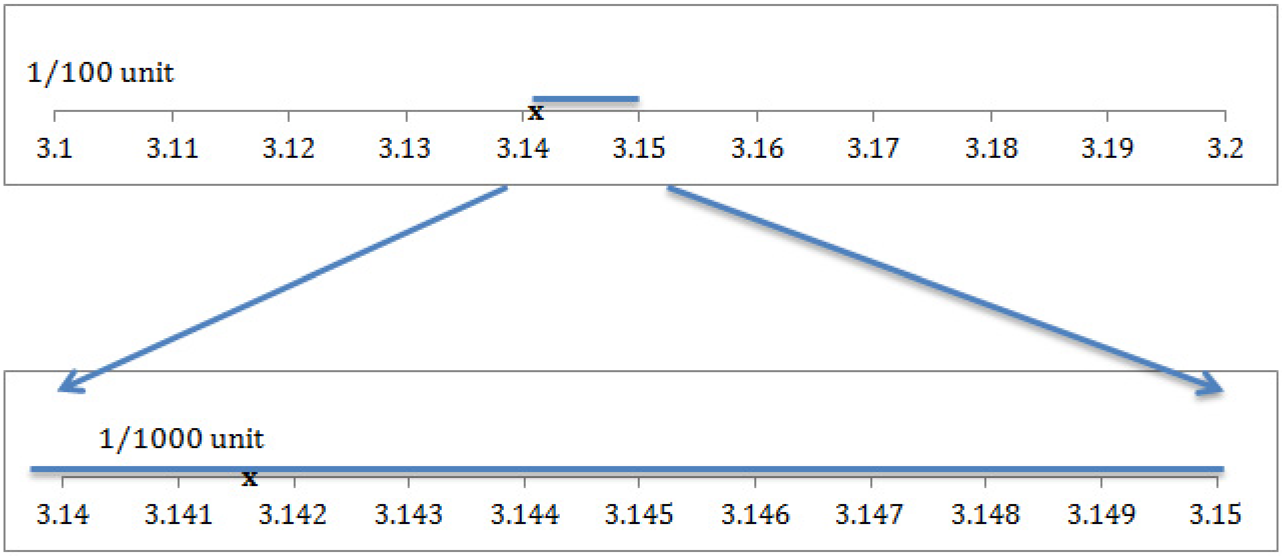
The real number π is placed in the hundredths place with ten smaller intervals, each 1/100 of one unit long.



Step 4) Each 1/100 unit is partitioned into ten one-thousandths.

Real number π satisfies $3.14 \leq \pi < 3.15$

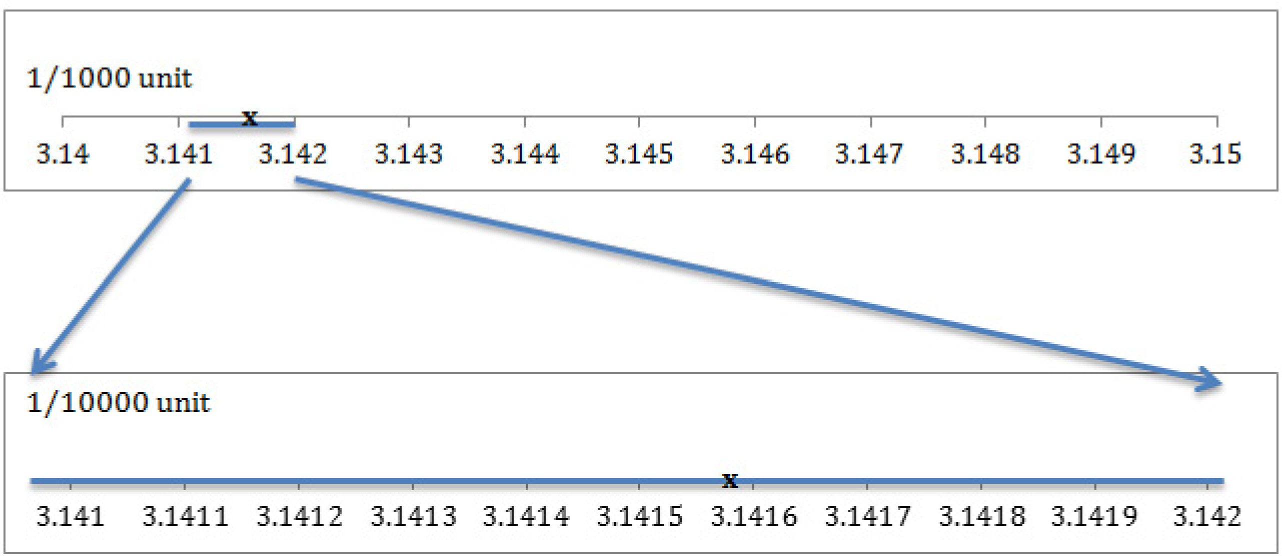
The real number π is placed in the thousandths place with ten smaller intervals, each 1/1000 of one unit long.



Step 5) Each 1/1000 unit is partitioned into ten one-ten-thousandths.

Real number π satisfies 3.141 ≤ π < 3.142, π = 3.14159...

