



Curriculum Units by Fellows of the National Initiative

2017 Volume V: From Arithmetic to Algebra: Variables, Word Problems, Fractions and the Rules

Formulating Algebraic Equations from Word Problems

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Content Objectives

Overview

*If you have 4 pencils and I have 7 apples, how many pancakes will fit on the roof? Purple, because aliens don't wear hats.*¹

This is a meme that I have encountered in various forms on social media, and even from students joking around in person. Word problems seem to be a commonly cited justification for hating math, particularly algebra. Jokes aside, in my experience as a secondary math teacher, even students who are skilled in solving given algebraic equations struggle in creating their own equation to solve a word problem. This is not so surprising, because the demands of solving an equation are very different from the demands of formulating an equation from a verbal or written context. To formulate an algebraic equation that correctly represents a situation, students must interpret a verbal or written statement, and symbolically represent the relationship of its parts. Thus, to improve student success with formulating algebraic equations, we must provide instruction in both the process of decoding a problem, and using mathematical notation that maintains the meaning of the original problem.

The purpose of this curriculum unit is to help students translate word problems into first order equations that can be written in the form $ax + b = c$. For purposes of this unit, a , b , and c will be whole numbers. To do this, I will help students identify and organize the given information, the unknown value, and the relationship of the values of the problem through verbal discussions and visual models. By organizing this information, students will have an easier time representing these quantities and relationships symbolically. In the long run, I hope this curriculum unit also gives my students confidence to approach more complicated word problems as they progress further in math.

This unit is the first of four complementary curriculum units developed in the 2017 Yale National Initiative (YNI) math seminar, "From Arithmetic to Algebra," about translating, simplifying, and solving algebraic expressions and equations. For the rest of the related curriculum units, please see the work of 2017 math Fellows Jeffrey Rossiter, Xiomara Pacheco, and Sally Yoo. (See Figure 1 below, which summarizes the connection between the concepts of our pieces.)

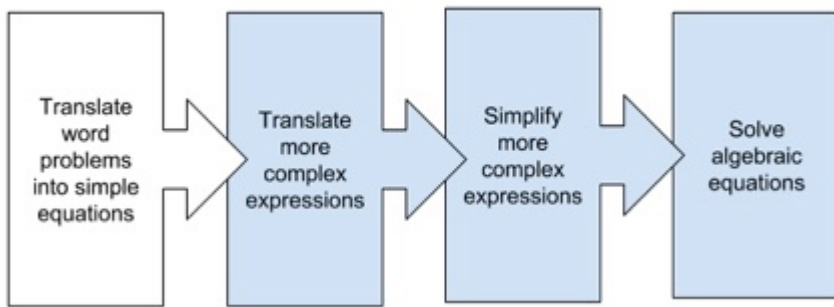


Figure 1

This order is surprising because, in most textbooks, translating and simplifying algebraic expressions comes first, followed by solving equations, and *ending* with word problems. However, starting with word problems provides the *motivation* for using algebraic notation to represent increasingly complex situations. While bar models are useful in representing, and even solving simple word problems, students will need more sophisticated skills in representing and manipulating equations with compound expressions. Therefore, the intention of this curriculum unit is to merely begin a broader investigation into algebraic expressions and equations, which may last for an entire semester.

Rationale

I teach at Peter Burnett Middle School in San Jose, California. Currently, Burnett Middle School's overall math performance on the California Assessment of Student Progress and Performance (CAASPP) falls well below the state average.² Last year, 13% of Burnett's students met or exceeded standards, compared to a 37% statewide average. Since CAASPP results are one of the measures used in course placement in my district, I worry that these scores will limit students' opportunities to take algebra 1, and serve as indicators of their likelihood to succeed in that course. Moreover, having taught or tutored every higher-level course from geometry to calculus, I know that algebra skills are necessary to take and succeed in each of these classes as well. So very few of Burnett's students are on track to take any of these higher-level math classes, which will limit their opportunities in college. Colleges will want students to take remedial math, which may lead to falling behind in their other courses, and hindering their path to a diploma. Even those who go directly into the workforce, or training outside of higher education, may be affected. Among high school graduates who don't attend college, "those with low levels of math are 50 percent more likely to be unemployed than those with higher levels of math, and they typically earn about \$ 1.30 less per hour."³ So while the CAASPP itself is a single test, what the scores reveal about math competence among Burnett students can have much longer-lasting repercussions. The goal of this unit is to improve my students' algebra skills, and consequently, their future opportunities.

This curriculum unit targets students in 7th grade math, which covers pre-algebra concepts and skills. Similar to whole school performance stated above, 13% of Burnett 7th graders met or exceeded math standards on the CAASPP, compared to 36% statewide. This number covers *all* 7th graders, including those who are in an additional math support course, those who are "on grade level", and those who are in advanced placement (7th grade accelerated, or Algebra 1). My curriculum unit will focus on students "on grade level" who, despite that classification, underperform on district benchmark exams. Fewer than 1 in 25 of my 55 "grade level" students were proficient on the 2016-2017 exam on my district's Algebraic Expressions and Equations, which assessed using variables to represent real-world quantities, and formulate simple equations to solve those problems. (See APPENDIX A for full list of standards.) The primary objective of this unit will be to equip

students with the tools to interpret a word problem, and to write an algebraic equation that accurately relates its quantities.

In the 2016-2017 school year, my district started using the *Springboard Course 2 Textbook* for 7th grade math; in this book, there is only one lesson on “Modeling and Writing Two-Step Equations”. After teaching this lesson, I have concluded that it has several deficiencies. First, the lesson over-scaffolds the translation process, by providing a numeric expression to represent the word problem, and by defining the variable for students. Then, the lesson over-simplifies the distinction between an equation and an expression. Specifically, equations “have an equal sign”, and expressions do not. Finally, the lesson introduces vocabulary, such as “coefficient” and “constant”, without connecting back to the word problem. As a result of these shortcomings, students do not get enough practice in creating the structures of the expression or equation themselves, including operations, or in creating appropriate notation for unknown values.

6th grade standards also cover some of the skills needed in translating situations into expressions or equations. By 7th grade, students should already know how to use variables to represent unknown numbers, and write expressions and equations to solve real world problems; specifically, in the 6th grade, students practice problems that can be written as one-step addition or multiplication equations. (See APPENDIX A for full list of standards.)

When I taught 6th grade math in 2015-2016, I used my own worksheets to instruct students on identifying “key words” when translating verbal sentences into algebraic expressions and equations. For addition, I gave the words *increase*, *more*, *sum*, and *deposit*. For subtraction, the words/phrases were *difference*, *decreased by*, *less than*, and *withdrawal*. For multiplication, I gave *times* and *product*. I indicated division by *quotient*. And for equal, I gave the word *is*, and the phrase *the same as*.

