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Understanding Problems: Using Bar Models with Common Core Taxonomies

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Introduction and Rationale

Next year I will teach third grade. This is a change from teaching fourth grade, where I have spent most of my teaching career up to now. Since I am switching grade levels, I recently reviewed Virginia's standards for third grade mathematics. I was not surprised to see that the importance of problem solving is emphasized, considering that problem solving is always the most difficult aspect of the math curriculum to teach. I believe that many factors combine to create the difficulty: the nuances of each problem, the reading comprehension component, the application of skills, and the need for deeper understanding of computational skills. These aspects correspond roughly to Polya's steps 1, 2, and 3.[1] In addition to the skills needed to understand and solve a problem, students need to have perseverance to bring it all together.

The Mathematics Standards of Learning for Virginia Public Schools[2] state: "Problem solving has been integrated throughout the six content strands. The development of problem solving skills should be a major goal of the mathematics program at every grade level. Instruction in the process of problem solving will need to be integrated early and continuously into each student's mathematics education. Students must be helped to develop a wide range of skills and strategies for solving a variety of problem types."

However, despite the emphasis placed on problem solving, there is little guidance or training on how to do that. In fact, as I read through the specific standards, one of the few utterances about problem solving that I found says the students will, "create and solve problems that involve multiplication of two whole numbers, one factor 99 or less and the second factor 5 or less." The standards do not mention any strategies or methods to use to solve problems.

When I go through the pacing guide provided by the school district, it skips around the textbook and provides random websites, some of which no longer exist, for additional practice. If a lesson is not directly tied to the standard, it is excluded. Fortunately for my students, I choose to follow the book more sequentially and intertwine Singapore math strategies, along with knowledge I have gained from working with Roger Howe through the Yale National Initiative.

Another issue that is promoted widely throughout resources and my school district is the strategy of key words to solve problems. By focusing on key words, children tend to spot the key word, then they choose what they deem to be the appropriate operation based solely on a word or two in the problem. Then they perform the "matching" operation on the numbers given in the problem. The key word strategy is limited, since a word problem with the word *more* may require addition or subtraction or, later in the curriculum, multiplication or division; and a word problem with the word *less* may require addition or subtraction depending on the given scenario. In order for our students to become more comfortable with problem solving, they need to comprehend the action and relationships described in the problem instead of relying on the key word strategy. To reiterate, the standards say, "The development of problem solving skills should be a major goal of the mathematics program at every grade level." For this reason, I believe our students deserve better. Teaching the key word strategy to children is a disservice, especially given the importance of problem solving as stated in the Virginia State standards.

In order to address the issues listed above, I am creating a curriculum unit that focuses on problem solving techniques. I wrote a curriculum unit for fourth grade in 2007, titled *Dr. Word Problem: Solving Word Problems with the Four Basic Operations of Mathematics Using Singapore Bar Models*.[3] I do not want to duplicate that unit. This unit will address some of what I consider to be the best problem-solving strategies. I will briefly touch on Singapore bar models, but if you are interested in learning more about them, my 2007 unit is listed in the resources, and includes its own list of resources.

The crux of this unit is understanding the Common Core taxonomy of problems,[4] while infusing Polya's steps,[5] and aspects of the Singapore model method.[6] I include a collection of word problems that identify the 14 types of one-step addition and subtraction problems and the nine one-step multiplication and division situations. The curriculum unit will include Polya's four-steps as he is still considered the "guru" of problem solving for his work from 1945.[7] It will also include some steps from Singapore that are likely based on Polya's four steps, as they seem to work hand in hand.

Polya's Method

George Polya was a mathematician from Hungary, who focused on problem solving. In his most prized publication, *How to Solve It* was published in 1945, Polya identifies four principles of problem solving.

Polya's First Principle: Understand the problem

Polya's Second Principle: Devise a plan

Polya's Third Principle: Carry out the plan

Polya's Fourth Principle: Look Back

The first principle is the most important for elementary students. The students should perform quite well with relatively simple word problems to which they are exposed if they can understand what is happening in the word problem.

A Brief History of Singapore Math

In 1965 Singapore became an independent Republic. Recognizing that there are no resources in Singapore except the people, this young nation focused on education as a means to transform its economy. In a report authored by Dr. Goh Keng Swee, the Ministry of Education Study Team identified weaknesses in Singapore's educational system. The findings led to the introduction of a New Education System (NES) in 1979. The Curriculum Development Institute of Singapore (CDIS) was established in 1980 and charged with creating a new math curriculum. In 1981, the first primary mathematics curriculum was introduced based on the concrete-pictorial-abstract approach. By 1983, the 'model method'[8] was introduced in primary 5 and 6 to address difficulties students had with word problems. Today the model method is first introduced to students in primary 1.[9]

Model Method

I have had much success with teaching my students how to use the model method. The training I have received regarding the Singapore model method teaches children to pull out and label the facts or the quantities from the problem. Then students write out the question, draw a model, and the final step is to refer back to the question and write the answer as a sentence that answers the question. The last step is not only helpful to make sure the students answered the question asked, but also because it essentially forces them to label the units. To reiterate, the steps I have my students follow are:

1. Pull out and label the facts (quantities).
2. Write the question.
3. Draw a model to show the relationship between the quantities in the problem.
4. Reread the question and write the answer as a sentence.

I have found that following these steps helps my students succeed. In my experience, it allows the students to focus more on the numbers, thus reducing the confusion the students face when they are mixed in with a bunch of words.

Because of my interest in and success with this method, I conducted further research into the history of the model method as well as the pros and cons of the technique, and the importance of identifying the unit to which the numbers in the problem refer.[10] It is informative and provides more insight into the technique.

The model method is a heuristic, or a practical method, used to solve whole number word problems representing part-whole and comparison problems. A before-after concept is used with both part-whole and comparison problems to solve more complex structures. The before-after concept involves drawing two bar model representations to represent different stages in the word problem. The model method allows students to solve higher level problems without the use of algebra. Additionally, the model method focuses on representation, which is the key to solving word problems.

According to Ng and Lee,[11] the model method can be broken into three phases. The first phase is the Text

Phase[12] , in which children read the problem that is presented in text form. The second phase is the Structural Phase,[13] where the students transform the text into a model. The third phase is the Procedural-Symbolic Phase[14] . In the final phase students have to use the model and develop the mathematical steps to solve it. Ng and Lee suggest that since primary students do not yet know how to work with equations, they must use the unitary method.[15] The unitary method is a technique where students undo operations to find the value of one unit. Typically, that unit can be used to figure out other chunks or pieces of the model, which lead to a solution to the problem. This strategy requires the students to develop arithmetic equations to stand for the unknown units, which is essentially what students do when solving with bar models. Due to the success of the Singapore math techniques and the model method, the curriculum has become more widely used across the world, and is popular in the United States.

Pros of the Model Method

There are many benefits to teaching the Singapore model method. Typically, students tend to see the numbers in a word problem and do the first calculation that pops into their heads. The model method helps teach students to slow down, read the problem several times, and think about what is happening. Once a student has gone through these steps, a model can be drawn to represent the situation. One could say, that it is a device that enforces Polya's four step method. I can attest to this from my own experience as I learned how to use the model method to solve word problems. I find myself reading the problem over slowly several times until I understand the problem, which is Polya's first step. Then, I extrapolate the key information that is both known and unknown, and restate the question in order to devise a plan as stated in Polya's second step. In this case, the plan is to draw a picture, in particular, a bar model. Then, I carry out the plan (step three) and figure out the answer. Following Polya's fourth step, I check over my work to make sure it makes sense. According to Englard, the model method "Puts the focus back on the relationships and actions presented in the problem and helps students choose both the operations and the sequence of steps that are needed to solve the problem."[16]

The steps required to understand and draw a bar model is the crux of the model method. Ng and Lee[17] refer to a number of studies that found that visual and concrete representations improve performance in solving word problems. "The model-method affords higher ability children without access to letter symbolic algebra a means to represent and solve algebraic word problems."[18] In one of the studies conducted by Ng and Lee, the teachers reported that because students represent the problem visually, it affords teachers the opportunity to inquire about difficulties students have with the representation. Another advantage of the model method is that it is not an "all or nothing process." The findings agree with other research that reports that children tend to make their errors in the representation of the model. The Singapore model method has become a popular math instruction method because it is proven to enhance problem solving skills.

Cons of the Model Method

Yan Kow Cheong with Math Plus Consultancy conducted a study of the model method. The difficulties noted pertain more to problems drawing the models. Cheong noted three main problems: 1) difficulty of an "accurate diagram," 2) division in a block diagram, and 3) inappropriate use of the model method.[19]

It can be difficult for students to draw models accurately. While the models do not have to be drawn to scale, the model does need to accurately represent the problem. The division of the blocks is problematic particularly when a bar model requires additional partitioning. Another problem is that some curriculum books are overusing the model method when it is not appropriate. Cheong explains that some writers of curriculum books, while pushing the use of the model approach, are using the method when it is not appropriate.