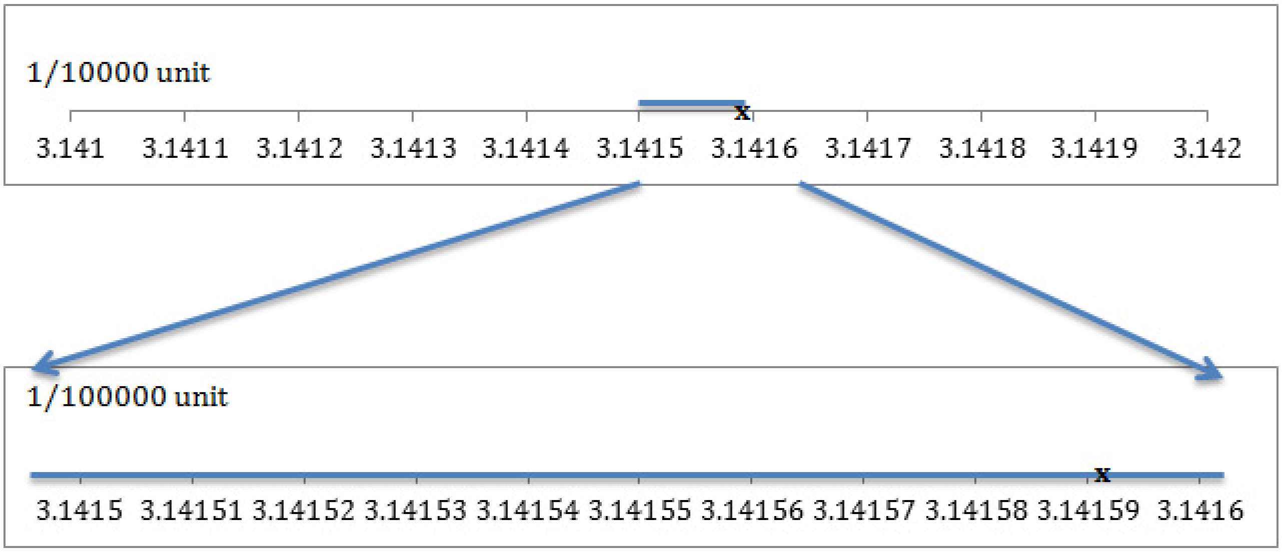
The real number π is placed in the ten-thousandths place with ten smaller intervals, each 1/10000 of one unit long.



Step 6) Each 1/10000 unit is partitioned into ten one-hundred-thousandths.

Real number π satisfies 3.1415 ≤ π < 3.1416, π = 3.14159...

The real number π is placed in the hundred-thousandths place with ten smaller intervals, each 1/100000 of one unit long.

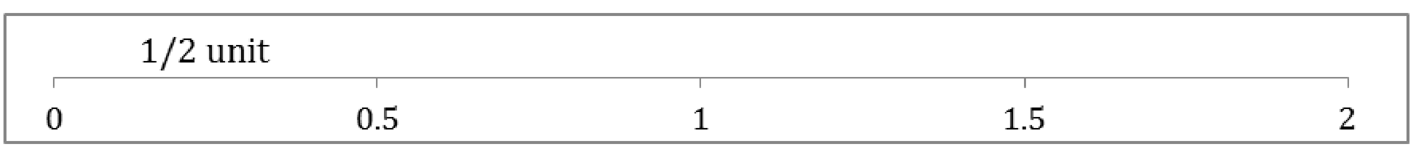


As the decimal expansion process repeatedly divides each interval into ten smaller intervals, all decimal numbers, including the ones that require infinitely many numbers on the right side of the decimal point, have a unique location, or *address*, on the number line.

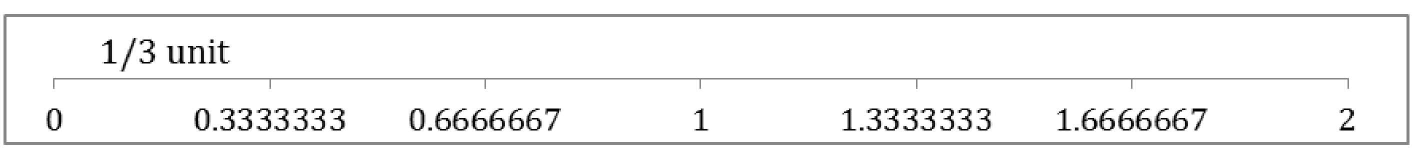
The notion of the decimal expansion can easily be applied to the American money system, as in dollars and cents.[10] A dollar bill can be treated as the unit, and then expand the decimal places by using dimes and pennies. A dime is worth 1/10 of a dollar, since ten dimes make one dollar (10 x 0.1 = 1), just as the intervals of on the number line between consecutive integers are broken into ten smaller intervals, each 1/10 of a unit long. A penny is worth 1/10 of a dime, and at the same time, worth 1/100 of a dollar. It is the same relation when the intervals on the number line of tenths units are broken into ten smaller units, resulting each new interval as 1/100. Thus, ten pennies make a dime (10 x 0.01 = 0.1) and one hundred pennies make a dollar (100 x 0.01 = 1). Although there are some examples of having an even smaller currency unit than a penny, such as gas price signs at gas stations (i.e., $3.99 9/10 per gallon), no smaller denominations than pennies are represented in actual coins. This is one example of the decimal system applied in real life.

Different sizes of interval scales, not only by tens, but also by other numbers, can also be presented to show how rapidly each unit segment gets divided and how each interval provides a different aspect of interval spacing.

