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Making Sense of Solving Equations Through Word Problems

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Overview

Based on my fourteen years of teaching Algebra to 7th and 8th graders in public middle schools in Oakland and Berkeley, I can confidently say that word problems are the part of the subject that is least liked by the students. Whenever word problems are introduced, many of my students react the same way: blank faces and groans. It feels as if a massive impenetrable steel wall separates my students from engaging with word problems. Many students shut down and literally say "I don't do word problems". Their avoidance of taking on the challenge and the reflexive belief of their inability of solving word problems are present at all skill levels. Neither good number sense, nor flexibility with number relationships exempts students from having negative views on word problems. Their apprehension does not dissipate as they enter higher math classes. In fact, many high school teachers in Algebra 2 and Calculus notice similar misgivings from their students. Thus elementary and middle school teachers have an important responsibility to teach students to feel more comfortable and gain confidence with solving word problems.

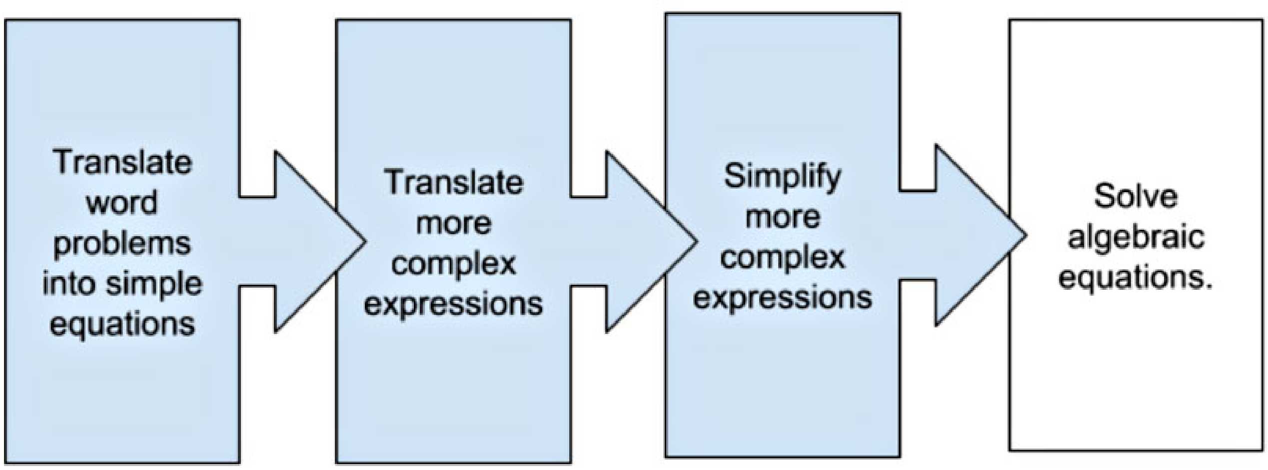
Another major concept that my 8th graders struggle with is solving multi-step linear equations.

Many of them lack the conceptual understanding from previous grade standards. Many students do not have a clear understanding of the many facets of variables. They lack a formal understanding of the Rules of Arithmetic, which interferes with simplifying expressions. They also lack a robust understanding of equality and equivalence. The combination of this confusion and misunderstanding comes to the surface in many different ways, expected and unexpected, when they are faced with multi-step one variable equations.

The purpose of this curricular unit is to improve my students' performance in solving linear equations and word problems by teaching the two topics simultaneously so that they can have a more positive and successful experience in high school math classes. This unit works to fully engage students from different cultures and levels of skill, using word problems to flush out and address the many misconceptions and gaps in prerequisite skills and at the same time to help students construct deep conceptual understanding of the procedures of algebra. The focus of the unit is on helping students realize the efficiency of using algebraic equations as they work through progressively more complex word problems using both an arithmetic approach and a procedural algebra approach. It is expected that their reliance on computational strategies will gradually transition to using procedural algebraic problem-solving strategies as problems increase in complexity and as they notice the limitations and benefits of both approaches. Furthermore I hope that facility with both approaches will deepen their understanding of algebra. Moreover I want my students to become flexible problem solvers, which means they can modify their approach when faced with challenges during the problem-solving process. In other words, they can start with an algebraic procedure then switch to an arithmetic approach if they realize their initial approach is too cumbersome or vice versa. Thus, they will effectively learn more about both approaches.

Scope and Sequence of Linear Expressions and Equations

This unit is the last of the four interrelated units developed in 2017 Yale National Initiative (YNI) Math Seminar. Due to the breadth and importance of learning to solve word problems with symbolic algebra, four national Fellows (2017 math Fellows Rachelle Soroten, Jeffrey Rossiter, Xiomara Pacheco, and I) built separate but related curricular units, for each section with word problems as the underlying theme. The first unit covers translating simple one-step and two-step word problems into equations. The second unit delves into translating complex phrases into expressions with many more terms in preparation for deriving multi-step equations. The third unit uses properties of the operation to simplify expressions. This fourth unit focuses on solving equations starting with one–step equations, going up to multi-step equations including variables on both sides of the equal sign. Below is a graphic that shows the scope and sequence of the four curricular units.