Difficulties with algebra

Solving word problems with algebra is particularly problematic due to:

1) understanding the meaning of letters used in symbolic algebra,

2) translating natural language into equations,

3) understanding the semantic structure of word problems, in particular the nature of relationships between quantities and how they are linked, and

4) using text-based semantic cues in the construction of equations."[20]

Singapore students who have been taught the model method have exhibited challenges transitioning to algebra. "Studies (e.g. Ng *et al.*)[21] have shown that poor bridging of students from the use of bar diagrams to the use of letter-symbolic algebraic methods can hinder their learning of algebra."[22] The bright students don't seem to have problems transitioning to algebra, but others tend to hold onto the bar model heuristic. A software tool called ALGEBAR[23] was designed in an effort to provide better bridging between the two subjects. Researchers conducted several studies to develop not only software, but a larger instructional package to address the algebra difficulties. The research determined that secondary problem solving requires mixed schemas. Students who used the bar models relied on these primary school schemas and forward calculations. Basically, prior knowledge influences new learning, and in this case, hinders it. According to this study, "students need to be taught the explicit structural homomorphism between bar model representation and algebraic equations."[24] Students tend to want to calculate instead of manipulating the letter-symbol algebraic equations. Teachers should provide scaffolding by using models followed by weaning once the students gain comfort with equations. Eventually, more difficult problems that can only be solved with algebra should be introduced, so the students will realize the importance of letter-symbol algebra and link it to their prior knowledge. While the ALGEBAR is not a part of my curriculum unit, this article provided insight into how the model method hinders students' algebra acquisition.

Whole Group Discussion

A strategy I will consciously infuse into my math block is more whole group discussion. I plan to use Math Talks throughout the year, so my students will use content specific vocabulary to explain their mathematical thinking. The other way I will employ whole group discussion is by affording time several times a week to discuss solutions and misconceptions about word problems. By sharing solutions and errors, the students will see that there are many ways to solve a single problem and that they can in fact learn from mistakes, even if they are not their own. The goal is that over time, students will not only become more comfortable with the idea of conversing about math, but will also learn from each other's brilliance as well as shortfalls.

Taxonomy of Word Problems

The Common Core uses a chart giving a taxonomy of one step addition and subtraction situations.[25] Three main types are identified: *change*, *comparison,* and *part-part-whole*. The *change* and *comparison* are then split into two main subtypes: *change plus* and *change minus,* and *more* and *less,* respectively. Each of these types is further subdivided into three more categories based on the unknown. For the change problems, these categories are: *result unknown, change unknown, start unknown.* For the comparison problems, the categories are: *difference unknown, b*igger *unknown*, and *smaller unknown*. The third main type, the *part-part-whole,* differs because there are only two subtypes: *total unknown (whole unknown)* or *addend unknown (part unknown).* There are interchangeable names for the various types of problems because the Common Core standards use names in the chart, but in literature other names are used to describe the problem types. I refer to them with the various names to build teacher background knowledge and hopefully to avoid confusion. These types will be discussed below with examples. I do not plan to have the students name the types, but it is really important to expose the students to all of the different kinds of problems.

As you explore these word problems, the natural connection between addition and subtraction will become more evident. For example, you will see that there are equations in which addition represents the problem, but subtraction is required to solve it.

As the different types of problems are introduced, the same numbers and situation will be explored. The numbers chosen for the initial explanation of the problem types will be small whole numbers. This is done purposefully, because the focus at this point is on introducing the problem types and evoking the similarities and differences.

Addition and Subtraction - Change

"Add to" or "Change plus"

For this category, the problem could have a *result unknown,* a *change unknown*, or a *start unknown.*

Eight birds are sitting on the fence. Three more land on the fence. How many are on the fence now? The bar diagram for this problem is in Figure 1.

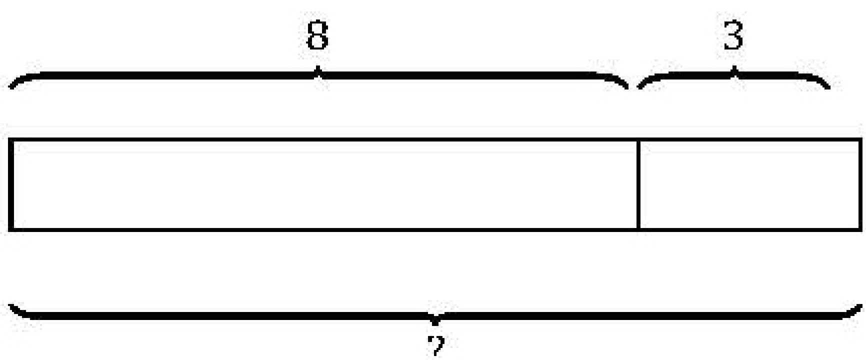

Figure 1

This is an *add to (change plus)/ result unknown* since there were eight birds on the fence and a positive change happened indicating addition. The total number of birds on the fence is unknown, therefore it is a *result unknown*. The equation to solve the problem is 8+3=∎.

The next example is an *add to (change plus)/change unknown.* Eight birds were sitting on the fence. Some more landed on the fence. There are now 11 birds on the fence. How many birds landed on the fence? The diagram for this is given in Figure 2.

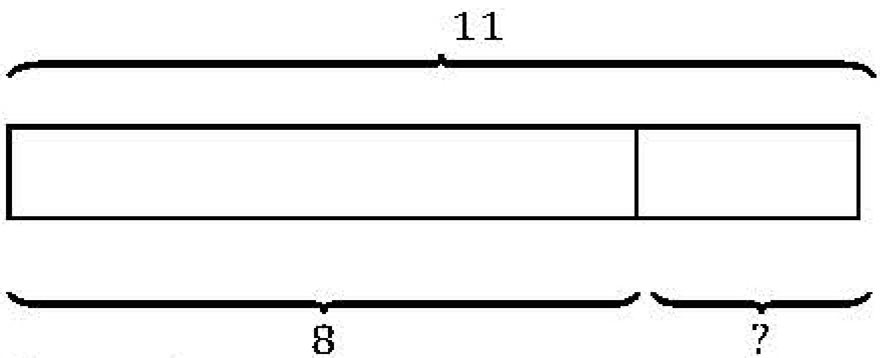

Figure 2

In this situation, the number of birds originally on the fence is known. Then some more birds join them, but the problem does not state how many more came. The result is 11 birds on the fence. The equation is now 8+∎=11. While the equation shows addition, this problem is solved by subtraction 8+∎=11. If students were using a key word strategy, they would see the word *more*, thus students may be inclined to add eleven and eight, which does not answer the question.

The third type is *add to (change plus)/start unknown.*

Some birds are on the fence. Three more land on the fence. There are now 11 on the fence. How many were on the fence in the beginning? Figure 3 shows the bar model for this problem.

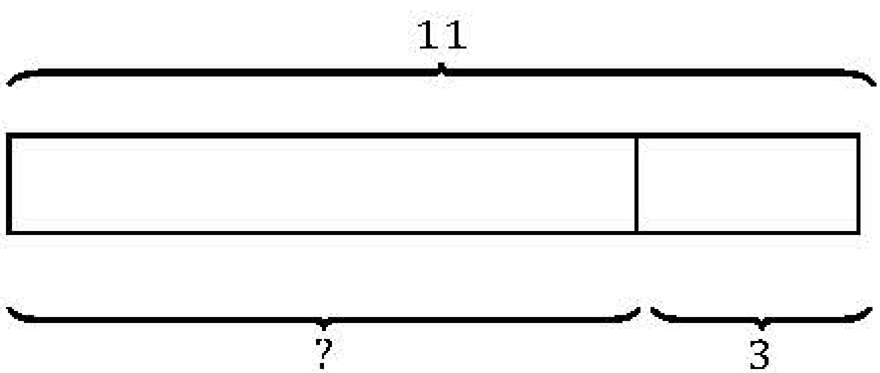

Figure 3

It would be expressed as ∎+3=11. The initial number of birds on the fence is unknown. The equation shows addition, but again, subtraction is required to solve for the answer; 11-3=∎. Once again, using the key word method, students would mistakenly be inclined to add eleven plus three since the key word *more* is used.

“Take from” or “Change minus”

These are also “add to” problems, but now the change is being “taken from.” There are the same three types:

result unknown, change unknown, and *start unknown.*

The first problem is a *Take from/result unknown*.

Six squirrels were in a tree. Two ran away. How many were left in the tree? The equation is 6-2=■ ,and it would be solved with subtraction. The bar model is shown in Figure 4.

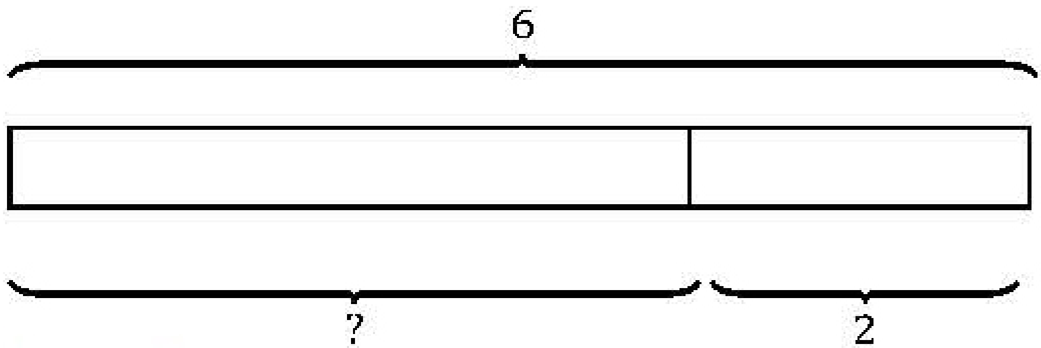

Figure 4

The second problem of this kind is a *Take from/change unknown*.

Six squirrels were in a tree. Some ran away. There were four left in the tree. How many ran away? Mathematically, the equation is 6-■=4, and the solution is ascertained by subtracting 6-4=2. The diagram for this is given in Figure 5.