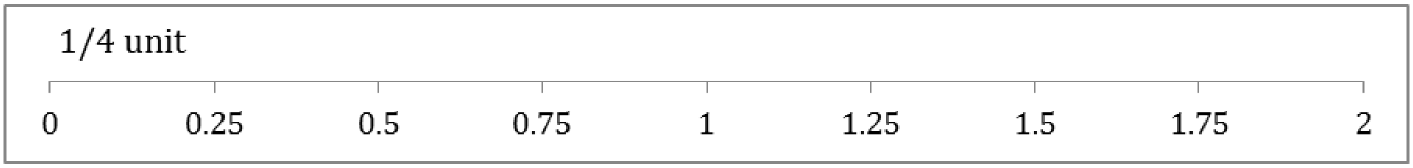
Example 3. Each unit is divided into one-half.



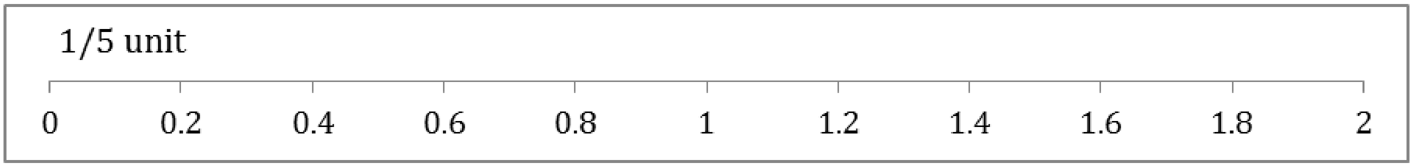
Example 4. Each unit is divided into one-third.



Example 5. Each unit is divided into one-quarter.



Example 6. Each unit is divided into one-fifth.



Looking at how the same number line can be divided into intervals of any given size is a good “overview” in making connections of “part-to-whole” relationships in decimal numbers. When we study fractions in later chapters, I also plan to have students study general fractions, and transfer area models of fractions to the number lines.

Classroom Activity 3 (one class period)

This third activity focuses on locating decimal expansions on the number line. To plot a number that is not an integer, such as 1.8 and 5.639, students need to learn how to break the number by different place values.

1.8 = 1•1 + 8 • 1/10, or 1.8 = 1•1 + 8• 0.1

5.63 = 5 •1 + 6 • 1/10 + 3 • 1/100, or 5.63 = 5 • 1 + 6 • 0.1 + 3 • 0.01

Students, then, expand three to five examples of decimal numbers on their notebook on their own. Students who need extra support in expanding the numbers will ask their team members to work together.

I will ask students to expand π = 3.14159… and I will dictate their answer on the board as follows:

π = 3.14159 = 3 • 1 + 1 • 1/10 + 4 • 1/100 + 1 • 1/1000+ 5 • 1/10000 + 9 • 1/100000 or

π = 3.14159 = 3 • 1 + 1 • 0.1 + 4 • 0.01 + 1 • 0.001 + 5 • 0.0001 + 9 • 0.00001

Once the numerical expansion is shared, I will demonstrate how to expand π = 3.14159 on the number line as Example 2 shows. Depending on how well students understand the concept of the number line expansion, I plan to model Steps 1 – 3, and ask students to do Steps 4 – 6 of Example 2. Students will continue to practice the number line decimal expansion by completing 1.8 and 5.63 expansions on their own. Students who need extra support will work together with their group members. I will walk around and monitor student work. Those who finish early can challenge themselves by expanding other well-known decimals, such as $\sqrt{2}$ = 1.4142135…, while waiting for others to finish their work.

Students will learn that every decimal, even a decimal that extends infinitely to the right, has a precise location on a number line.

As for applying the notion of the decimal system to the American dollar system, we will review the relationships of pennies, dimes, and dollars and transfer the structure to the number line. I will first ask my students if they could think of any base-10 scale system we use in everyday life. It will be the most ideal if the students' response comes out as "our money system." An example of the whole class discussion will be as follows:[11]

T: How many pennies are in 1 dollar?

S: 100 pennies.

T: A dollar is equal to how many cents?

S: 100 cents.

T: (Write 100¢ = 1 dollar.)

T: We can write 1 hundredth dollar using a decimal. Write in decimal form.

S: (Write 0.01.)

T: Place the dollar sign before the ones. (Write 1¢ = (1/100) dollar = .01 dollar =

$0.01.) We can read $0.01 as 1 cent.

T: (Show 7 pennies.) 7 pennies is how many cents?

S: 7 cents.

T: What fraction of a dollar is 7 cents?

S: (Write on their individual whiteboard) 7/100 dollar.

T: Write a number sentence to show the value of 7 pennies as cents, as a fraction of

a dollar, and in decimal form. S: (Write 7¢ = $0.07.) Repeat writing equivalent

number sentences for 31, 80, and 100 pennies.

T: A dime also represents a fractional part of a dollar. How many dimes are in a

dollar?

S: 10 dimes.

T: What fraction of a dollar is 1 dime? How do you write in decimal form?