However, both in my experience, and according to cognitive research,⁴ this type of *syntactic*—or word-by-word—translation directly leads to many misconceptions and mistakes. Students often made reversal errors, as when translating “four less than a number” to “4-x,” rather than “x-4.” Moreover, as my lists were not exhaustive, students struggled when translating word problems with unfamiliar words indicating operation, such as “each” in, “There were 56 students in a room, arranged in 4 rows. How many students were in *each* row?” Even the key words that I have listed can have multiple meanings, depending on the context. If I say, “Andrew had three cookies, then bought two *more*,” this indicates addition, whereas, “Andrew has three cookies, which is two *more* cookies than Robert has,” indicates subtraction. These examples illustrate Polya’s⁵ comparison of translating a word problem into an equation, to translating between English and French, which requires understanding the English sentence, knowing the forms of expression specific to French, and being familiar with idiomatic expressions that cannot be translated word-for-word. Similarly, we must be familiar with rules and notation specific to mathematical representation, and focus on whole meaning rather than word-for-word translation from a word problem. I am confident that I have not been alone in using the key word approach. Thus, 7th graders in 2016-2017 may have come in with prior misconceptions about translating word problems to expressions or equations.

For the 2017-2018 school year, my rising 7th graders will be the first class that used the *Springboard* curriculum in 6th grade. Their Course 1 textbook has more material on formulating algebraic expressions and equations on “Writing Equations”, “Representing Situations with Equations”, and “Modeling and Solving Addition, Subtraction, and Multiplication Equations”.

“Writing Expressions” has the same syntactic format as my own lesson on identifying key words, thus

enforcing the same misconceptions. However, the given examples focus on translating verbal phrases such as “12 less than twice a number,” without instruction on how to transition to meaningful word problems. So students lack guidance in considering how context and phrasing of a real world problem affect the structure of the equation. This lesson also asks students to “simplify” or show “equivalence” between expressions, prior to any lessons on how to simplify, or what equivalence means. It would be better to discuss these terms when students actually encounter a situation that can be represented by multiple expressions; this way, both the process of simplifying, and the concept of equivalence will have applied meaning to the students.

“Representing Situations with Equations”, like the lesson in Course 2, over-scaffolds by providing the structure of the equation, rather than having students derive it themselves. Again, there is also little instruction about transitioning from purely verbal phrases to word problems that represent an actual situation. Thus, students lack guidance in *semantic*, or conceptual, understanding of word problems. Similar issues repeat themselves in the remainder of the lessons named above, although they do provide actual contexts to translate.

Key Ideas

In the 2017 YNI math seminar, we examined the Common Core State Standards (CCSS) taxonomies for one-step equations⁶ by looking at a given context, examining which problem type is represented, and how the equation for that context changes, depending on what is given, and what is unknown. However, I worry that having students memorize the problem types will essentially be the same as having them memorize key words. Instead, I want the students to learn how to carefully read each problem, and critically think about the data, the unknown values, and the relationship between the values. To scaffold this process, I will use visual models, based on the bar models from the Singapore Bar Method, as described below. While some students will be able to translate a word problem directly into symbolic terms, visual models may assist struggling students, and may help all students with more complex problem types. Additionally, students who are able to write an equation directly from a word problem may use Singapore bar models to check consistency between the relationship represented in the equation and the relationship represented in the word problem.

Singapore Bar Models to Categorize Problems

The Singapore Bar Method scaffolds the process of translating word problems by requiring students to identify and organize the unknown information, the given data, and the condition(s) relating these values in a single model. Thus, students are using a visual means of answering Polya’s guiding questions in understanding “Problems to Solve.”⁷ Furthermore, the process of answering these questions pushes students to conduct multiple reads of a word problem. “The first reading provides the context of the story. In subsequent readings of the text, information is further processed to ascertain what the givens are and what is to be found.”⁸ Additionally, a collaborative paper between a neuroscientist and math educators references multiple studies, anecdotal information from teachers, and neuroimaging to link math performance and use of visuals.⁹ So along with the specific information bar models prompt and convey, using visual representations may help students, and raise math achievement, in general.

The way that students will set up a bar model will depend on the type of problem to be represented. Practice with providing the models for these one-step equations should prepare students to use them to then model multi-step equations. I can assign the various one-step problems to four types of bar models: Part-Whole, Comparison, Multiplication and Division, and Multiplicative Comparison. The first three models are those identified in resources I have encountered on the Singapore method.¹⁰ The last is a combination of the Comparison and Multiplication and Division models. As I describe the problem types that apply to each bar

model, I will refer to the CCSS taxonomies, which I have adapted in the table below. However, as I mentioned above, I do not plan to teach the students this language.

Selected CCSS Problem Taxonomies

I have modified some of the terminology of the taxonomies about the problem types and their parts. Instead, I will use alternative language we used during the YNI seminar, as it better describes the situations that apply to a specific problem type. For example, the problems I label as “Part-Part-Whole” are called “Put Together/Take Apart” in the CCSS guide. However, in the example situation about the red and green apples, you really aren’t putting anything together, as both types of apples are already there. It is more accurate to think of red apples and green apples as being *parts* of a *whole* group of apples. Also, the CCSS guide identifies the components of “Compare (Multiplication/Division)” problems as *Number of Groups • Group Size = Product*. However, I find that the language of *Factor • Smaller = Bigger* better matches the “Compare (Addition/Subtraction)” language. (For a more in-depth explanation, see *Multiplication & Division Models* section below.)

Addition/Subtraction Problem Types

Problem Type	Overarching Equation ¹¹	Example Situation
Part-Part-Whole	$Part + Part = Whole$	There are some red apples (<i>Part</i>), and there are some green apples (<i>Part</i>). There is a total number of apples (<i>Whole</i>).
Add To	$Start + Change = Result$	Some bunnies sat on the grass (<i>Start</i>). Some more bunnies hopped over (<i>Change</i>). There is a new number of bunnies on the grass now (<i>Result</i>).
Take From	$Start - Change = Result$	Some apples were on the table (<i>Start</i>). I ate some of the apples (<i>Change</i>). There is a number of apples on the table now (<i>Result</i>).
Compare	$Smaller + Difference = Bigger$	Lucy has some apples (<i>Smaller</i>), and Julie has some more than Lucy (<i>Difference</i>) so that Julie has a (<i>Larger, Bigger</i>) number of apples.

Multiplication/Division Problem Types

Problem Type	Overarching Equation	Example Situation
Equal Groups	$Number\ of\ Groups \cdot Group\ Size = Product$	There are some bags (<i>Number of Groups</i>) with some number of plums in each bag (<i>Group Size</i>), which gives a total number of plums (<i>Product</i>).
Compare	$Factor \cdot Smaller = Bigger$	A blue hat costs some number of dollars (<i>Smaller</i>), and a red hat (<i>Bigger</i>) is some number of times as much (<i>Factor</i>) as the blue hat.

I am purposely skipping problems based on the Array category in the taxonomy, because I did not come across many problems that require arrangement into rows and columns. Also, since the Singapore Bar Method relies on a linear model, rather than an area model, it is not ideal when approaching these types of problems.

(For the example situations with various unknown and given quantities, please see the CCSS guide for math, pp.88- 89.)

Addition/Subtraction Models

Part-Whole Models show a bracket as equal in length to a pair of bars, which indicates that the large one has the same value as the two small ones combined. This is how addition is realized in the domain of length

measurement. These can, of course, be applied to “Part-Part-Whole” problems, which can be written as $Part + Part = Whole$. With this first model, I will discourage my students from making equal parts, unless this is specifically stated in the problem, as they may be tempted to simply divide a quantity by two if they draw the small bars the same length. Alternatively, introducing obviously different quantities may help students see this themselves. (See Figure 2 below.)