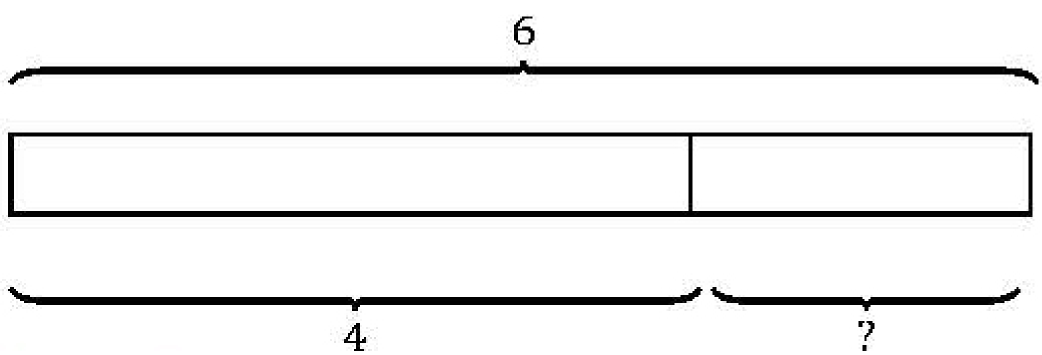

Figure 5

The last one in this series is a *Take from/ start unknown*.

Some squirrels were in a tree. Two ran away. There were four left. How many were in the tree at the start? The bar model for this problem is shown in Figure 6.

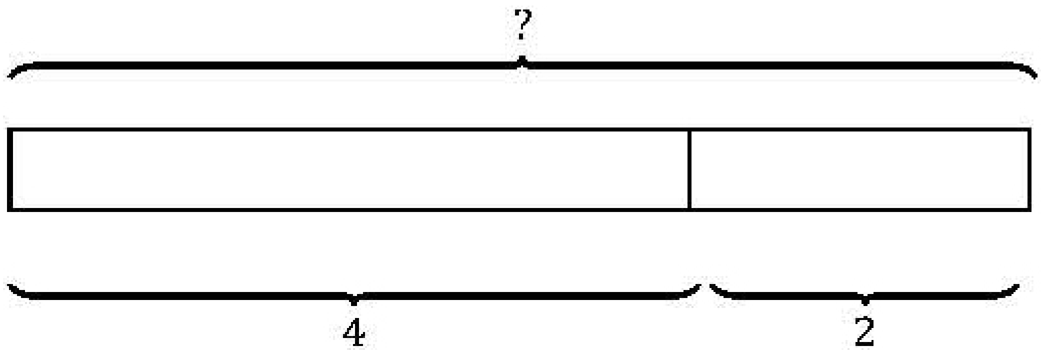

Figure 6

The equation is ■-2=4 , but the solution is found by computing 4+2=6. Once again, the key word method would be problematic. The problem says *left*, which typically means to subtract, but in fact addition is required.

Addition and Subtraction - “Put together” or “Part-part-whole”

For these, it is either an *unknownpart* or *unknown addend* or an *unknownwhole* or *unknowntotal.*

There are six green apples and four red apples are in a bowl. How many apples are in bowl?

The number sentence is 6+4=■, and the answer is 10. The diagram for this problem is given in Figure 7.

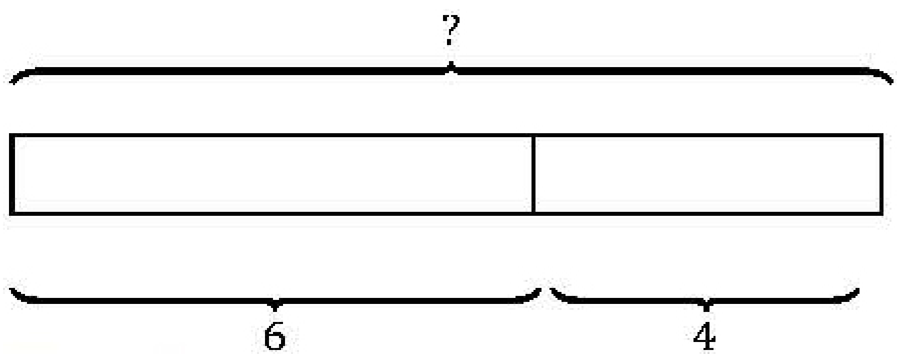

Figure 7

An example of an *unknown part* or *unknown addend* is:

There are 10 apples in the bowl. There are six green apples and the rest are red apples. How many apples are red? Figure 8 shows the bar model for this problem.

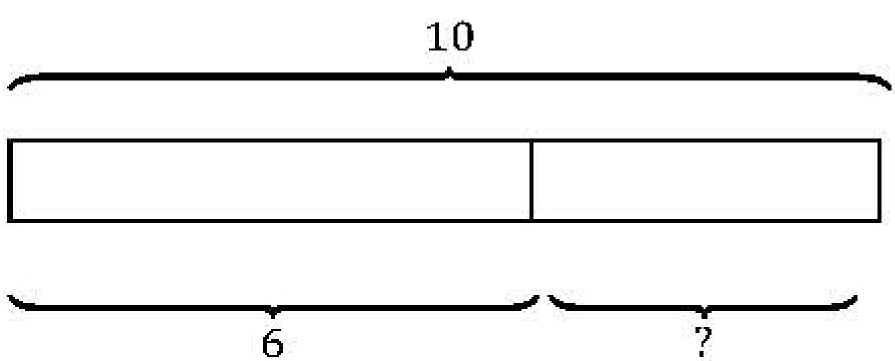

Figure 8

The equation, 6+∎=10, is used to represent the problem, but to compute the answer students need to compute 10-6=4.

Addition and Subtraction - Compare

The compare problems are probably the trickiest of the addition and subtraction types. There are two of each kind: *difference unknown*, *bigger unknown*, and *smaller unknown*, for a total of six. Part of the confusion stems from the interchangeable vocabulary used to describe these types.

Difference unknown/more

Jaquan has six apples. Devin has two apples. How many more apples does Jaquan have than Devin? The model is shown in Figure 9.

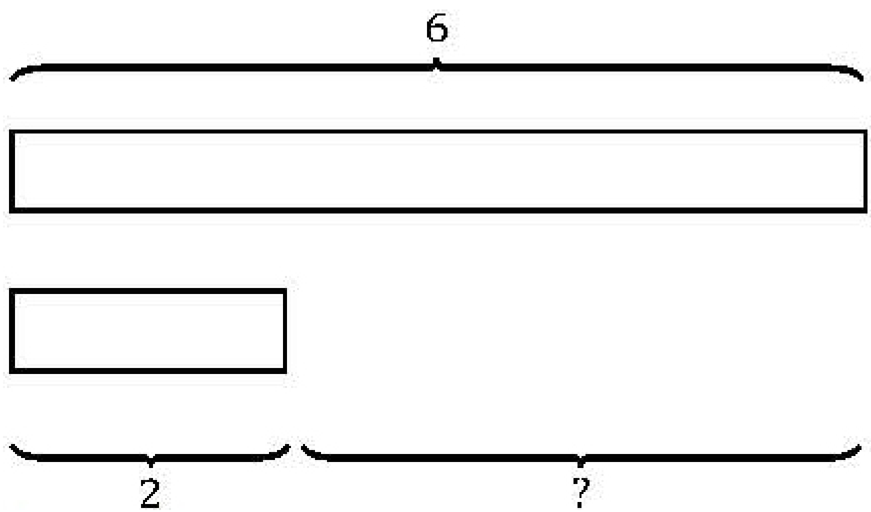

Figure 9

It is a *difference unknown,* as the answer to the subtraction problem is unknown. It falls into the *more* category because the question asks, "*How much more?"* The number sentence 2+∎=6 represents this problem. However, the operation needed solve the problem is 6-2=∎.

Difference unknown/fewer (less)

Jaquan has six apples. Devin has two apples. How many fewer does Devin have?

The sentence 6-2=∎ will provide the solution. Figure 10 models this problem.

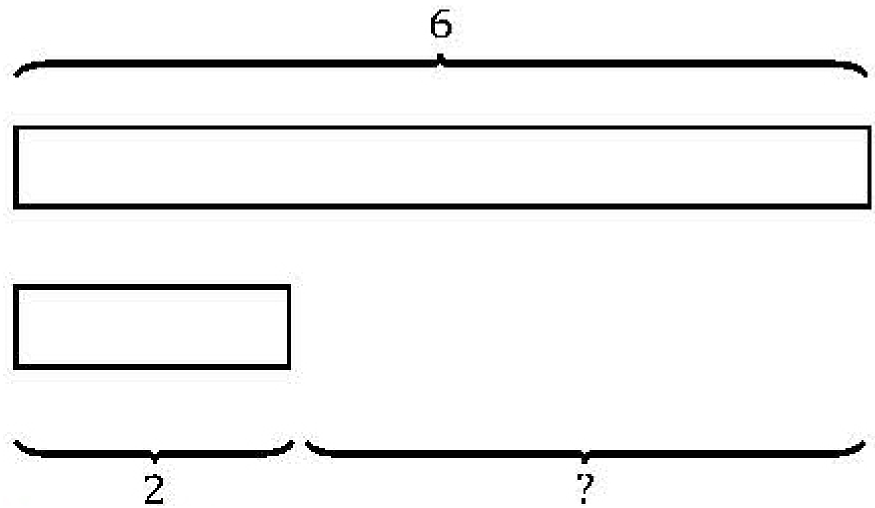

Figure 10

The two problems above sound quite different, but are answered with the same computation.

Bigger unknown/more

Jaquan has 4 more apples than Devin. Devin has 2 apples. How many apples does Jaquan have?

The equation 2+4=∎, leads to the correct answer of 6 apples. The bar model to represent this problem is shown in Figure 11.