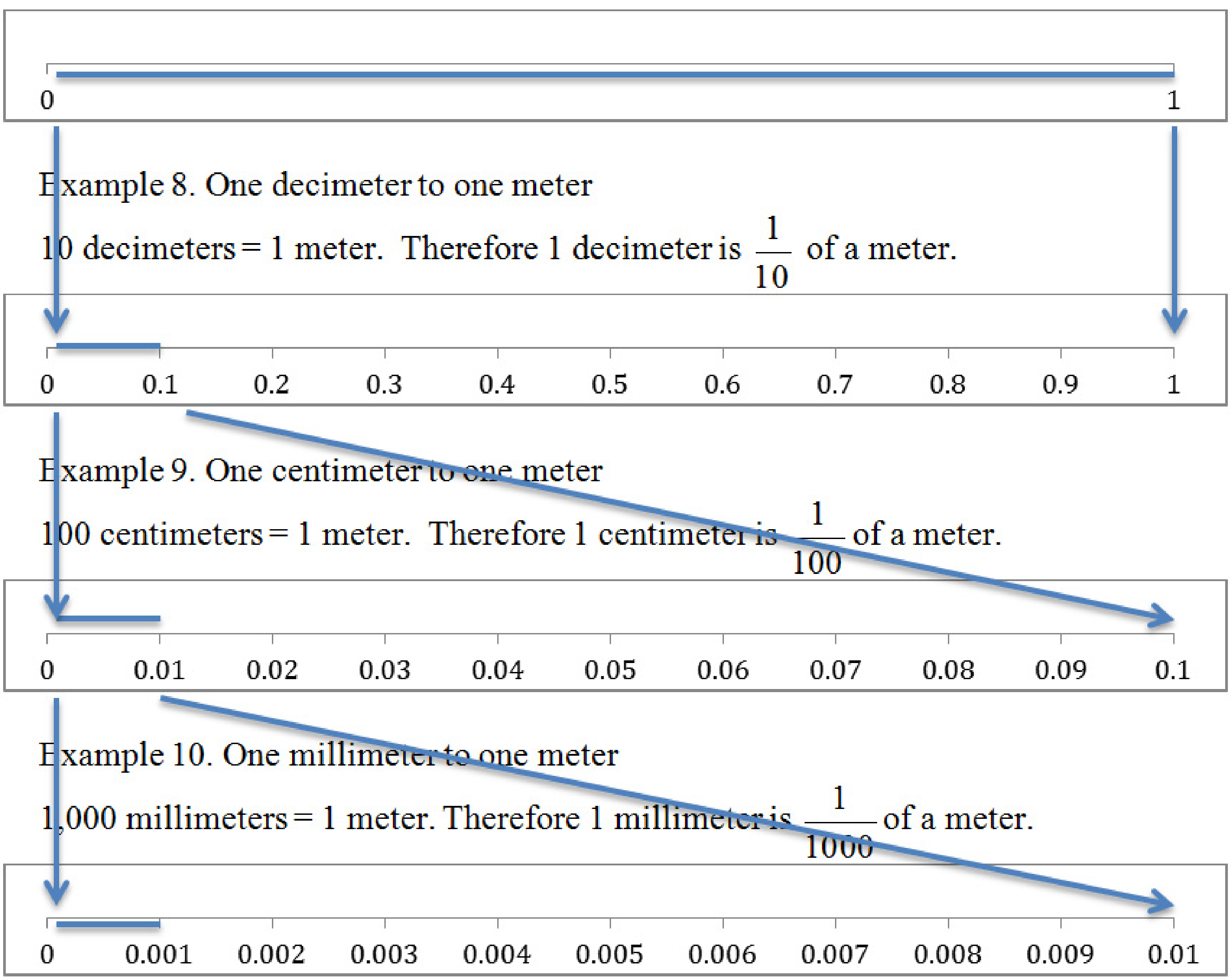
S: 1/10 dollar and 0.1 dollar.

When the initial prior knowledge check is done, students will demonstrate their understanding of decimal expansion of the money system on a number line. For example, $7.83 will be plotted on a dollar-scaled number line, a dime-scaled number line, and a penny-scaled number line to see how what we learn in the classroom is directly related to our daily life. Well, dollars are not unit of length. Nevertheless, it is true that people use length models to represent many quantities. This is the basis for graphing various functions of various variables. The dollar expansion activity can also be used as a quiz problem to check for understanding.

1. The Metric System

Students can also apply their knowledge of decimal expansions to the metric system, a decimal system of base-ten applied to measurements, by using a centimeter to represent 1, 10 for a decimeter, and 100 for a meter. This can be done on a different scale; for example, a meter can represent 1, a decimeter as 0.1, and a centimeter as 0.01. Students can choose the scale according to how they rescale the number line. It clearly shows how rapidly base 10 expansions approximate the numbers they represent, or how rapidly this possible error shrinks.