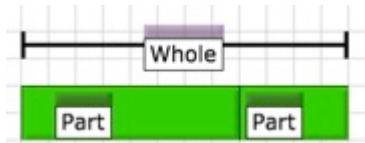


Figure 2

For example, “Sandy’s high school played 14 hockey games this year. The team won most of their games. They were defeated only in 4 games. How many games did they win?” (The model for this is in Figure 3 below.)

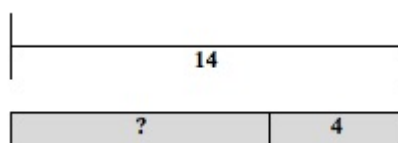


Figure 3

The bracket above indicates the total number of hockey games, the “?” represents the number of games that her team won, and the 4 represents the number of games that the team lost. (See APPENDIX B and Teacher Resources for more example problems.) Although this model follows the order of the parts listed in the word problem—that is, “games won” in the leftmost bar, and “games lost” in the rightmost—this type of problem may be useful in introducing the commutative property of addition. This discussion would come naturally from students who draw the bars in the opposite order.

The diagram for the Part-Whole Model can also be applied to “Add to” and “Take from” problems, labeled appropriately for each problem type. Although these are distinguished in the taxonomy from “Put Together/Take Apart” problems, this model captures well any one-step problem with an additive relationship. It can also be applied to comparison problems.

For “Add to” problems, we will use the convention of labeling the leftmost bar as the *Start*, the rightmost bar as the *Change*, and the bracket as the *Result*. For example, “There were 8 roses in the vase. Melanie cut some more roses from her flower garden and added them to the vase. There are now 17 roses in the vase. How many roses did she cut?” (See Figure 4 below.)

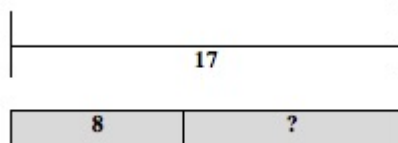


Figure 4

The bracket above indicates how many roses are in the vase now, the 8 represents the roses that were in the

vase before Melanie cut roses, and the “?” represents how many roses Melanie cut from her garden.

For “Subtract from” problems, we will label the bracket as the *Start*, the leftmost bar as the *Change*, and the rightmost bar as the *Result*. For example, “Tom received 80 dollars for his birthday. He went to a sporting goods store and bought a baseball glove, baseball, and bat. He had 39 dollars left over, how much did he spend on the baseball gear?” (See Figure 5 below.)

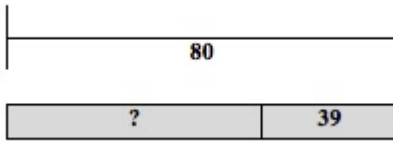


Figure 5

The bracket above indicates the amount of money Tom received for his birthday, the “?” represents how much he spent on baseball gear, and the 39 represents how much money he had left. (See APPENDIX B and Teacher Resources for more example problems.)

Comparison Models apply to “Compare (Addition/Subtraction)” problems, which can be written as *Smaller + Difference = Bigger*. (See Figure 6 below.)

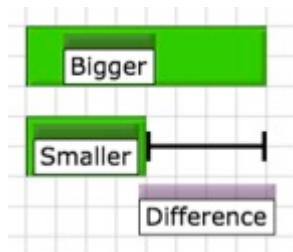


Figure 6

The bars in this model are stacked vertically in order to show the size of one value in relation to the other values. For example, “Connie has 15 red marbles and 28 blue marbles. How many more blue marbles than red marbles does Connie have?” (See Figure 7 below.)

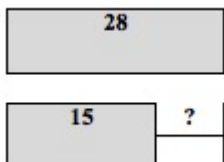


Figure 7

Although 15 red marbles are mentioned first in the word problem, students should figure out that the bigger amount, 28 blue marbles, should be the top bar, with the “?” symbol labeling the bracket, and representing how many more blue marbles Connie has, compared to the number of red marbles she has. Deciding on the correct relationships is the core of understanding the problem. The construction of bar models provides a structure to help students work through this process. (See APPENDIX B and Teacher Resources for more example problems.)

Multiplication/Division Models

Just as multiplication can be viewed as repeated addition of equal parts, the Multiplication and Division Model can be viewed as an extension of the Part-Whole Model. This model applies to “Equal Group” problems that can be written in the form $Number\ of\ Groups \cdot Group\ Size = Product$. (See Figure 8 below.)

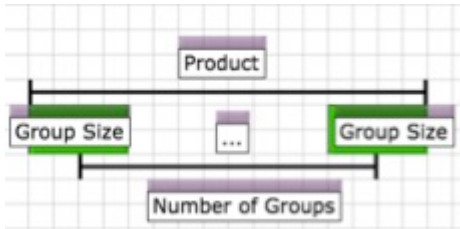


Figure 8

As you can see, the generalization of this model is a bit more abstract than the Part-Whole Model. This is because, without knowing how many groups there are, we are unable to draw the correct number of equal parts. Instead, we use ellipses to represent a continuation of bars of equal size between the first and last bars that are drawn in. We can also use ellipses to represent a large number of groups that would be too tedious to draw in our model. A “large number” is subjective, and will be different according to the person using the model, but one book recommends to use “this approach when multiplying by a factor greater than 10.”¹²

Here is an example of the same situation, with different unknown values. Notice how the model changes, depending on whether *number of groups* is unknown or *group size* is unknown.

“At a restaurant, Mike and his three friends decided to divide the bill evenly. If the total bill was \$ 52, how much did each person pay?” (See Figure 9 below.)

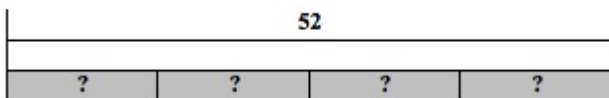


Figure 9

In this example, the unknown value is the *group size*, as indicated by the “?” inside each of the equal groups. This is fairly straightforward to illustrate using a bar model.

In contrast, consider the situation written as, “At a restaurant, Mike and some of his friends decided to divide the bill evenly. If the total bill was \$ 52, and each person paid \$ 13, how many people divided the bill?” (See Figure 10 below.)

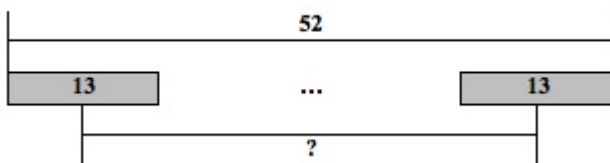


Figure 10

Here, we know that the group size is 13, representing the \$ 13 that each person paid, but we don't know the number of groups, which represents the number of people who split the bill. As mentioned in the general model, since we don't now how many equal groups to draw, we use ellipses to indicate the continuation of the same sized group between the two that are illustrated. (See APPENDIX B and Teacher Resources for more example problems).