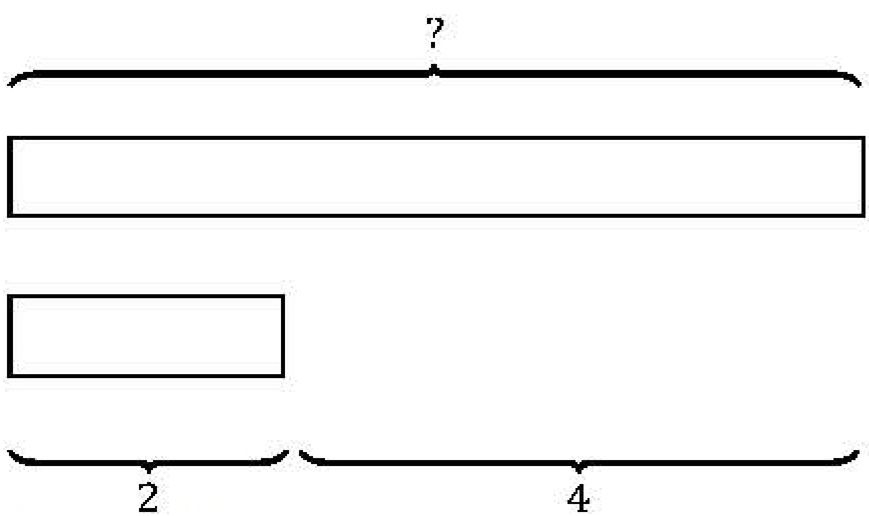

Figure 11

Bigger unknown/fewer (less)

Devin has 4 fewer apples than Jaquan. Devin has 2 apples. How many apples does Jaquan have?

In order to determine the number of apples Jaquan has, use the equation 2+4=∎. Figure 12 provides the model for this problem.

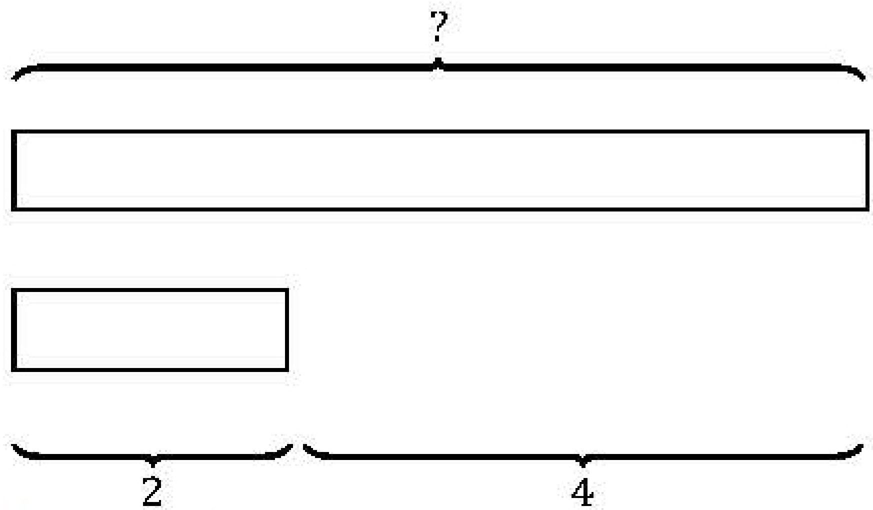

Figure 12

The two problems above, again, sound quite different, but are answered with the same computation.

Smaller unknown/more

Jaquan has 4 more apples than Devin. Jaquan has 6 apples. How many apples does Devin have? The statement ∎+4=6 represents this scenario, but the subtraction sentence 6-4=∎ is needed to solve this problem. Once again, the word *more* that indicates addition when using the key word strategy would lead students astray. The model in Figure 13 represents this problem.

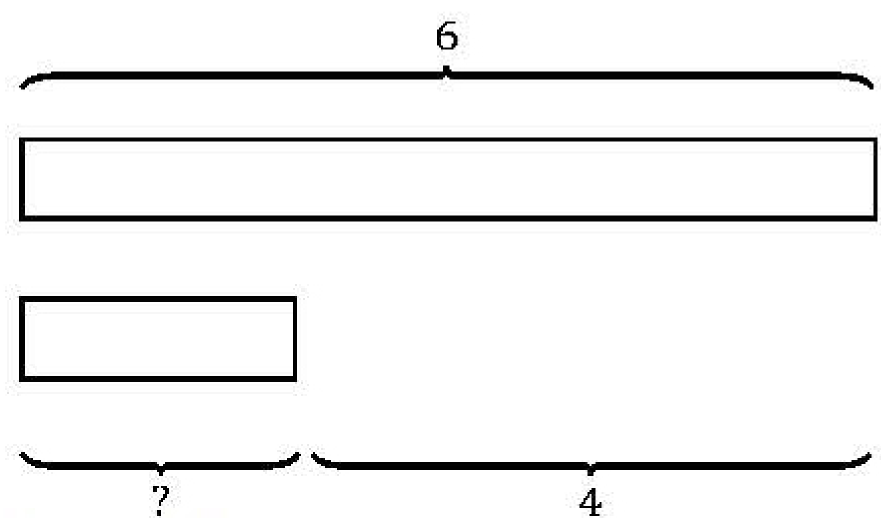

Figure 13

Smaller unknown/fewer (less)

Devin has 4 fewer apples than Jaquan. Jaquan has 6 apples. How many apples does Devin have? 6-4=∎ will solve this problem. Figure 14 shows the bar model for this problem.

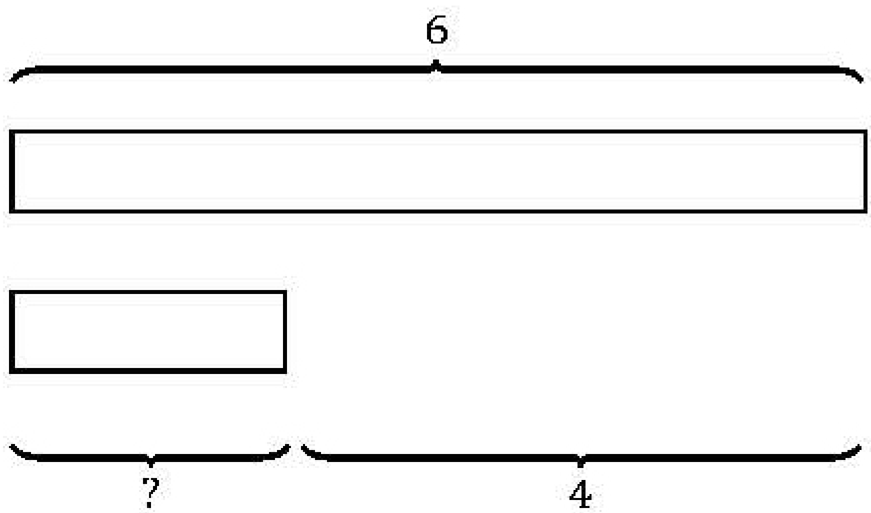

Figure 14

The two comparison problems above are answered with the same computation, and sound quite different.

As with addition/subtraction problems, there are designated problem types for multiplication and division with three main categories: *equal groups, comparison*, and *array/area*. Just as the main types of addition/subtraction problems addressed different important contexts where addition and subtraction are used, the categories of multiplication/division problems point to the main interpretations or applications of these operations. The *equal groups* and *comparison* ones are closely related. The *number of groups* is analogous to the comparison factor. In fact, when a collection is made out of a number of equal groups, the comparison factor between the big collection and one of the groups is exactly the number of groups. Another similarity between the *compare* and the *comparison* problems are that they both can be represented through linear measurement. The *equal groups* and the *comparison* problems have three subcategories: *unknown product, groups size unknown,* and *number of groups unknown.* The *arrays/area* problems are different because they represent problems in two-dimensions. The Singapore bar models are a linear representation of a problem, and would not be an effective strategy for *array/area* models. As you read these examples, notice how multiplication equations are often solved with division and vice versa. I will now explain each type as I did for the addition and subtraction ones.

Multiplication/division – equal groups

Equal groups/unknown product

At the toy store, cars come in packs of six. If Jose buys four packs, how many toy cars will he buy in total? 6×4=∎ is the equation and the product is 24. The representation is shown in Figure 15.

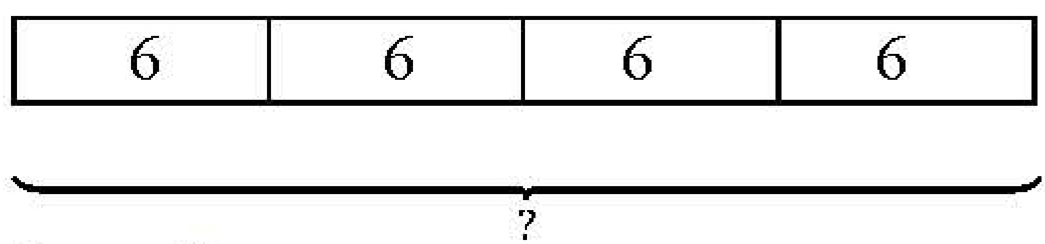

Figure 15

Equal groups/group size unknown

At the toy store, Jose buys 24 cars. Jose buys four packs. How many are in each pack?

In this situation, $4\times\blacksquare=24$ the number of groups (packs) and the product is known, but the number in each group or the groups size is unknown. The equation sets up as a multiplication problem, but it uses division $24\div4=6$ to find the solution. The bar model in Figure 16 shows this problem.