Example 7. One meter

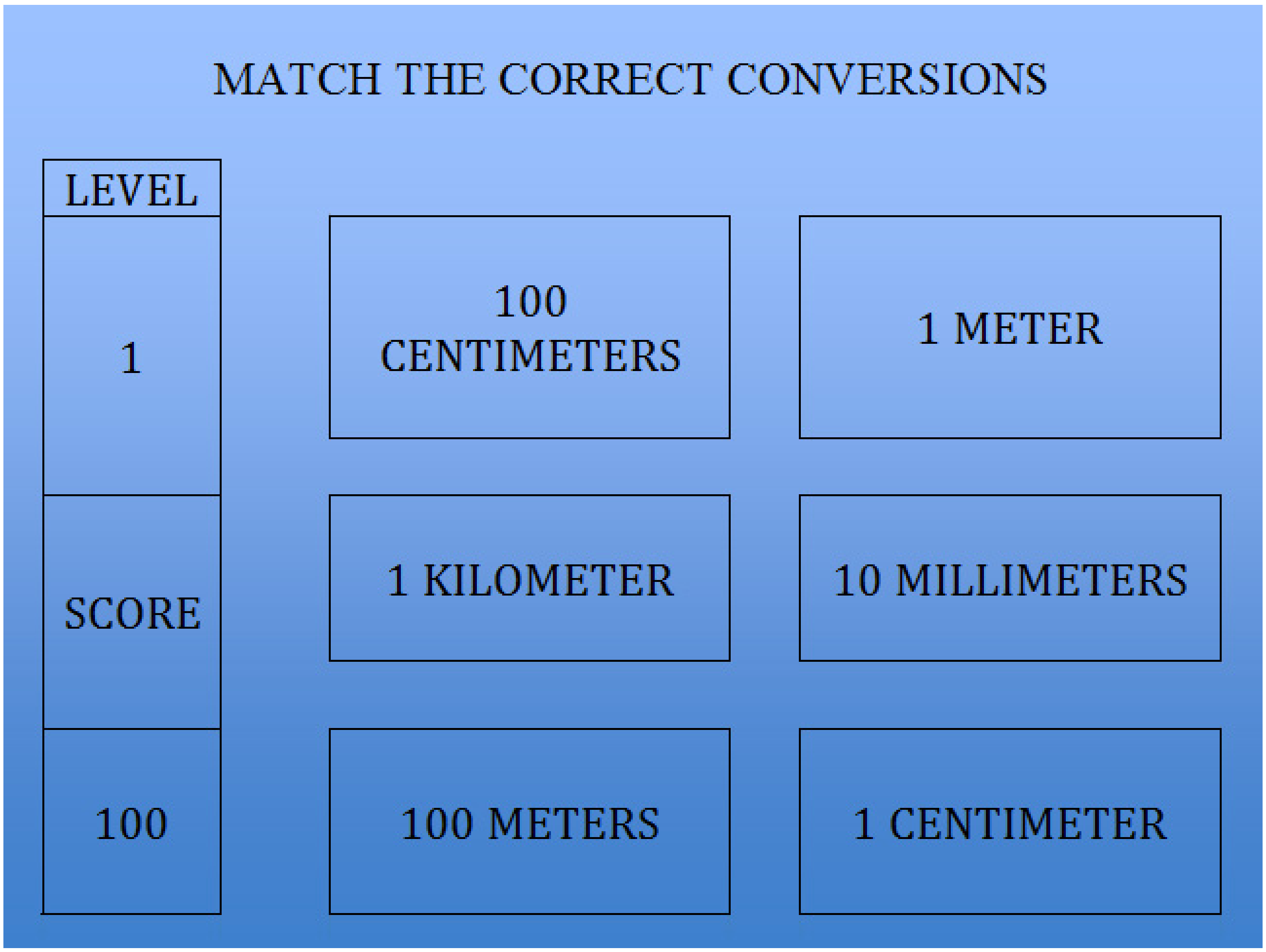


Metric system calculations involve the simple process of moving the decimal point to the right when multiplying by ten or to the left when dividing by ten (i.e., 5.284 meters is 528.4 centimeters and 5284 millimeters).

Classroom Activity 4 (one class period)

The objective of this exercise is to understand the metric system as an example of base-ten scales and connect relationships of a millimeter, centimeter, decimeter, and meter as a set of consecutive powers of ten. I will start off the activity by asking students to share what they already know about the metric system. We will gather prior knowledge and I will organize student responses on the board. Students then play a metric length measurement game on their iPads (Sheppard Software Games). The goal of the online game is to match up equivalent metric measurements, such as 10 millimeters = 1 centimeter and 1000 millimeters = 1 meter. An example picture of the game page is as follows:

Picture 1. Metric Length Matching Game [12]



When the initial prior knowledge check is done, students will demonstrate their knowledge of decimal expansion of the metric system on a number line. Students will measure their height with a metric ruler and plot the number on three different number lines as they did for the money system. For example, 1 meter and 53 centimeters will be plotted on a meter-scaled number line as 1.53, a centimeter-scaled number line as 153, and a millimeter-scaled number line as 1530. They will use Example 1 number lines as a supporting material to create three number lines in their notebook.

2. Scientific Notation

In the world of science, scientists encounter measurements that are so large or small that it would be cumbersome to write them with dozens of trailing or leading zeros.[13] For example, the mass of the sun in kilogram is 1,988,920,000,000,000,000,000,000,000,000 and the mass of a dust particle is 0.000000000753 kg. It is tedious to write out all those numbers repeatedly. Scientific notation is to express very large or small numbers in a constructed system so that it is easier to read and understand such long numbers. The mass of the sun can be rewritten as 1.98892×10^{30} and the mass of a dust particle can be rewritten as 7.53×10^{-10} in scientific notation. The purpose of scientific notation is to display the fundamental properties of a measured number in two ways:

(1) To express how large or small it is in the power of ten

(2) To express how accurately it is known by indicating the number of digits given

in the mantissa

It is another form of an approximation of properties of decimal expansion in the base-ten system to compare the relative size of unthinkably large or small scales. In scientific notation, a number *x* is rewritten as a simple, but precise decimal between one and ten, multiplied by a power of ten raised to a power, n:

$1 \le x < 10 \; \times \; 10^{n}$

Some of the powers of ten are as follows:

$10^{0} = 1$	$10^{-1} = 0.1$
$10^{1} = 10$	$10^{-2} = 0.01$
$10^{2} = 100$	$10^{-3} = 0.001$
$10^{3} = 1,000$	$10^{-4} = 0.0001$
$10^{4} = 10,000$	

For example, 1879 in decimal form is rewritten as 1.879×10^{3} in scientific notation since 1.879 multiplied by 1000 (moving the decimal points three times to the right) is equal to the original number 1879. 0.01879 in decimal form is rewritten as 1.879×10^{-2} in scientific notation since 1.879 multiplied by 0.01 (moving the decimal point two times to the left) is equal to the original number 0.01879.