Finally, the Multiplicative Comparison Model combines aspects of the Comparison Model the Multiplication and Division Model. This model applies to "Compare (Multiplication/Division)" problems, which can be written in the form

$Factor \cdot Smaller = Bigger$, for whole number factors. However, if a factor is between 0 and 1, then $Factor \cdot Bigger = Smaller$ would be an accurate statement. For example, "Tanya had half as many red toy trucks as Al" could be represented in the equation, $\frac{1}{2} a = t$, where a is the number of red toy trucks Al has (the *Bigger* amount), and t is the number of red toy trucks Tanya has (the *Smaller* amount). However, this curriculum unit will focus on cases wherein the multiplicative factor is a whole number. So, to attack this problem, I would ask my students to convert it into a comparison of Al to Tanya, rather than the other direction, and to write the equation $a = 2t$. This can then be handled nicely with a bar model similar to the one illustrated in Figure 9 above.

Although the CCSS guide uses the same language as the "Equal Group" problems, I am using the more descriptive names, "smaller" and "bigger" to mirror the language in the "Compare (Addition/Subtraction)" problems. (See Figure 11 below.)

Figure 11

As with the Comparison Model, the bars in this model are stacked vertically in order to show the size of one value in relation to the other value. Further, as with the Multiplication and Division Model, in order to represent the number of groups, we use ellipses in the general model, when the number of groups is unknown, or when the number of groups is too large.

For example, "A bus is 45 feet long. It is 3 times as long as a car. How long is the car?" (See Figure 12 below.)

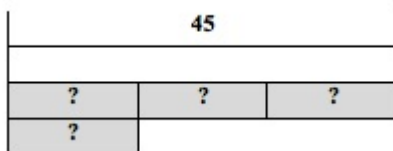


Figure 12

The 45 represents the length of the bigger vehicle, and as you can see, the bar is three times the length of the smaller bar, the length of which is represented by "?". The quantities throughout the model are in feet. (See APPENDIX B and Teacher Resources for more example problems.)

It is worth noting that this type of word problem is another common source of reversal errors. For example, for the situation "6 times as many students as professors," students are likely to write the equation " $6S=P$,"¹³ with S representing the number of students, 6 as the *Factor* relating the number of students and professors, and P representing the number of professors.

However, we could construct a bar model. (See Figure 13 below.)

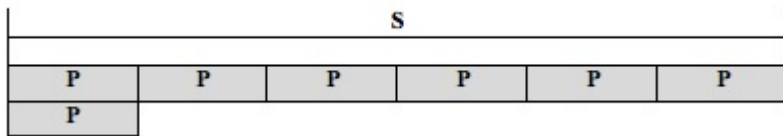


Figure 13

Here, the number of students, represented by S , is equivalent to 6 times the number of professors, represented by P . Also note here that, rather than using a “?” to represent an unknown value, I used the variables themselves. As students become more comfortable with the use and meaning of variables, I will encourage them to make that transition as well.

However, this model can be a source of confusion in and of itself. There is only one S , and 6 P s; does this mean, therefore, that there are more professors than students? Before I answer this question, it may be useful to review some common proportional relationships—or multiplicative comparisons—through the lens of unit conversions.

Proportions as Multiplicative Comparisons

There are 3 feet in 1 yard, which can be written as the ratio, $3 \text{ feet}/1 \text{ yard} = 1$. However, you want to know the number of yards you have, Y , if given F number of feet, it is better to write this relationship as a multiplicative comparison. It would be tempting to write $3F=Y$, as this equation matches almost word-for-word with the sentence, “there are 3 feet in 1 yard.”

Yet, if we add units into the equation, we can see why it is incorrect.

$3 \text{ feet}/\text{yard} \cdot F \text{ feet} = Y \text{ yards}$ simplifies to $3F \text{ feet}^2 / \text{yard} = Y \text{ yards}$.

Therefore, the units are not compatible.

Instead, if we make 3 the coefficient for Y , we get $3 \text{ feet}/\text{yard} \cdot Y \text{ yards} = F \text{ feet}$, which simplifies to $3Y \text{ feet} = F \text{ feet}$, which will make sense in standard numerical form, $3Y=F$.

To clarify this point further, it may be helpful to repeat the reasoning above using other commonly known unit conversions, such as “12 inches in 1 foot”, “60 minutes in 1 hour”, and “7 days in 1 week”. All of these examples illustrate how objects in multiplicative comparison problems have an *inverse relationship* between the number of units and the size of the unit. That is, because there are 3 feet in a yard, you must multiply 3 by the *number of yards* (the larger unit) to get the *number of feet* (the smaller unit); this means that there is a *larger number of the smaller unit*, and a *smaller number of the larger unit*.

Thus, revisiting the student and professor example in the previous section, “6 times as many students as professors” can be also thought of as the ratio 6 students/1 professor. The correct equation, therefore would look like:

$(6 \text{ students}/\text{professor}) \cdot (P \text{ professors}) = S \text{ students}$

This simplifies to $6P \text{ students} = S \text{ students}$. Since the units are the same, the equation will still make sense in

standard numerical form, or $6P=S$. Saying that there is one professor for 6 students is parallel to saying that there is one yard for 3 feet. "Professor" is functioning as the larger unit, and "student" as the smaller. Hence the *number* of students must be 6 times the *number* of professors.

Transitioning from Models to Algebraic Equations

The next step for students using a visual model to transition into writing an equation is to introduce suitable notation.¹⁴ The two parts that students often struggle with are where to place the "=" symbol, and how to use a variable.

Another benefit of using visual models prior to writing equations is that it emphasizes the relational meaning of the "=" symbol, as opposed to the operational meaning. "*Operational meaning* stands for the...asymmetric use: 'operation equals answer,' which is often described for elementary arithmetic...*Relational meaning* focuses on a symmetric use of the equal sign."¹⁵

Since the sizes of bars and brackets show the relationship between the values in the models, it is easier to understand why a given situation represents an *equation* rather than an *expression*. Looking at the bus example for the Multiplicative Comparison Model, each bar is its own expression. The top bar represents one quantity (the length of the bus, in feet), and the bottom bar represents another quantity (the length of the car, in feet). Since the top bar is comprised of three copies of the bottom bar, the length of the bus is *the same as* three times the length of the car, or the length of the bus "=" three times the length of the car. Therefore, the relationship of the quantities in the bar models helps you can see an *equation* relating two *expressions*, or the lengths of the different vehicles

Regarding variables, there is a common misconception of "symbols as labels that can shift within an activity; for example... $b = \text{books}$, which can be interpreted later as the number of books, the value of the book, the value of the total number of books, etc."¹⁶ To avoid this, I will model and reinforce the need to identify the unknown value as a *number* with appropriate units.¹⁷ I will make sure that my students develop the habit of specifying the units of their variables.

It may be useful for struggling students to identify the units of *all* of the parts of a problem. Additive problem types should have the exact same unit for each of its parts, in order for the addition to make sense in standard numerical form. Multiplicative problem types are a bit more challenging, as each factor may have different units, but, once multiplied, should result in the same unit on both sides of the equation. The models themselves are one means of checking that units correspond. In the additive case, in the Part-Whole Model example on Sandy's hockey team, although one part is "number of games lost," and another part is "number of games won," these parts are both a certain number of the unit "game," which is also the unit of the "total number of games." Meanwhile, in the Multiplication and Division example on plums divided among bags (see Classroom Activities), if you multiply the number of groups "number of bags of plum," by the group size, "number of plums per bag", the product will be the total number of plums.