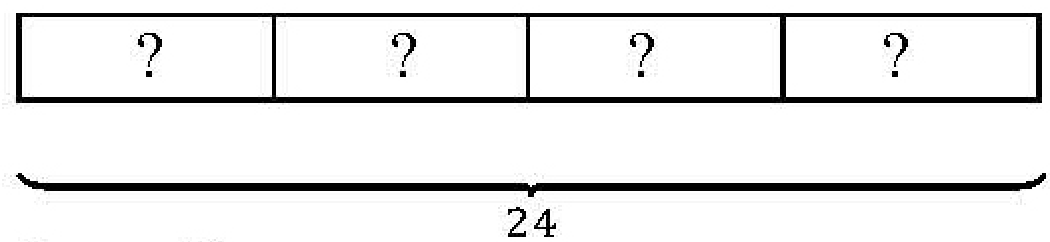

Figure 16

Equal groups/number of groups unknown

This next problem type is about equal groups. It is like the previous one but instead of group size, the number of groups is unknown.

At the toy store, Jose buys 24 cars. There are six in each pack. How many packs does Jose buy? The equation $6\times\blacksquare=24$ represents the scenario. The number in each group is known as is the product or total that Jose bought. The unknown is the number of groups or how many packs Jose buys. Once again division is used: the equation $24\div6=4$ determines that the answer is four groups. The bar model in Figure 17 shows this problem. Notice the ellipses, which are used when the number of groups is unknown or when there are a large number of groups.

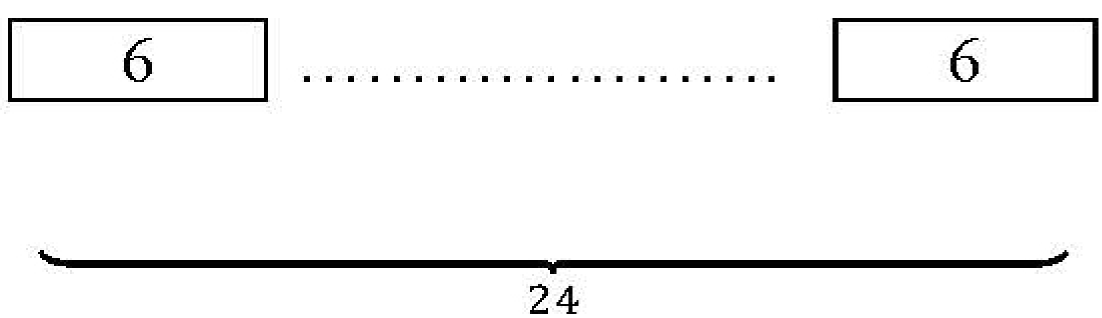

Figure 17

Multiplication/division – arrays/area

These problems use situations with objects arranged into rows or columns, representing an array, or they use measurement to show area. These are essentially equal groups problems, with the rows being the groups, and the columns being the number of groups. The point of making them a distinct category is to connect with the area model for multiplication. The arrays and area problems do not align well with Singapore bar models; therefore, I did not include images of bar models in this section.

Arrays/area – unknown product

Some chairs are arranged in the classroom. There are three rows of seven chairs. How many chairs are in there? The multiplication sentence $3\times7=\blacksquare$ corresponds to this problem, and the answer is 21 chairs.

Arrays/area – group size unknown

Twenty-one chairs are arranged into three equal rows. How many chairs are in each row? $3\times\blacksquare=21$ depicts this situation, but division is required to determine the group size of each row. So, $21\div3=7$ results in the solution to this problem. In this problem, the number of equal groups is known, but the group size is unknown.

Arrays/area – number of groups unknown

Twenty-one chairs are arranged into equal rows. There are seven chairs in each row. How many rows will there be? The number sentence $7\times\blacksquare=21$ depicts this problem, but students must divide $21\div7=3$ to ascertain the answer. In this similar situation, the number of chairs in each group is known, but the number of groups or rows is unknown.

Multiplication/division – comparison

The multiplication comparisons are also used to express measurement problems. An example and explanation of one with and without measurement is listed below.

Compare/unknown product

A pack of gummy bears cost $ 2 and a sandwich costs three times as much. How much does the sandwich cost? The equation $ $2\times3=\blacksquare$ denotes the scenario, and multiplication is used to determine that the cost of the sandwich is $ 6. The bar model in Figure 18 represents this problem.

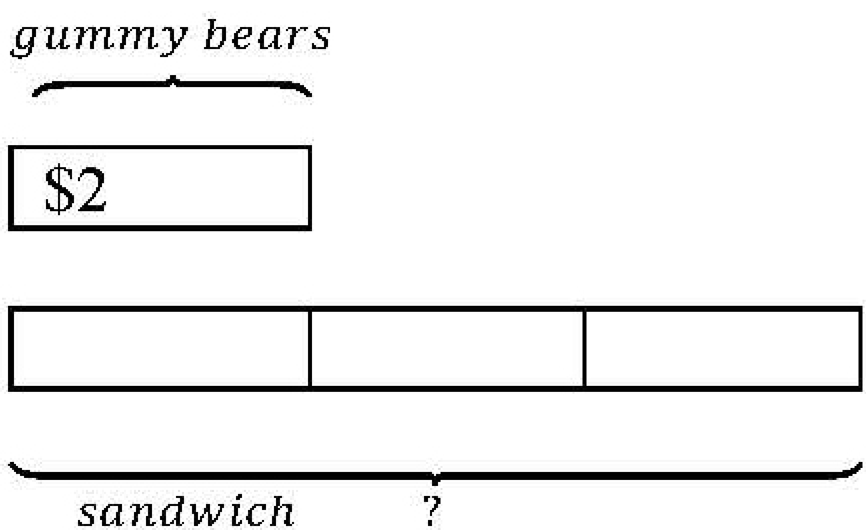

Figure 18

Jamie wants to measure the length of her desk using a large paper clip. She lays 24 paper clips end to end. Each paper clip is two inches long. What is the length of her desk? This would be represented mathematically as $24\times2=\blacksquare$. Figure 19 models this problem.

Figure 19

Compare/smaller quantity unknown

A sandwich costs $6. It costs three times as much as gummy bears. How much do the gummy bears cost? Therefore 3 × ∎ = $ 6 , but the operation of division $ 6 ÷ 3 = $ 2 is necessary to find the answer. The smaller quantity, or in this case, the price of the gummy bears is unknown. Three groups of $2 equals $6. The bar model is shown in Figure 20.

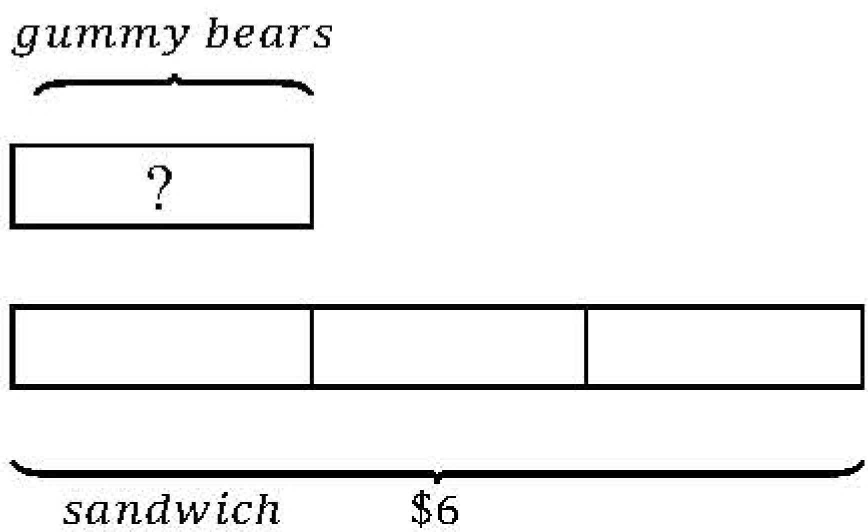

Figure 20

Compare/ Comparison factor unknown

A sandwich costs $6 and gummy bears cost $2. How many times as much does the sandwich cost? The statement $ 2 × ∎ = $ 6 represents the situation, but the division sentence $ 6 ÷ $ 2 = 3 is needed to determine the comparison factor. The bar model for this problem is shown in Figure 21.

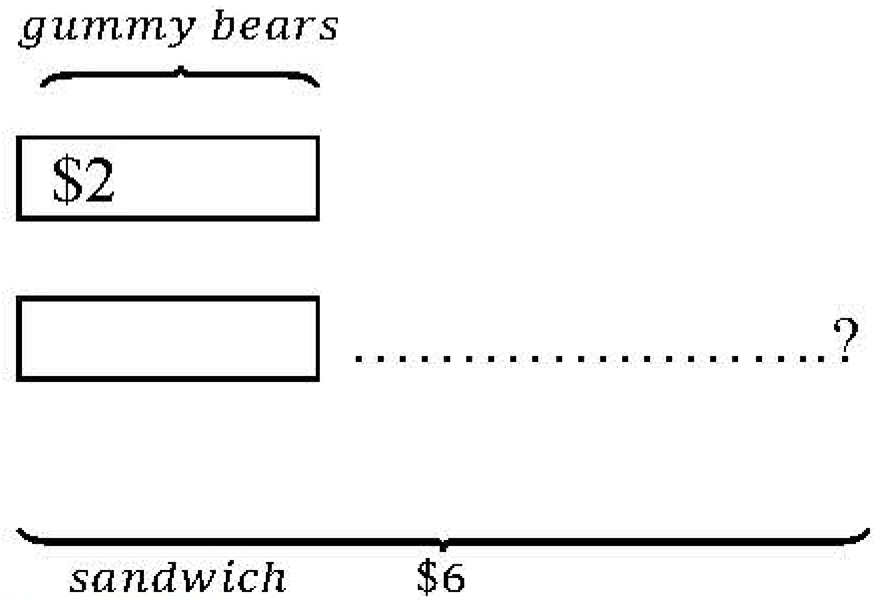

Figure 21

Multi-step word problems

All of the above-mentioned types of problems are one-step problems: they require only a single computation to arrive at an answer. One-step problems can be combined to make problems with several steps – two, three, or even more. One could say, they are the nuts and bolts that make up multi-step problems. I expect that, exposing my students to the component parts and developing a level of comfort with the various kinds will provide a strong foundation for them to be more successful with multi-step problems. Now, let’s look at a few examples of multi-step problems.