Students will watch a short movie, “Powers of Ten” by Charles and Ray Eames (1977) as an introductory clip for scientific notations. The film starts with a square image, showing a one-by-one meter section of an ordinary picnic blanket, then, the camera quickly pulls back until the screen shows an area ten meters square. Thereafter, every ten seconds the camera is ten times farther away, showing an area ten times as wide. In the language of scientific notations, each of those jumps is one “power of ten.”[14] It is difficult to grasp an extremely vast scale of distances, such as the magnitude of the universe, and an extremely small scale of sizes, such as the scale of molecules and atoms. The movie is designed to help us see the scale of the universe, from the very large clusters of galaxies to the unimaginable tiny components of a single proton.[15] Students will discuss how distance and size rapidly change when 10 is multiplied or divided in measuring different matters. It will provide another perspective for understanding the decimal system applied to measurements and a general appreciation of decimal usage in the field of science. The theory of measurement is mathematical, but its application is the real world of science and engineering.[16]

Students will make connections and express length measurement in the metric system in terms of scientific notation since both systems are fundamentally based on the decimal system with a base-ten scale.

Example 11. Metric system in scientific notation

1 kilometer = 1 x 10^{3}

1 meter (base) = 1 x 10^{0}

1 decimeter = 1 x 10^{-1}

1 centimeter = 1 x 10^{-2}

1 millimeter = 1 x 10^{-3}

There are also micron (10^{-6} meters), the nanometer (10^{-9} meters), and the Angstrom unit (10^{-10} meters), which is used in atomic and molecular physics.

Classroom Activity 5 (one class period)

The focus of this activity is for students to make connections on how the base-ten scale is used in different forms of measurements. Since teaching scientific notation involves a new concept of using integer exponents to my 8th graders, I will not emphasize on mastering the computation of scientific notation in this unit lesson. Students will focus on scientific notation in more detail at a later time of this year.

Students will watch "Scientific Notation (Math Song)" by Colin Dodds on YouTube. It is important to introduce what scientific notation is and how it is used since some students may have never heard of scientific notation. We will watch this video clip two times. When we watch it for the second time, students will be asked to take notes on the following question: "In your own words, explain (1) what scientific notation is, (2) why it is useful for scientists to use, and (3) write one large number and one small number in scientific notation. Students will share their answers with their team members first and we will collect group answers as whole class.

Students, then, watch "Powers of Ten" by Charles and Ray Eames (1977). As students watch the movie, they will find answers to the following questions:[17]

At what power of ten do we see Lake Michigan?

At what power of ten do we see the solar system?

What do we see when we are at the 10^{-2} power?

At what power of ten do we begin to see DNA?

At what power of ten are an atom's outer electrons?

Finding answers to those questions will give students a general idea of what positive and negative powers of ten means in measurements. I will make a transition to connect the metric system to scientific notation by presenting the following chart:

Example 11-1. Rewrite the following metric system in scientific notation.

(Students will fill out the Scientific Notation sections)

Distance	Meter	Scientific Notation
1 kilometer	1000	1×10^{3}
1 meter	1	1×10^{0}
1 decimeter	0.1	1×10^{-1}
1 centimeter	0.01	1×10^{-2}
1 millimeter	0.001	1×10^{-3}
1 micron	0.000001	1×10^{-6}
1 nanometer	0.000000001	1×10^{-9}
1 Angstrom unit	0.0000000001	1×10^{-10}

For a homework assignment, students will visit different websites to find the radius of the Earth (we can use iPads in class if we have enough time) and rewrite it in scientific notation. We will compare our answers to discuss the range of the measurement and see how different websites state different answers.

3. Repeating decimals

A rational number is a number that can be written as a fraction *a/b* where *a* and *b* are whole numbers. It is a theorem that numbers with finite, i.e., terminating decimal expansions, as well as those numbers that are infinite with repeating blocks, are rational numbers.[18]

There are two prerequisite lessons that must be reviewed and covered prior to working with repeating decimals.

Prerequisite Lesson 1: *Converting terminating decimals into fractions.*

Objective: Students understand that every rational number can be converted to a fraction and the relationships between fractions and decimals by converting one to the other. Depending on where the decimal point is placed in a number, students must figure out what goes into the denominator; 1/10, 1/100, 1/1000, etc. The method of converting decimals to fractions is about understanding the meaning of place value in referencing decimals.

Prerequisite Lesson 2: Long Division

Objective: Students complete and explain the different steps in long division calculations. Students can find finite or infinite decimal expansions using long division. Converting fractions to decimals requires dividing the numerator by the denominator and the final product of the decimal value is sometimes terminated exactly, or repeats infinitely into a rational number. It is important not to leave any reminder as a whole number when performing long division. The remainder must be treated as part of the numerator with zeroes after the decimal point. For those who are not at grade level, I will provide extra support during after school intervention sessions and go through each step in long division. Long division is needed to show that a rational number is represented by a repeating decimal.