I expect that even students who can directly translate word problems into equations without the help of bar model(s), by explicitly identifying units for variables and known values alike, will have an easier time verifying that their equation makes sense when they solve a problem arithmetically, using the corresponding model. Can you combine the units in the equation in a way that is compatible with the situation? Does the solution match the magnitude and the units of the context? This last check also reinforces students' conceptual understanding of the word problem as a whole, and the connection between the word problem, visual model,

and equation.

Once students understand how to categorize one-step word problems by bar model type, represent the problem using the appropriate bar model, and translate those models into equations, they should be ready to put those pieces together to categorize and symbolically represent multi-step situations.

Introducing Inverse Operations

Along with providing a scaffold for translating word problems into equations, Singapore Bar Models can help with introducing the concept of inverse operations. “By a process of undoing, the value of the unknown [quantity] is found. Hence, children [using bar models] circumvent the cognitive demands that are inherent in solving algebraic equations, which normally require the construction of a system of equivalent equations.”¹⁸ In the Part-Whole Model examples in the previous section, the models represent additive relationships. However, to figure out the missing value in each of these examples, students must actually *subtract*, which is the inverse operation of adding. The diagram helps them understand what to do. Similarly, the Multiplicative Comparison Model example represents a multiplicative relationship. But in order to solve this problem, students should *divide*, which is the inverse operation of multiplying. By providing a visual wherein students can literally *take away* part of an amount, or *split up* an amount, use of bar models makes students less likely to perform the wrong operation, even when it is suggested by a “key word” (for example, getting $r = 25$ for $8 + r = 17$ by adding $8+17$).

Similarly, a bar model representing a two-step equation can help justify the convention of removing the constant prior to the coefficient. Below, I am using a combination of a Part-Whole Model and a Multiplication and Division Model to represent the general form of a first order equation in one variable, $ax + b = c$. Using the structure we have established for the Part-Whole Model, ax will be the first *Part* and b will be the second *Part* of the model. Then, using the structure of the Multiplication and Division Model, a will represent the number of groups, and x will be the group size. (See Figure 14 below.)

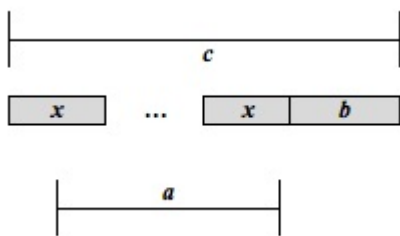


Figure 14

Using the model, you would first subtract b from c to be left with a groups with equal, unknown size, x . To figure out the value of x , divide the new amount $(c-b)$ by a .

Symbolically, this looks like

$$ax + b = c$$

$$ax + b - b = c - b$$

$$ax + (b - b) = (c - b)$$

$$ax + 0 = (c - b)$$

$$ax = (c - b)$$

$$ax/a = (c-b)/a$$

$$1x = (c-b)/a$$

$$x = (c-b)/a$$

Thus, bar models can help students both translate word problems into equations, and introduce them to aspects of solving these equations algebraically.

Teaching Strategies

Assessing through an International Baccalaureate Lens

As an International Baccalaureate (IB) school,¹⁹ Burnett grades students in each subject on a scale from 1-8 across four different criteria in a subject. Therefore, rather than having a single grade that summarizes a students' mastery of math as a whole, I can assign *four* scores on their knowledge of math skills and concepts, investigation of patterns, communication, and application of math in real-life contexts. In addition to breaking down the subject into four criteria, each score comes with a descriptor that provides qualitative feedback for the student and parents. For this curriculum unit, I will assess students using Criterion C and Criterion D. (See Teacher Resources for criteria rubrics.)

Criterion C, Communicating: Students use appropriate mathematical language and different forms of representation when communicating mathematical ideas, reasoning and findings, both orally and in writing.

For Criterion C, students will represent the information in a word problem using models, symbols, and words. Within their equations, I will check that students clearly define their variables as numbers, with appropriate units. I will also give students opportunities to create a bar model, an algebraic equation, or a word problem when given one of the other representations, as well as the opportunity to create their own original word problem, using *all* of the mentioned representations. Throughout the curriculum unit, I will require students to write and/or discuss how the model, equation, and word problem all show the same values and relationships.

Criterion D, Applying Mathematics in real-life contexts: Students transfer theoretical mathematical knowledge into real-world situations and apply appropriate problem-solving strategies, draw valid conclusions and reflect upon their results. For Criterion D, students will be engaging with real-life contexts presented in word problems. Applying appropriate problem-solving strategies will require writing a correct equation, which may include choosing the correct bar model to represent a situation. Then, after solving either arithmetically or algebraically, students will reflect whether their solution is reasonable for the given context. This includes checking that the units of each part of the equation correspond, and that they make sense after calculation.

Thus, assessing students using Criteria C and D will push me to frame each class activity around using multiple mathematical representations to understand and solve word problems from a variety of contexts.

Additionally, having students use the criteria rubrics to self-assess and assess and their peers, they will gain practice on critiquing their own representations—models and/or equations—and the reasonability of their solutions in context of a given word problem.

Explicit Direct Instruction (EDI) with Graphic Organizers

When introducing each new model, and the idea of combining models, I plan to use EDI, consisting of orientation, gradual release (teacher-led, collaborative, then independent problem-solving), and closure. As students will be responsible for multiple representations of a problem, I will provide graphic organizers for students to record each word problem’s model type, the bar model itself, and the resulting algebraic equation with a defined variable and unit. This graphic organizer will mirror the questions that students should ask themselves when translating between representations. (See Figure 15 below.)

Word Problem		
Val has three more apples than Pat. Val has five apples. How many apples do they have all together?		
Model Type(s)	Bar Model	Algebraic Equation
Comparison Model and Part-Whole Model		$? =$ the number of apples they have all together = a $5 + (5 - 3) = a$
State & Check your Solution		
If Val has 5, and that is 3 more than Pat, then Pat has $5 - 3 = 2$, and together they have $5 + 2 = 7$.		

Figure 15

(See APPENDIX C for a blank template of the graphic organizer.)

Just as the bar method itself is merely a scaffold as opposed to an end in and of itself, the graphic organizer is merely a helpful tool that students should not rely on. Instead, I hope that students can find their own way to identify important aspects of a word problem, and to organize their multiple representations. The graphic organizer is just to get them started.

Think-Write-Pair-Share

A strategy I will use within the EDI framework is giving students individual time to work on a translation problem, record their thoughts, and then discuss with a partner. One thing that I do, in particular, is create different student pairings so that students can hear from multiple perspectives. There is the usual pairing of speaking to the student sitting next to you (“elbow partners”) or across from you (“face partners”), but I have also had students go to a corner of the room, corresponding with the colored sticker on their desks, and other pairings that will also require students to get up and move. To keep each student accountable, I will then call on a non-volunteer to share his or her solution to the whole class, and the rest of the class would verify this solution. Peers may ask specific clarifying questions, or they may help the student sharing if they believe that the solution was incorrect. This would also be an opportunity for broader whole-class discussion on the translation process.