Rationale

In the United States today, teaching students the skills to truly understand Algebra is one of the most important ways to improve their economic status. Bob Moses, civil rights activist and the founder of Algebra Project, states, "In today's world, economic access and full citizenship depends crucially on math and science literacy" (1). Algebra is a "gate keeper" because it is a requirement for students to enroll in higher math classes and science classes. Therefore, if students are considering careers in the fields of science and technology, they need to pass Algebra sooner rather than later. Unfortunately according to David Foster, executive director of Silicon Valley Math Initiative (SVMI), the state of California is experiencing an "Algebra Crisis". One piece of evidence for this claim is that three out of four students scored below proficiency in the 2008 Algebra 1 California Standards Test (CST) (2). Even more troubling, according to the California Dropout

Research Project 2008 of Los Angeles Unified School District, the high school dropout rate is approximately double for students who do not pass Algebra 1 by 9th grade (70% versus 35%) (3) Consequently it has been said that "Algebra is the #1 trigger of dropouts in high school" (4). Thus we can see some ways that Algebra performance can affect opportunities for all students regardless of career choices.

In response to the "Algebra Crisis" and student underperformance on international math assessments, many states, including California, adopted Common Core Standards. My district, Berkeley Unified School District (BUSD), implemented Common Core Math five years ago. Unfortunately too many of our students, especially students of color, have had limited success in math from early school years. In fact the achievement gap has become wider and wider in higher grades. In BUSD on the 2016 Smarter Balance Assessment for Math, Black/African American students scoring at or above proficient decreased significantly from 26% in 6th grade to 11% in 8th grade. Similarly, Latino students also decreased from 43% in 6th grade to 34% in 8th grade. This is in stark contrast to the situation for White students of whom 82% were proficient in 6th grade, and also 82% in 8th grade, no change at all. I predict that this downward trend for students of color and the achievement gap between white peers will only get worse in Algebra 1 and in other high school math classes.

Knowing the importance of passing Algebra 1 in our students' career choices and their future economic standings, many elementary and middle school math teachers are preoccupied with the question of how to make Algebra concepts more accessible to all of our students. Like my peers, I have tried many different instructional strategies and pedagogies in my 8th grade classroom. However, a much lower percentage of my students are proficient in Algebra strands than in Geometry strands. On 2016 District Assessments, 90% of my 8th grade students were proficient in Geometry (Congruency and Similarity) strands, but only 71% were proficient in Algebra (Expressions and Linear Equations and word Problem) strands. Even worse, only 38% percent of students of color and English learners were proficient, which means nearly two thirds are not mastering these essential concepts. I find this troubling, because these students will fall farther and farther behind in high school math classes, which can close doors to more lucrative career choices.

In every one of my fourteen years of teaching, there have been a number of students who think the value of x is 1 because x represents an unknown value in the equation and they hear that "x is a shortcut for 1x." When solving 4x = 10, some students subtract 4 from both sides of the equation instead of dividing by 4. Every time I notice such misconceptions as above, I ask the following questions: *What does 4x mean? What are we looking for? What operation do we need to do to find the value of one x?* Through this process of questioning, I can get students to understand what to do with the equation we are discussion. Unfortunately these "aha" moments do not stick. One of the goals of this unit is for students to see concretely the meanings of equations such as 4x = 10 and ¼ - x = 10 by referring the symbols to the situations described in the word problems.

Although my own student work analysis and research show that misconceptions around variables is one of the key reasons for mistakes in solving equations, most of my 8th graders would say that they know all they need to know about variables. In reality most students understand variables only in the form of labels representing objects: t for table, c for centimeters, and x for a number. Phillips (5) discusses the many different roles the variables can take on (labels, constants, unknowns, generalized numbers, varying quantities, parameters, abstract symbols) depending on the situation. Studies (6) show students of all mathematical level, even in college, have misconceptions regarding how to think of variables due to the multiple roles they play. By intentionally drawing connections between the context in the word problem and the given equation, I hope that my students should arrive at the broad conception that a variable is a place holder for any number in some set of numbers, but that it can seem to play the various roles mentioned above, depending on text.

Context

Martin Luther King Jr. Middle school is a diverse public urban school in Berkeley, CA. We have about 1100 students from 6th through 8th grade. They are from a variety of economic, social, and cultural backgrounds. Some students live in multi-million dollars houses on the hills, while others live in dilapidated apartments on the flatlands, or even in the streets, homeless. A significant number of our students are on free or reduced lunch. Of our total enrollment, 35% are identified as Socioeconomically Disadvantaged. The ethnic breakdown of our student population is 47% White, 13 % Black or African American, 18% Hispanic or Latino, 7% Asian, 2% American Indian or Alaska Native or Filipino or Native Hawaiian or Pacific Islander, and 13% two or more races. The families of our students speak over thirty languages. During our 8th grade promotion ceremony, there were 27 languages spoken by the students to welcome their families in their home language. Of the total student population about 10% are English Learners.