There are 37 children in the swimming pool. There are nine fewer adults than children in the pool. How many adults and children are in the pool? One way to solve this problem is to think 37-9=■ to determine the number of adults in the pool. This first part is an addition/subtraction comparison, which will guide us to the number of adults in the pool. If 37-9=28, then there are 28 adults in the pool. The second step is to combine the number of adults and children with the equation 37+28=65. The second step is a put together or part-part-whole with the total unknown. As you can see, this two-step problem is comprised of situations that have already been discussed. Figure 22 shows the model for the first step in this problem.

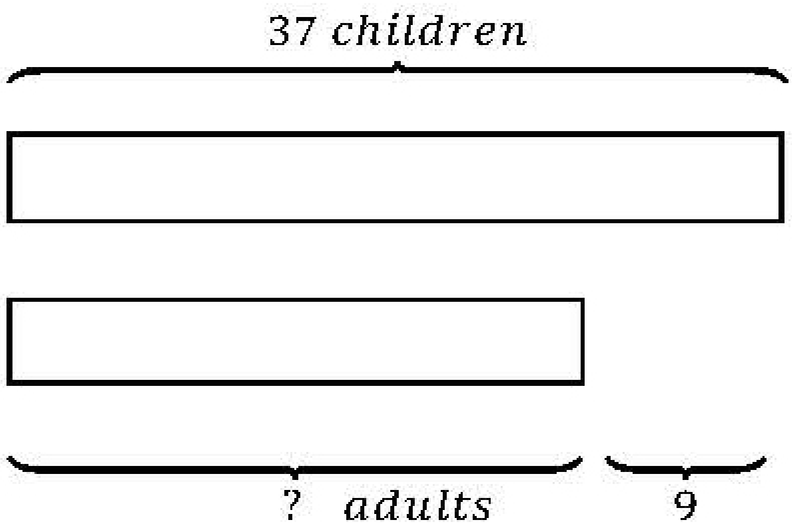

Figure 22

37-9=28 adults, then a second bar model would represent the second step. The model below represents the second step which is an addition/subtraction part-part-whole problem with the whole unknown. The second step is modeled in figure 23.

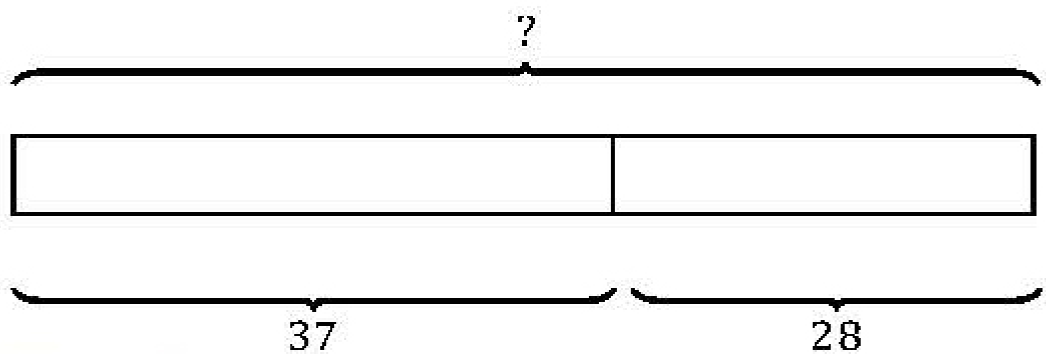

Figure 23

Let’s look at one more example. Five children are on the playground. Then three times as many join them. How many children are on the playground now? The first step is *multiplication/division comparison with the result unknown.* This step is solved by calculating 5×3=∎. So, 15 children joined them. The second part is a *change plus/result unknown* which would use the addition statement 15+5=20 to figure out the result. Figure 24 shows both steps using a single bar model.

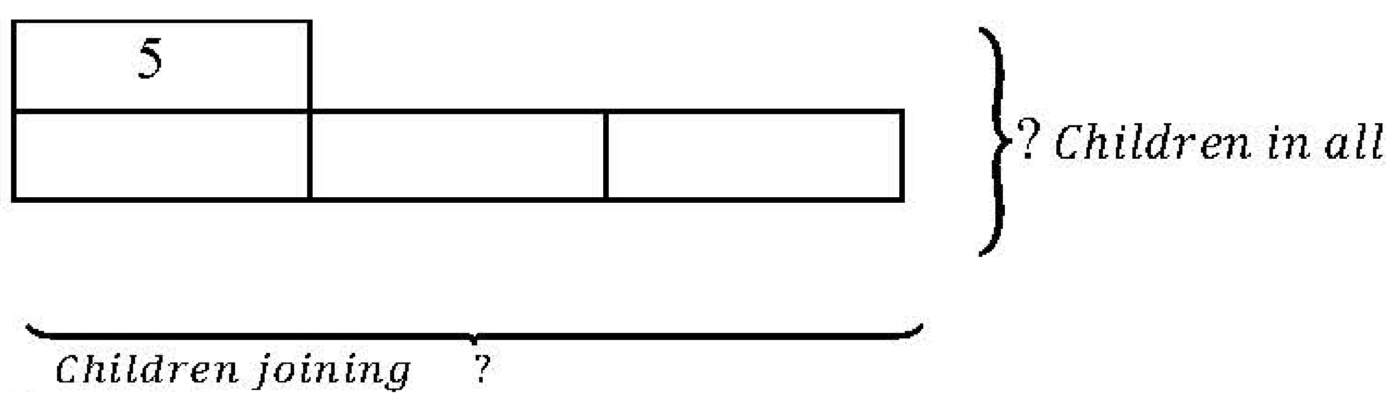

Figure 24

The one-step addition/subtraction and multiplication/division problems can be combined into about 400 types of two-step problems. It would be challenging to expose your students to all the different combinations of two-step problems. It would be impractical to teach all the combinations of three-step problems, as there are about 8,000 variations. Thus, the strategy of introducing students to the 23 parts that compose all multi-step problems is important to lay a strong foundation and should be taught with fidelity. An important takeaway, as demonstrated through the examples of word problems, is that inverse relationships between operations are not discrete. Therefore, addition and subtraction should be dealt with together, and likewise and multiplication and division should be dealt with at the same time.

Activities

The Virginia Standards of Learning state that the problem-solving process should be integrated early and continuously. So, I plan to work through the Common Core taxonomy of problem types right from the start of the school year. I plan to spend about 15 minutes of the daily math period on problem solving, introducing the 23 types of problems.

I will slowly introduce the types (i.e. Change Plus) one subcategory (i.e. result unknown, change unknown, and start unknown) at a time. I anticipate using the I do, we do, you do approach for each subcategory. So, the teacher will model, students will do one with me, and then students will do one alone. After a type has been fully introduced, I will allow time for a mixed review and students will engage in a Math Talk so students can share their solutions and misconceptions.

After students have worked with all 23 types, the focus will shift to multi-step problems, and will continue throughout the year. Below is the schedule I plan to follow beginning the second week of school.

Day 1 – Change Plus +/- result unknown

Day 2 – Change Plus +/- change unknown

Day 3 – Change Plus +/- start unknown

Day 4 – Mixed Change +/- plus problems and Math Talk

Day 5 – Change Minus +/- result unknown

Day 6 – Change Minus +/- change unknown

Day 7 – Change Minus +/- start unknown

Day 8 – Mixed Change Minus +/- problems and Math Talk

Day 9 – Mixed Change Plus +/- and Change Minus +/- problems

Day 10 – Part-Part-Whole +/- whole unknown

Day 11 – Part-Part-Whole +/- part unknown

Day 12 – Mixed Part-Part-Whole +/- practice and Math Talk

Day 13 – Mixed Change Plus, Change Minus, & Part-Part-Whole +/- practice and Math Talk

Day 14 – Compare Difference Unknown +/- more

Day 15 – Compare Difference Unknown +/- less

Day 16 – Compare Bigger Unknown +/- more

Day 17 – Compare Bigger Unknown +/- less

Day 18 – Compare Smaller Unknown +/- more

Day 19 – Compare Smaller Unknown +/- less

Day 20 – Mixed Compare Practice +/- and Math Talk

Day 21 – Mixed Change +/-, Part-Part-Whole +/- , Compare Practice +/- and Math Talk

Day 22 – Equal Groups ×/÷ – Product Unknown

Day 23 – Equal Groups ×/÷ – Group Size Unknown

Day 24 – Equal Groups ×/÷ – Number of Groups Unknown

Day 25 – Mixed Equal Groups ×/÷ Practice and Math Talk

Day 26 – Mixed Review Addition/Subtraction Change, Part-Part-Whole, Compare & Multiplication/Division Equal Groups ×/÷ practice and Math Talk

Day 27 – Area/Arrays ×/÷ Product Unknown

Day 28 – Areas/Arrays ×/÷ Group Size Unknown

Day 29 – Areas/Arrays ×/÷ Number of Groups Unknown

Day 30 - Areas/Arrays ×/÷ Mixed practice and Math Talk

Day 31 - Mixed Review Addition/Subtraction Change, Part-Part-Whole, Compare & Multiplication/Division Equal Groups and Areas/Arrays practice and Math Talk