Structured Student Groups

In addition to collaborating during direct instruction lessons through Think-Write-Pair-Share, students will have opportunities to collaborate for more sustained periods through structured group work. To facilitate team interdependence, I will assign group roles²⁰ of Facilitator/Timekeeper, Recorder, Errand Monitor/Presenter, and Summarizer (see Teacher Resources for link to role descriptors.) To balance teamwork and individual accountability, I will assess students as a group, as well as individually; students will have an opportunity to grade themselves, as well as their group mates.

I will use structured group work for activities that require a final product that will be shared either as a presentation, or through a gallery walk²¹. This may include activities where students must create their own word problem, given either a model or an algebraic equation, or showing how changing what is unknown in a context will change the model and algebraic equation.

Classroom Activities

Preparing for the Unit

Prior to teaching the different models to students, I plan to give students a short pre-assessment on simple word problems. A sample task on this pre-assessment is this matching activity.

Word Problem	Equation
1) Four puppies were sitting on the rug. Some more puppies ran there. Then there were six puppies. How many puppies ran over to the first two?	A) $4+2 = ?$
2) Four puppies were sitting on the rug. Two more puppies ran there. How many puppies are on the rug now?	B) $?+2 = 6$
3) Some puppies were sitting on the rug. Two more puppies ran there. Then there were six puppies. How many puppies were on the rug before?	C) $4+? = 6$

Students should correctly identify:

1. C
2. A
3. B

I will then ask students to verbally explain how they matched each word problem with its respective equation. In this way, I can see what intuition students already have about translating real world situations to symbolic representations.

Sequence of Content

I will instruct students in the different bar models and their corresponding problem types, according to complexity of the model.

1. Part-Whole Model

2. Comparison Model
3. Multiplication and Division Model
4. Multiplicative Comparison Model
5. Combining Model Types

I will teach the Part-Whole Model first, as it is needed to later understand the Multiplication and Division model. Similarly, both the Multiplication and Division Model, and the Comparison Model are needed to represent “Compare (Multiplication/Division)” problem types. Of course, it is necessary to learn all of the models for the one-step problem types, before representing multi-step situations.

For each new bar model, and when we first combine models, I will introduce a word problem that will produce the desired model. For each problem type applied to the model, I will ask: *How do you set up the model? What is the unit of each piece of the model? What is the unknown value, and its unit, and what variable will you use? How can you represent each part of the model in an equation? What is the solution to the situation (either using the model or the equation)? Check that the solution makes sense for the units and the magnitude of the situation. If you change what is unknown in the word problem, how does that change the model and the equation?*

Below, find a one-step example, a two-step example, and a multi-step example. (See APPENDICES B-F for more examples.) Note that I will not have students change the unknown in the multi-step word problem.

One Step Equations

1) *Word Problem:* If 18 plums are shared equally into 3 bags, then how many plums will be in each bag?

Model Type(s): Multiplication and Division Model

Bar Model: (See Figure 16 below.)

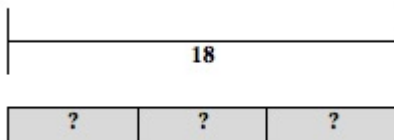


Figure 16

Algebraic Equation:

? = the number of plums in each bag = p

$$3p = 18$$

State & Check your Solution:

There will be 6 plums in each bag, because 3bags of plums•(6 plums per bag) =18 plums.

2) *Word Problem:*

If some plums are shared equally into 3 bags, and there are 6 plums in each bag, how many plums are there

in total?

Model Type(s): Multiplication and Division Model

Bar Model: (See Figure 17 below.)

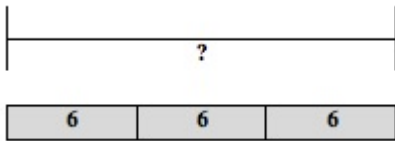


Figure 17

Algebraic Equation:

? = the total number of plums = t

$$3 \cdot 6 = t$$

State & Check your Solution:

There will be 18 plums total, because 3 bags of plums \cdot 6 plums per bag = 18 plums.

3) *Word Problem:*

If 18 plums are shared equally into some bags, and there are 6 plums in each bag, how many bags are there?

Model Type(s): Multiplication and Division Model

Bar Model: (See Figure 18 below.)

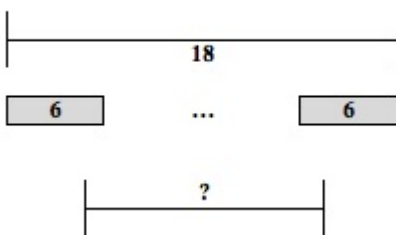


Figure 18

Algebraic Equation:

? = the number of bags of plums = b

$$b \cdot 6 = 18$$

State & Check your Solution:

There will be 3 bags of plums, because (3 bags) \cdot 6 plums per bag = 18 plums.

Two Step Equations

1) *Word Problem:* Oceanside Bike Rental Shop charges a 12 dollar fixed fee plus 7 dollars an hour for renting a bike. Fred paid 54 dollars to rent a bike. How many hours did he pay to have the bike checked out?

Model Type(s): Part-Whole Model, Multiplication and Division Model

Bar Model: (See Figure 19 below.)

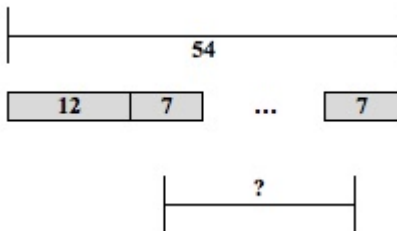


Figure 19

Algebraic Equation:

? = the number of hours he had the bike checked out= h

$$12 + 7h = 54$$

State & Check your Solution:

Fred paid to rent the bike for 6 hours, because $\$12 + \$7 \text{ per hour} \cdot (6 \text{ hours}) = \$12 + \$42 = \54 .

2) *Word Problem:* Oceanside Bike Rental Shop charges a fixed fee plus 7 dollars an hour for renting a bike. Fred rode for 6 hours, and paid 54 dollars to rent a bike. How much did Fred pay for the fixed fee?

Model Type(s): Part-Whole Model, Multiplication and Division Model

Bar Model: (See Figure 20 below.)

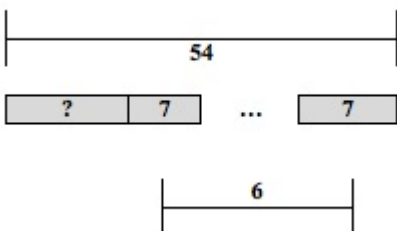


Figure 20

Algebraic Equation:

? = the amount Fred paid for the fixed fee, in dollars= f

$$f + 7 \cdot 6 = 54$$

State & Check your Solution:

Fred paid a fixed fee of \$ 12 because $(\$ 12) + \$ 7 \text{ per hour} \cdot 6 \text{ hours} = \$ 12 + \$ 42 = \$ 54$.

3) *Word Problem:* Oceanside Bike Rental Shop charges a 12 dollar fixed fee plus some dollars an hour for renting a bike. Fred rode for 6 hours, and paid 54 dollars to rent a bike. How much did Fred pay per hour of renting the bike?