Students in small groups will find decimals with a periodically repeating expansion and investigate systematic rules associated with different patterns when converting fractions to decimals. This exercise is for students to develop a convincing argument establishing that every real number with a repeating decimal eventually finds its own address on the number line and therefore it is a rational number. Each student team’s conclusion will be justified with supporting examples.

Example 12. How to express a repeating decimal as a fraction:

0.121212… = x

0.121212… = 1/100 (12.1212…) = 12/100 + 0.1212…

x =12/100 + x/100

Therefore, subtracting x/100 from both sides, we get

x (1 – 1/100) = 12/100, or x (99/100) = 12/100

Dividing by 99/100, we get

x = (12/100)/(99/100)

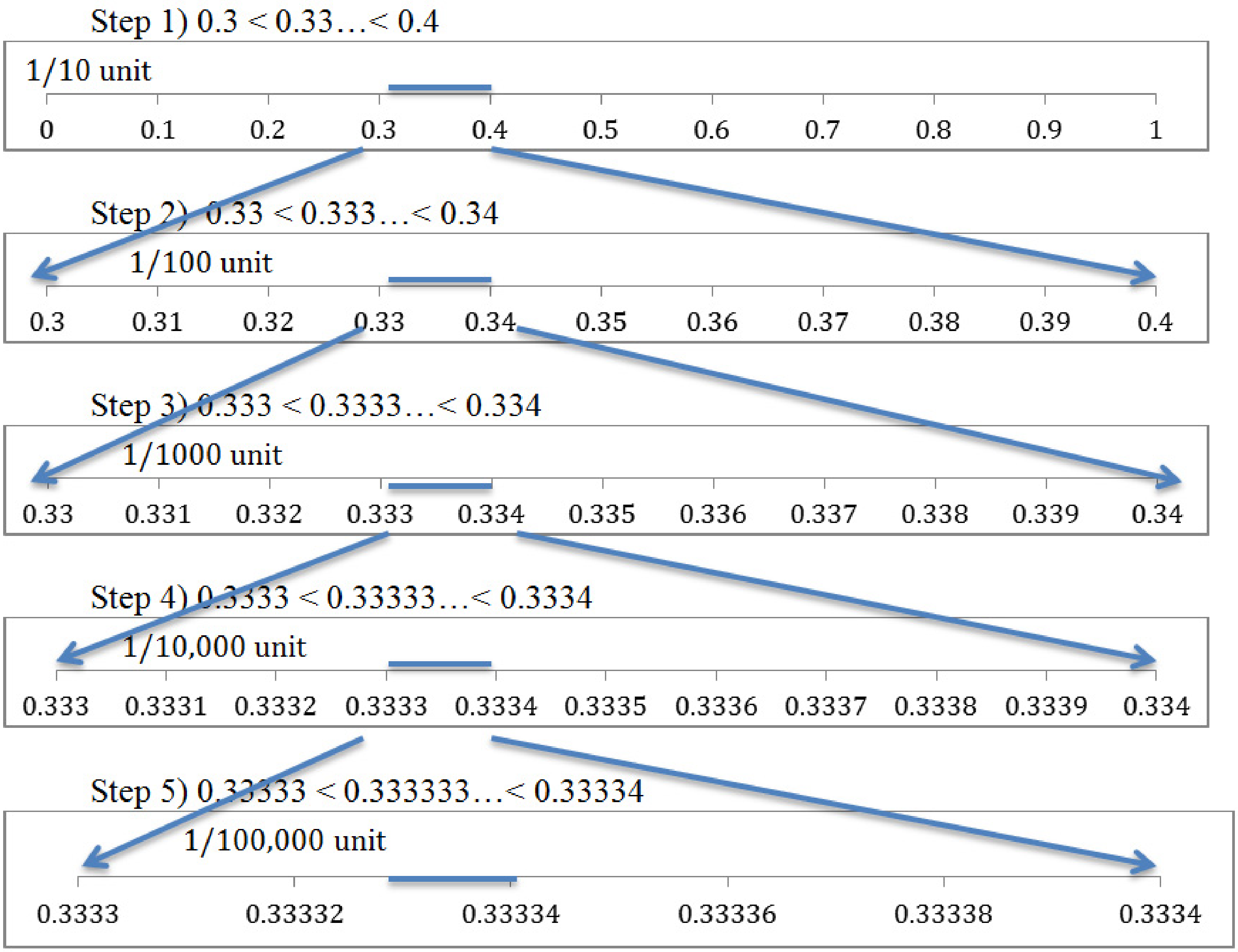
After simplifying, we finally see that

x = 12/99 = 4/33

Therefore, 0.121212… = 12/(100-1) = 12/99 = 4/33

Students can see that this technique applies to any infinite repeating decimal to show that every real number that as a repeating decimal expansion is a rational number. It is proven that the set of real numbers with repeating decimal expansions precisely finds their number line addresses.[19] Students can also use decimal expansions (“the address system”) to closely place the precise location of a repeating decimal.

Example 13. Placing 0. 3333…



We will also discuss that 0.33333… is 1/3 because 3(0.33333) = 0.99999 and 1 is between 0.99999 and 1.00002, which is 4(0.33333), or 0.99999 < 1 < 1.00002. When it is translated in fraction, 3 x 1/3= 1, therefore 0.99999… is close enough to be equal to 1.