Day 32 – Compare ×/÷ Unknown Product

Day 33 – Compare ×/÷ Smaller Quantity Unknown

Day 34 – Compare ×/÷ Comparison Factor Unknown

Day 35 - Mixed Compare ×/÷ practice and Math Talk

Day 36 – Mixed Review Addition/Subtraction Change, Part-Part-Whole, Compare & Multiplication/Division Equal Groups, Areas/Arrays, Compare practice and Math Talk

Day 37 - Mixed Review Addition/Subtraction Change, Part-Part-Whole, Compare & Multiplication/Division Equal Groups, Areas/Arrays, Compare practice and Math Talk

Day 38 - Multi-step Problems

Day 39 - Multi-step Problems

Day 40 – Multi-step Problems and Math Talk

Day 41 – Multi-step Problems and Math Talk

Day 42 - Multi-step Problems and Math Talk

Day 43 - Multi-step Problems and Math Talk

Day 44 - Multi-step Problems (ongoing throughout the year) and Math Talk

Day 45 – Problem Solving Assessment

I am selecting four activities from the above schedule to highlight and flesh out in more detail. I have selected Day 1, Day 4, Day 27, and Day 40 purposefully. Day 1 will introduce how I plan to go through each type of problem. Day 4 provides an example of the end of a type of problem review and lays out the format for the Math Talk. Day 27 is the first day of area/array problems and since this type of problem does not lend itself to bar models, it is important to lay out an approach. Finally, I chose Day 40 to share a Math Talk from the multi-step section. The problems will initially come from the collection of problems, and I will categorize and make up more as needed. I also will likely utilize some problem sets from curriculum units developed in the Yale National Initiative ® seminars with Roger Howe in 2015, *Problem Solving with the Common Core,* and 2017, *From Arithmetic to Algebra: Variables, Word Problems, Fractions, and Rules.*

Day 1 – I plan to kick off the unit by letting my students know that we will focus on problem solving this year, and will spend about 15 minutes each day working on problem solving strategies. We will also learn to solve many word problems using Singapore bar models because Singapore is ranked relatively high in math in the world. I will show them where Singapore is on a map.

Then I will introduce a Change Plus problem with a result unknown. I will not categorize the problem type, but I will provide three examples of a change plus result unknown problem. The first one I will model how to read the problem carefully, how to pull the facts out, how to write the question, how to draw a bar model, and how to go back to the question to write the answer in a statement. Then we will reflect as a class if the answer seems reasonable and makes sense.

Next, we will do another Change Plus problem with the result unknown. This time, the students will help me. I will ask them what the facts are. As they share a fact, I will record it and project it for the class to see. Once the facts are pulled out, a student will tell me the question to write down. Next the students will guide me to draw a bar model, solve the problem, and write the answer as a statement. I will ask students to share their thoughts on the reasonableness of the answer.

With the third problem, again a Change Plus result unknown, students will solve it independently following the steps that were modeled with the previous two problems.

Day 4 – For the mixed review, I plan to use one of each type of Change Plus problems. Students will solve three problems on a handout (using the steps learned in class.) Once students have completed the problems, we will share solutions, and students will have the opportunity to share different solutions or bar models. The Math Talk will begin organically as students share solutions, but I will work to guide students to understand the importance of providing a safe environment for sharing both correct and incorrect solutions. I also will stress the importance of learning from mistakes, yours and others, and how we can all benefit from talking about and analyzing misconceptions.

Day 27 – This is the first day area/array problems are addressed. Since these problems do not lend themselves to a bar model, I plan to point this out to students so they understand that a bar model is a linear representation and an array or area problem is two-dimensional. I intend to have students follow the same procedure as explained above, but instead of a bar model, students will draw a picture of the array or an area model using a rectangle.

Day 40 – Now that all of the problem types have been taught, students will now tackle two-step problems. Students should know that these are two-step problems, but that each step is similar to problems they have already solved. I anticipate these problems will lend themselves to a more robust class discussion as students have various solutions, particularly regarding the order in which the two steps are solved. The drawings may include two distinct bar models as shown in the multi-step problems previously described.

Sample Collection of Problems

Addition

Add to/result unknown - very basic to introduce third graders to bar models

1) Eight birds were sitting on the fence. Three more landed on the fence. How many are on the fence now?

Add to/change unknown

2) Eight birds were sitting on the fence. Some more landed on the fence. There are now 11 birds on the fence. How many birds landed on the fence?

Add to/start unknown

3) Some birds were on the fence. Three more landed on the fence. There are now 11 birds on the fence. How many birds were on the fence in the beginning?

Add to/result unknown – using money

4) Josie had $17 and her grandma gave her $25 for her birthday. How much money does she have now?

Add to/ change unknown

5) Josie had $17. Her grandma gave her some money for her birthday. She now has $42. How much money did her grandma give her?

Add to /start unknown

6) Josie had some money. Her grandma gave her $25 for her birthday. She now has $42. How much money did Josie have at the start?

Add to/result unknown – three addends

7) If 234 people attend a concert on Thursday night, 367 attend on Friday night, and 329 people attend on Saturday night. How many total people attended the concert?

Add to/ change unknown – three addends

8) 234 people attend on Thursday night, and 367 attend on Friday night. A total of 930 people attended on Thursday, Friday, and Saturday. How many people attended on Saturday?

Add to/ start unknown

9) After buying 62 stamps, Mr. Jones now has 418 stamps. How many did he have at first?

Take from, Change unknown

10) Mr. Sims had 245 stamps. Then Mr. Jones gave him some. He now has 312. How many did Mr. Jones give him? (Add to /change unknown)

Put together/total unknown

11) What is the total length of the sides (perimeter) of a triangle if one side is 23 inches, another is 19 inches, and the third side is 32 inches?

Compare less, larger unknown

12) Paul is 21 years old. He is 23 years younger than his mother. How old is his mother?

Compare more, bigger unknown

13) There are 342 trucks parked outside the football stadium. There are 147 more cars than trucks. How many cars are there?

Subtraction

Take from/result unknown - A really simple problem to introduce bar models

14) Six squirrels were in a tree. Two ran away. How many were left in the tree?

Take from/change unknown

15) Six squirrels were in a tree. Some ran away. There were four left in the tree. How many ran away?

Take from/ start unknown

16) Some squirrels were in a tree. Two ran away. There were four left. How many were in the tree at the start?

Put together/addend unknown or Part-Part-Whole/ part unknown

17) A total of 39 dogs and humans were at the dog park. If there were 22 dogs, how many humans were there? **

Put together/total unknown or Part-Part-Whole/ total unknown

18) There were six green apples and four red apples are in a bowl. How many apples are in bowl?

Put together/part unknown or Part-Part-Whole/ part unknown

19) There are 10 apples in a bowl. Six of the apples are red and the rest are green. How many are green?

Part-Part-Whole/ part unknown or Put together/addend unknown

20) There are 263 people watching a play. If 79 of them are children, how many are adults?

Part-Part-Whole/ part unknown or Put together/addend unknown

21) There were a total of 406 people at the symphony concert. There were 214 children and the rest were adults. How many were adults?

Part-Part-Whole/whole unknown

22) There are 402 boys and 369 girls watching a parade. How many children are watching the parade in all?

Compare/difference unknown/more

23) Sam has $28. Jaquan has $31. How much more money does Jaquan have than Sam?

Take from/Result Unknown

24) Ryan had $134. He spent $28 at the Sporting Goods Store. How much money does he have left?

Take from/result unknown

25) Twenty-two dogs were playing at the dog park. Three left with their owners. How many dogs are at the dog park now?

Take from/Start unknown

26) Ryan went to the Sporting Goods store with his money. Ryan spent $46 in the Sporting Goods Store. He had $78 left. How much money did he have at first?

Take from/change unknown

27) Mia had $124. She purchased some spinners. She now has $88. How much did the spinners cost?

Multiplication

Equal groups/product unknown

28) At the toy store cars come in packs of six. If Jose buys four packs, how many toy cars will he buy in total?

Equal groups (measurement)/product unknown

29) Mrs. Bronto has 14 cans of Saurus Soup. If each can is three inches high, and she piles them one on top of another to make a Brontosaurus Soup Tower, how high will the tower be?

Equal groups/ product unknown

30) Jordan is having a birthday party. He wants to have a can of soda for all of his guests. If there are 18 people at the party, how many six-packs of soda will his mom have to buy, so everyone can have one can of soda?

Equal groups (measurement)/ group size unknown

31) Sami has Fruit by the Foot and it is 36 inches long. She cuts it into equal pieces so she and her three friends will each have a piece. How long will each piece of string be?

Equal groups (measurement)/ number of groups unknown

32) Mr. Davis, the art teacher, has a piece of yarn 56 inches long. He cuts it into pieces that are seven inches long. How many pieces will he get?

Arrays/area, group size unknown

33) Ben draws a rectangle with an area of 24 square centimeters. If one side is eight centimeters long, how long is a side perpendicular to it?

Array/area, number of groups unknown

34) Ms. Carpenter cuts a rectangle out of fabric. It is 48 square inches. The length is eight inches. How long is the width?

Compare/group size unknown

35) Aidan has three times as much money as Ella. They have a total of $36. How much money does Ella have?

Equal groups/number of groups unknown

36) If Mrs. Pencil goes to the store to buy erasers for the class. There are 24 students in the class. Erasers are sold three in a pack. How many packs of erasers must she buy to have one for each student?

Arrays/area, product unknown

37) Naomi is making an array with two-sided counters. She puts seven in a row and makes six rows. How many two-sided counters are in her array?