Model Type(s): Part-Whole Model, Multiplication and Division Model

Bar Model: (See Figure 21 below.)

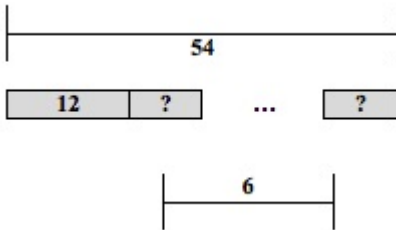


Figure 21

Algebraic Equation:

? = amount Fred paid per hour of renting the bike, in dollars = p

$$12 + p \cdot 6 = 54$$

State & Check your Solution:

Fred paid \$ 7 per hour to rent the bike, because $\$ 12 + (\$ 7 \text{ per hour}) \cdot 6 \text{ hours} = \$ 12 + \$ 42 = \$ 54$. (See APPENDIX B and Teacher Resources for more two-step word problems)

Multistep Equations

1) *Word Problem:* Divide \$ 79 among three people so that the second will have three times the amount of the first, and the third will have two dollars more than the second

Model Type(s): Part-Whole Model, Multiplicative Comparison Model, Comparison Model

Bar Model: (See Figure 22 below.)

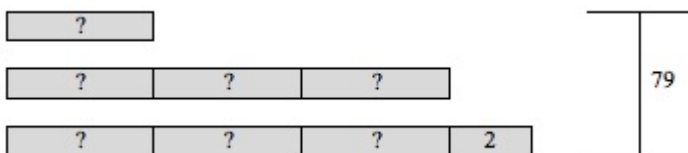


Figure 22

Algebraic Equation:

?=the amount of money the first person gets, in dollars=f

$$f+(3f)+(3f+2) = 79$$

$$7f+2 = 79$$

State & Check your Solution:

The first person will get \$ 11, because $(\$ 11)+(3 \cdot \$ 11)+(3 \cdot \$ 11+\$ 2)=\$ 11+\$ 33+(\$ 33+\$ 2)=\$ 44+\$ 35=\$ 79$.

It is worth noting that this last problem is quite complex, as it requires adding three unknown values. Using bar models makes this problem more manageable, and should help students see that they need 7 copies of the unknown, and then \$ 2 more, even if they are not comfortable with “adding like terms”. However, as mentioned in the overview, students will later encounter problems that require more sophisticated methods of translating and solving. (See APPENDIX B and Teacher Resources for more multi-step word problems).

Once students have shown proficiency in problems of at least two model types, I will provide a mix of word problems. Students will then determine:

- What is the structure of this problem, and what is it asking?
- What model do you need?

Then they will then continue the process outlined in the examples above to represent each word problem as algebraic equations.

Student-Created Word Problems

Up to this point, I have given students a word problem, and they have been creating a model—or combination of models for multi-step problems—and an algebraic equation that represent the situation. Next, I would give students either an algebraic equation or a visual model, and ask them to create a word problem that matches the given representation. Then I will have them explain why the equation or model accurately represent the word problem they have created.

A given equation: Create a word problem that can be represented as $2x+3=9$

A given model: Create a word problem that can be represented by the model (See Figure 23 below.)

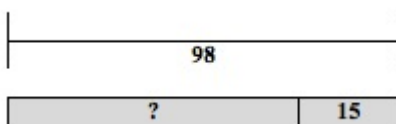


Figure 23

Once students have practiced creating a word problem that matches a specific model or equation, I will have them come up with an original word problem. Then, they will

1. Represent the word problem using an appropriate bar model (or combination of bar models);
2. Define the variable, with appropriate units, and write a corresponding algebraic equation;

3. Explain how the word problem, model, and algebraic equation match; and
4. Rewrite the word problem with the same situation and amounts, but with a different quantity for the unknown value;
5. Determine how the new unknown value effects the other representations of the situation.

This will be one of their summative assessments, completed in groups, in addition to a more traditional individual paper-and-pencil test.

Resources

Annotated Bibliography

Ciobanu, Mirela, "In the Middle-Misconceptions About the Equal Sign in Middle School," *OAME/AOEM Gazette* (2014), 14-16.

This article defines the "operational" meaning of the "=" sign (*calculation equals answer*), and describes how this interpretation causes errors, especially when solving algebraic equations.

Fuchs, Lynn S. et al., "The Effects of Schema-Broadening Instruction on Second Graders' Word-Problem Performance and Their Ability to Represent Word Problems with Algebraic Equations: A Randomized Control Study," *Elem Sch J.* (2010), 446-463.

This is a study on teaching students to representing the structural, defining features of word problems with the schema of "overarching equations." Essentially, this unit is doing something analogous, but using the "problem types" and their corresponding bar models rather than "overarching equations."

Howe, Roger, "From Arithmetic to Algebra," *Mathematics Bulletin—A Journal for Educators*, 49(2010), 13-21.

This is a paper on teaching algebra through analyzing and discussing both arithmetic and algebraic word problems. In particular, I found his discussion on units of variables, and of terms in an addition equation, to be helpful in my unit.

James, Jonathan "The Surprising Impact of High School Math on Job Market Outcomes," *Economic Commentary*, (2013).

This is a study on the effect of high school math courses on unemployment rates and income, both for high school graduates who attend college, and those who go straight into the workforce. I was surprised at the effect for those who don't attend college!

Kinzel, Margaret Tatem, "Understanding Algebraic Notation from the Students' Perspective", *The Mathematics Teacher*, Vol. 92, No. 5 (1999).

This paper examines students' various struggles with representing and interpreting word problems using algebraic symbols (variables).

MacGregor, Mollie and Stacey, Kaye "Cognitive Models Underlying Students' Formulation of Simple Linear Equations", *Journal for Research in Mathematics Education*, Vol. 24, No.3 (1993), 217-232.

This paper shows students' struggles in representing compared unequal quantities. Most notably, reversal errors can occur even if the student is not translating word-for-word (syntactically).

National Governors Association Center for Best Practices, Council of Chief State School Officers, *Common Core State Standards for Mathematics*, (Washington D.C.: National Governors Association Center for Best Practices, Council of Chief State School Officers, 2010), 88-89.

This is the list of the one-step problem taxonomies. In my unit, I changed the labels used in the table to ones we used in seminar, because I found them to be more descriptive than the ones in the CCSS guide. For example, the unknowns in the “Compare (Multiplication/Division)” problems changed from “number of groups” to “factor”, “group size” to “smaller”, and “product” to “bigger.” I also took some of my example problems in Appendices B-F from this document.

Ng, Swee Fong and Lee, Kerry, “The Model Method: Singapore Children’s Tool for Representing and Solving Algebraic Word Problems,” *Journal for Research in Mathematics Education*, Vol 40, No. 3 (2009), 282-313.

This paper studies the effect of using models to represent key information in a problem, in order to solve both arithmetic and algebraic problems. What I found most helpful were both the general, and applied examples of the Part-Whole model, the Comparison model, and the Multiplication and Division model.

Polya, G. *How to Solve It: A New Aspect of Mathematical Method* (Princeton and Oxford: Princeton University Press, 2004).

This is a book on methods of problem solving, broken up into four phases. In my curricular unit, I focus on the first phase, which is “understanding the problem.” The rest of Polya’s phases are devising a plan, carrying out the plan, and reviewing/extending the plan.