Classroom Activity 6 (one class period)

This activity is for students to see that all rational numbers, even if the decimal numbers have an infinitely repeating pattern, has a precise place on the number line. I will ask students to find a fraction that is equivalent to $0.\overline{12}$ = 0.121212…without showing them the formula to convert repeating decimals to fractions. Students will be given a small group teamwork time to figure out how to find the equivalent fraction to the given repeating decimal. I will then model how to derive the repeating decimal formula by showing them Example 12. We will do one or two more repeating decimal problems together and students will solve one problem independently.

Example 14. Students show how to express a repeating decimal as a fraction:

0.3333… = x

0.3333… = 1/10 (3.333…) = 3/10 + 0.333…

x = 3/10+ x/10

Therefore, subtracting x/10 from both sides, we get

x (1 – 1/10) = 3/10

Or x (9/10) = 3/10

Dividing by 9/10, we get

x = (3/10) / (9/10)

After simplifying, we finally see that

x = 3/9 = 1/3

Therefore, 0.3333… = 3(10-1) = 3/9 = 1/3

At this point, students will plot $0.\overline{3}$ on the number line by demonstrating the decimal expansion method as Example 13 exhibits. Example 13 will be done individually, instead of working in groups since students should feel comfortable doing decimal expansions by now since we have practiced together for many times by now. This assignment will reconfirm the fact that every decimal number, even if it repeats infinitely, has one unique address on the number line. Students now know that every decimal that has a repeating pattern is a rational number. In order to check if their answer is correct, students will use long divisions to see if the fraction matches to the repeating decimal. Students will trade their work with others to check their answers.

4. Patterns in Repeating Decimals

Students will examine some repeating decimals in different repeating lengths and place those decimals on the number line.

Example 15. Repeating length of different digits.

a. 1/9 (Repeating length of one digit)

Long division $9\overline{)1} = 0.111\ldots = 0.\overline{1}$

b. 1/11 (Repeating length of two digits)

Long division $11\overline{)1} = 0.090909\ldots = 0.\overline{09}$

c. 1/37 (Repeating length of three digits)

Long division $37\overline{)1} = 0.027027027\ldots = 0.\overline{027}$

d. 1/7 (Repeating length of six digits)

Long division $7\overline{)1} = 0.142857142857142857\ldots = 0.\overline{142857}$

When having rational numbers, you can predict the address system by analyzing decimal patterns where irrational numbers cannot be predicted. Students will verify these repeating decimals by the method of Example 12. Any number that has an infinitely long decimal expansion with no repeating pattern cannot be rational; therefore it is an irrational number.

Classroom Activity 7 (one class period)

We will continue to problem solve the repeating decimal conversion problems. In this exercise, students will be introduced to repeating decimals with special patterns, such as the examples shown in Example 15. As we do the long divisions of the given fractions, it is important to make sure students bring down another zero from the decimal form of 1 and not leaving the remainder from the previous step in order to keep dividing. Although doing the long division of 1/9 (Example 15-a) is rather simple, the rest of Example 15 fractions will carry longer repeating decimal blocks because the denominator will generate all possible reminder to complete the repeating block. This activity will be assigned as a group task since it can be time consuming for some of the 8th graders.

Once students complete delivering each fraction’s repeating decimal values, they will check their answers by using the conversion method of Example 12 and 14. This is a formative assessment for me to check where my students are and decide if I need to provide extra support on this lesson.

Further in the future : Placement of irrational numbers on the number line

Students will use the Pythagorean theorem to find the location of some irrational numbers on the number line. For example, students can draw a line (1 inch) and mark it as “1” on the number line. Then put another 1-inch line that is perpendicular to the line that’s already drawn. As students complete drawing the right triangle by putting the hypotenuse, the length of the hypotenuse will be Ö2, according to the Pythagorean theorem rule. By using a compass, students will transfer the same length of the hypotenuse on the number line and mark it as “Ö2.” Students are required to explain their thinking process by making a flow map. Each team will also be asked if there is any other way, other than using the geometric way, of locating irrational numbers on the number line without using the Pythagorean theorem. Teacher can introduce students to the method of rational approximations using a series of rational numbers to get closer and closer to a given number if no one comes up with an alternate way of placing an irrational number on the number line.[20] Students use the method of rational approximations to determine the decimal expansion of an irrational number. As students plot irrational numbers on the number line, it will help reinforce the idea that irrational numbers fill in the holes that rational numbers leave on the number line and it is a measuring tool rather than a counting system.

Appendix

Common Core 8th Grade Mathematics

Number Sense

Know that there are numbers that are not rational, and approximate them by rational numbers.

1. Know that numbers that are not rational are called irrational. Understand

informally that every number has a decimal expansion; for rational numbers show that the decimal expansion repeats eventually, and convert a decimal expansion that repeats eventually into a rational number.

Expressions and Equations

4. Perform operations with numbers expressed in scientific notation, including problems where both decimal and scientific notation are used. Use scientific notation and choose units of appropriate size for measurements of very large or very small quantities (e.g., use millimeters per year for seafloor spreading). Interpret scientific notation that has been generated by technology.

Standard for future lessons:

Number Sense

1. Use rational approximation of irrational numbers to compare the size of irrational numbers, locate them approximately on a number line diagram, and estimate the value of expressions. For example, by truncating the decimal expansion of Ö2, show that Ö2 is between 1 and 2, then between 1.4 and 1.5, and explain how to continue on to get better approximations.

Resources for Teachers and Students

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