Arrays/area, group size unknown

38) There are three equal sections in the theater. Each section has eight rows with the same number of seats. There are 72 seats in each section. How many seats are in each row?

Arrays/area, size of group unknown

39) Mr. Rebmann is setting up chairs for a band concert. He needs 32 seats for the band members. He decides to put eight chairs in each row. How many rows does he need?

Compare (measurement)/product unknown

40) A Slinky is four inches long when in its box. How long will it be when it is stretched five times its length?

Compare (measurement)/smaller quantity unknown

41) Jennie has a jump rope. It is 48 inches long. This is six times the length of Sophia's shoe. How long is Sophia's shoe?

Compare (measurement)/comparative factor unknown

42) Shannon can spin her spinner on her finger for 42 seconds. Will can only spin his spinner for seven seconds. How many times longer can Shannon spin her spinner than Will?

Compare/smaller quantity unknown

43) Ebony has a collection of seashells. She has 32 seashells. This is four times as many as Shannon has. How many seashells does Shannon have?

Compare/smaller quantity unknown

44) In a parking lot, there are four times as many cars as buses. There are 356 cars. How many buses are there?

Compare/smaller quantity unknown

45) Justin has some dogs. He goes to the pet store and sees four times as many. There are eight dogs at the pet store. How many dogs does Justin have?

Compare/smaller quantity unknown

46) James buys packs of pencils for her classroom school supplies. She buys 192 pencils, which is eight times the number of students in her class. How many students are in her class?

Multi-step

Compare, smaller unknown and *Put together/Part-part-whole, total/whole unknown*

47) There are 37 children in the swimming pool. There are nine fewer adults than children in the pool. How many adults and children are in the pool?

Compare, bigger unknown and *Put together/Part-part-whole total/whole unknown*

48) Jaden has $37. If Nico has $6 more than Jaden, how much do Jaden and Nico have in all?

Change minus/Take from, result unknown and *Change plus/Add to, result unknown*

49) Jazmin has 47 stickers. She gives eight to Sammy. Then she gets 26 more for her birthday. How many stickers does she have now?

Compare, fewer unknown and Put together/Part-part-whole, total/whole unknown

50) Children are at a concert. There are 3,402 boys, and there are 987 fewer girls than boys. How many children are there in all?

Put-together/ Part-Part-whole, total/whole unknown and *Compare, smaller unknown*

51) At a show, there are 6,020 spectators - men, women and children. There are 3,860 men and 2,020 women. How many children are there?

Multiplication/division compare, unknown product and *Put together/Part-part-whole, total/whole unknown*

52) Joe has a stamp collection. He has 135 U.S. stamps, and three times as many foreign stamps. How many foreign stamps does he have? How many stamps does he have all together?

Change minus/take from, result unknown and *Equal groups, unknown product*

53) Haskell earns $2,395 per month. He spends $1,780 per month, and saves the rest. How much does he save in six months?

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This article speaks to the power of the model method as a technique to teach problem solving.

Englard, Lisa. "Raise the Bar on Problem Solving." *Teaching Children Mathematics*17, no. 3 (October 2010): 156-63. Accessed May

14, 2017. JSTOR.

This article is easy to read and reveals the results of teaching bar models to students.

Fan, Lianghuo, and Yan Zhu. "Representation of problem-solving procedures: A comparative look at China, Singapore, and US mathematics textbooks." *Educational Studies in Mathematics*66, no. 1 (2007): 61-75. Accessed June 09, 2017. doi:10.1007/s10649-006-9069-6.

This article compared problem solving in text books from three countries. It was not very helpful, but provided a little insight.

Fong, Ng Swee, and Kerry Lee. "Model Method: A Visual Tool to Support Algebra Word Problem Solving at the Primary Level." *Series on Mathematics Education Mathematics Education*, 2009, 169-203. Accessed June 09, 2017. doi:10.1142/9789812833761_0008.

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This report describes how the education in Singapore has evolved. The report was insightful.

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This article provides an overview of the model method for primary students.

Looi, C.-K.,-S. Lim. “From bar diagrams to letter-symbolic algebra: a technology-enabled bridging.” Journal of Computer Assisted Learning 25.4 (2009): 358-74. Web. 5 June 2017. JSTOR. This article addresses the difficulty students in Singapore have moving from bar diagrams to algebra. The ALGEBAR computer technology helps to bridge the gap.

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This article examines the model method and emphasizes aspects to look for when teaching students the method.

Xin, Yan Ping, and Dake Zhang. "Exploring a Conceptual Model-Based Approach to Teaching Situated Word Problems." *The Journal of Educational Research*102, no. 6 (2009): 427-42. Accessed June 09, 2017. doi:10.3200/joer.102.6.427-442.

This article supports the use of the model method to teach students in grades 3 through 5.

Additional Resources for Teachers and Students

A., Van De Walle John. *Elementary school mathematics: teaching developmentally*. London: Longman, 1994. An excellent resource for teaching all aspects of elementary math in a concrete way.

Bluman, Allan G. *Math word problems demystified*. New York: McGraw-Hill, 2011. This book is full of word problems at all levels as well as word problem strategies. The problems would need to be carefully selected for the appropriate grade level.

Chew, Terry. *Singapore math challenge: grade 3*. Columbus, OH: Frank Schaffer Publications, 2013. A wonderful challenging book to develop math thinking for students in grades 3-5 in the U.S.

Forsten, Char. *Step-by-step model drawing: solving word problems the Singapore way*. Peterborough, NH: Crystal Springs Books, 2010. Another good source for word problems that are made for the model method.

Primary mathematics. Singapur: Marshall Cavendish Education, 2003. A source of word problems for bar modeling appropriate for high third graders or fourth grade students in the U.S.

Primary mathematics 2A. Singapore: Marshall Cavendish Education, 2006. A good source of word problems that lend themselves to bar models. The level 2 books are appropriate for grade 3 in the U.S.

Primary mathematics 2B. Singapore: Marshall Cavendish Education, 2006.

Primary mathematics 3B. Singapore: Marshall Cavendish Education, 2007. This is also a resource for high third graders or four grade students in the U.S.

Appendix A: Implementing District and National Standards

This curriculum unit is designed to provide a framework for teaching word problems focused on building students' understanding of the word problem. It utilizes the taxonomy of situations from the Common Core standards, which the state has added to Virginia's math standards of learning curriculum framework for 2016. The 2016 standards will be phased in during the 2017-18 school year and tested in 2019. This unit aligns to the new guidelines from the state and also provides another proven tool with the bar model.

Virginia's Standards of Learning (2009)

3.2 The student will recognize and use the inverse relationships between addition/subtraction and multiplication/division to complete basic fact sentences. The student will use these relationships to solve problems.

3.4 The student will estimate solutions to and solve single-step and multistep problems involving the sum or difference of two whole numbers, each 9,999 or less, with or without regrouping.

3.6 The student will represent multiplication and division, using area, set, and number line models, and create and solve problems that involve multiplication of two whole numbers, one factor 99 or less and the second factor 5 or less.

The above standards are for 2009. This year the 2016 standards are being implemented. The end- of-the-year test will cover 2009 standards and will include field test items from the 2016 standards.

Regarding problem solving, the only difference between 2009 and 2016 is that in 2016 students will solve problems with elapsed time. The standards are pretty vague. When I went to the Curriculum Framework document, I discovered charts with problem types. They are practically identical to the Common Core chart, with a few different names. These are new additions to the Virginia Standards.

Virginia's Standards of Learning (2016)

3.3 The student will

b) create and solve single-step and multistep practical problems involving sums or differences of two whole numbers, each 9,999 or less

3.4 The student will

b) create and solve single-step practical problems that involve multiplication and division through 10 x 10;

d) solve single-step practical problems involving multiplication of whole numbers, where one factor is 99 or less and the second factor is 5 or less.

3.9 The student will

b) solve practical problems related to elapsed time in one-hour increments within a

12-hour period;

c) identify equivalent periods of time and solve practical problems related to equivalent periods of time.

Common Core Standards

CCSS.MATH.CONTENT.3.OA.A.3 Use multiplication and division within 100 to solve word problems in situations involving equal groups, arrays, and measurement quantities, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem.

This standard relates directly to the taxonomy chart and the types of problems discussed in my curriculum unit.

CCSS.MATH.CONTENT.3.OA.D.8 Solve two-step word problems using the four operations. Represent these problems using equations with a letter standing for the unknown quantity. Assess the reasonableness of answers using mental computation and estimation strategies including rounding.

By learning all of the types of one step problems, students will combine them to solve two-step problems. By utilizing Polya's steps, students will be taught how to check answers for reasonableness.

The following three standards relate to area and perimeter problems. A few are included in the collection of problems.

CCSS.MATH.CONTENT.3.MD.C.7.B Multiply side lengths to find areas of rectangles with whole-number side lengths in the context of solving real world and mathematical problems, and represent whole-number products as rectangular areas in mathematical reasoning.

CCSS.MATH.CONTENT.3.MD.C.7.D Recognize area as additive. Find areas of rectilinear figures by decomposing them into non-overlapping rectangles and adding the areas of the non-overlapping parts, applying this technique to solve real world problems.

CCSS.MATH.CONTENT.3.MD.D.8 Solve real world and mathematical problems involving perimeters of polygons, including finding the perimeter given the side lengths, finding an unknown side length, and exhibiting rectangles with the same perimeter and different areas or with the same area and different perimeters.

Endnotes

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1. G. W. Melvin, "Polya's Problem Solving Techniques," https://math.berkeley.edu/~gmelvin/polya.pdf.
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