Powell, Sarah, “The Influence of Symbols and Equations on Understanding Mathematical Equivalence,” *Interventions in School and Clinic (ISC)*, Vol. 50(5) (2015), 266-272.

This article focuses on the “relational” meaning of the “=” sign, where one side of the equation is “the same as” the other side of the equation. The article goes on to explain how instructing on this relational meaning of equality improved student outcomes in setting up and solving word problems.

Teacher Resources

http://www.mathplayground.com/ThinkingBlocks/thinking_blocks_modeling%20_tool.html

This is a tool for creating bar models. Requires Adobe Flash. My diagrams in the “Key Ideas” section for this unit plan were screen-grabs from this tool.

Forsten, Char, *Step-by-Step Model Drawing: Solving Word Problems the Singapore Way* (Peterborough, NH: Crystal Springs Books, 2010), 30-32. I only used this book to see how she treated an unknown number of groups in the multiplication/division model. However, she is very step-by-step in explaining the process for beginners of the Singapore Bar Method. While this may be helpful as an entry, it is a bit prescriptive and too procedural.

The following are places where I got most of my word problems featured in the appendices.

http://www.math-aids.com/Algebra/Algebra_1/Word_Problems

<http://cdn.kutasoftware.com/Worksheets/PreAlg/One-Step%20Word%20Problems.pdf>

<http://www.onlinemathlearning.com/comparison-word-problems.html>

https://learnzillion.com/resources/11807?card_id=68694

<https://www.ixl.com/math/algebra-1/solve-linear-equations-word-problems>

<https://www.khanacademy.org/math/algebra/one-variable-linear-equations/alg1-linear-eq-word-probs/e/linear-equation-world-problems-2>

http://www.algebra-lab.org/lessons/lesson.aspx?file=Algebra_OneVariableWritingEquations.xml

<https://www.cliffsnotes.com/study-guides/algebra/algebra-i/word-problems/age-problems>

<https://www.algebra.com/algebra/homework/word/age/Solving-Age-Problems.lesson>

<http://www.purplemath.com/modules/ageprobs.htm>

Here is one version of group role cards you can give to students, including sentence starters for each group member.

http://www.readwritethink.org/files/resources/lesson_images/lesson277/cooperative.pdf

Appendix A

The CCSS Standards I am addressing in this unit are:

7th Grade Standards

- MA.7.EE.B.A: Use variables to represent quantities in a real-world or mathematical problem, and construct simple equations and inequalities to solve problems by reasoning about the quantities.
- MA.7.7.EE.B.4.A: Solve word problems leading to equations of the form $px+q=r$ and $p(x+q)=r$, where p , q , and r are specific rational numbers. Solve equations of these forms fluently. Compare an algebraic solution to an arithmetic solution, identifying the sequence of the operations used in each approach.

6th Grade Standards

- MA.6.EE.B.6: Use variables to represent numbers and write expressions when solving a real-world or mathematical problem; understand that a variable can represent an unknown number.
- MA.6.EE.B.7: Solve real-world and mathematical problems by writing and solving equations in the form $x + p = q$ and $px = q$.

Appendix B

Part-Part Whole Model Problems

“Put Together, Take Apart” Problem

Alyssa is baking a cake. The recipe calls for 8 cups of flour. She already put in 2 cups. How many more cups

does she need to add?

“Add To/Take From” Problem

Tom received 80 dollars for his birthday. He went to a sporting goods store and bought a baseball glove, baseball, and bat. He had 39 dollars left over, how much did he spend on the baseball gear?

Comparison Model Problem

Bob ate 14 crackers. Steve ate five fewer than Bob. How many did Steve eat?

Multiplication and Division Model Problem

At a restaurant, Mike and his three friends decided to divide the bill evenly. If each person paid \$ 13 then what was the total bill?

Multiplicative Comparison Model Problems

The giraffe is 20 feet tall. The kangaroo is 5 feet tall. The giraffe is how many times taller than the kangaroo?

Combining Bar Models -Multistep Word Problems

Part-Whole Combination Problem

When ringing up a customer, a cashier needs 17 seconds to process payment as well as 4 seconds to scan each item being purchased. How long does it take to ring up a customer with 6 items?

Comparison Combination Problem

Jose has a board that is 44 inches long. He wishes to cut it into two pieces so that one piece will be 6 inches longer than the other. How long should the shorter piece be?

Appendix C

Blank Graphic Organizer

Note: Fill in the word problems yourself to save time for the students, OR have students only write down the important information (data, unknown values, relationship) in the top row of the organizer.

Word Problem

Model Type(s) Bar Model Algebraic Equation

State & Check your Solution

Endnotes

1. "How I see math problems," *someecards*, July 16, 2016
<https://www.someecards.com/usercards/viewcard/MjAxMi1mOThmZjNhOGJiYmE5YzMy/>
2. "Peter Burnett Middle School," *Great Schools*, May 30, 2017
<http://www.greatschools.org/california/san-jose/5654-Peter-Burnett-Middle-School/>
3. Jonathan James, "The Surprising Impact of High School Math on Job Market Outcomes," *Economic Commentary*, (2013)
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5. G. Polya, *How to Solve It: A New Aspect of Mathematical Method* (Princeton and Oxford: Princeton University Press, 2004), 173-174
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7. Polya, *How to Solve It*, 7-8
8. Swee Fong Ng and Kerry Lee, "The Model Method: Singapore Children's Tool for Representing and Solving Algebraic Word Problems," *Journal for Research in Mathematics Education*, Vol 40, No. 3 (2009), 291
9. Jo Boaler, Lang Chen, et al. "Seeing as Understanding: The Importance of Visual Mathematics for our Brain and Learning," (July 16, 2017)
<https://bhi61nm2cr3mkdggk1dtaov18-wpengine.netdna-ssl.com/wp-content/uploads/2016/04/Visual-Math-Paper-vF.pdf>
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11. Lynn S. Fuchs, et al., "The Effects of Schema-Broadening Instruction on Second Graders' Word-Problem Performance and Their Ability to Represent Word Problems with Algebraic Equations: A Randomized Control Study," *Elem Sch J*. (2010), 4.
12. Char Forsten, *Step-by-Step Model Drawing: Solving Word Problems the Singapore Way* (Peterborough, NH: Crystal Springs Books, 2010), 30-32
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16. Margaret Tatem Kinzel, "Understanding Algebraic Notation from the Students' Perspective", *The Mathematics Teacher*, Vol. 92, No. 5 (1999), 437.
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18. Ng and Lee, "The Model Method," 293.
19. International Baccalaureate Organization, *Mathematics Guide, For use from September 2014/January 2015* (Cardiff, Wales: International Baccalaureate Organization, 2014)
20. "Cooperative Group Role Cards," *Read, Write, Think*, (July 14, 2017)
http://www.readwritethink.org/files/resources/lesson_images/lesson277/cooperative.pdf
21. Wanda Collins, "Gallery Walk," *Math Concentration*, (July 15, 2017)
<http://www.mathconcentration.com/profiles/blogs/using-the-gallery-walk-as-a-teacher-strategy>

<https://teachers.yale.edu>

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