All of the classes are extremely heterogeneous. 11% of our students are designated as Students with Disabilities. Our school has a Counseling Enriched Classroom for students designated as emotionally disturbed, where they have access to counseling services five days a week, and are mainstreamed into some classes depending on their needs. In addition we are a full inclusion school, which mean that many students with moderate to severe disabilities are included in general education classrooms on a full-time basis. Furthermore, even among the students without Individualized Educational Plans (IEP), there is a wide range of academic preparedness. Some students are reading at a third-grade level, while others are at a college level. In math classes, some are struggling with multiplication and division facts, while others are ready for Algebra 2 and beyond. This extremely wide range in academic skills poses huge challenges for teachers to differentiate as much as possible so that all students are engaged and continuing to learn.

Mathematics Background

Prerequisite Skills

Conceptual understanding of the prerequisite skills such as simplifying expressions and the roles of the variable can affect student learning in solving algebraic equations. The paper, "Key Misconceptions in Algebraic Problem Solving" (7) discusses the importance of students having deep conceptual understanding of features in an equation such as the equal sign, variable, like terms, negative signs, and more. The article states that "misconceptions or gaps in conceptual knowledge of relevant features inhibit students' performance and learning" of procedural algebra. Thus, students with the necessary prerequisite concepts will likely learn the concepts of procedural algebraic solving, while others may have "shallow" learning, which can lead to the mistakes I have seen in my classrooms year after year. Fortunately one of the key findings of the study is that if these same prerequisite skills are taught during the lesson, those students can also learn the skills for solving equations.

In this unit I will address the prerequisite skills by continually connecting manipulation of the algebraic equation to the computation used, and also referring back to the context of the word problem the equation came from.

Example: *Sum of two consecutive numbers are 31. Find the numbers.*

Arithmetic approach: Since the numbers are consecutive, the larger number is one more than the smaller one. If we subtract 1 from it, we will get the smaller one. This will also subtract 1 from the sum, so the double of the smaller number is 31 – 1 =30. Dividing by 2 then gives the smaller number. 30/2 = 15 Hence the two consecutive numbers are 15 and 16.	Algebraic approach: Let x represent the smaller of the two numbers. Since the numbers are consecutive, the other number is (x + 1). Therefore, the word problem is translated to x + (x + 1) = 31. (x +x) + 1 = 31 Associative Property. 2x + 1 = 31 Combine like terms. 2x + 1 + -1 = 31 +-1 Addition Property of Equality. 2x = 30 Combine like terms. (1/2) 2x = (1/2) 30Multiplication Property of Equality ((1/2)2)x=15 Associative Property of Multiplication 1x = 15 Inverse Property of Multiplication x =15 Identity Property of Multiplication The smaller of the two numbers is 15. Verify in the original equation. 15 + (15 + 1) = 31 15 + 16 = 31 31 = 31 verified

Through discussions, I plan to activate prior knowledge on combining like terms by showing that the *x* in the above equation is referring to the same number *15*, the smaller number of the two numbers. Thus, we can combine the *x* and *x*, which results in *2x* or two times the value of *x*, since both *x*s represent the same value: *15*. But the number *1* cannot be combined with *x* since they do not have the same value. In addition, the idea of relational equality, the left side of the equation having the same value as the right side, will be addressed in similar manner. In the above example, by substituting *15* for both variables *x*, the left side of the equation is equal to 31 just as the right side is equal to *31*. Thus, relational equality is emphasized.

Reverse Engineering

"Reverse engineering" in this unit refers to connecting features of the given equation with the word problem it models. Students will make sense of the structure of the algebraic equation by examining each feature in context of the word problem before transforming the equation. When faced with word problems, students intuitively slow down and try to make sense of them. However students do not have the same expectations toward algebraic equations. They do not recognize the importance of investing time into understanding all the features of the equation. Instead they race through them until they come upon an equation with the same structure but which is written slightly differently. For those students, the slight difference in the way the problem is written makes it a totally new problem. Even though the problem is essentially the same as the problems they have successfully solved previously, these students do not know how to approach it.

I will use reverse engineering to explicitly teach students to examine the structure of the equation before transforming it to a simpler form so that they can generalize the concepts, instead of memorizing infinitely many procedures for all the different ways equations can be written. Since translating words to symbols was covered in the two previous curricular units, in this unit students will take the time to make sense of the equation by connecting all the relevant features of the equation to the text in the word problem, instead of deriving the equation from the word problem. For example: *The sum of two consecutive numbers is 31. Find the numbers.* The equation $x + (x + 1) = 31$ will be given to the students as a model for the above word problem. Students will focus on each feature of the equation through following questions: *What does x represent? Why are there three terms (x, x, 1)? Why are all the terms being added? Why do they equal to 31?* By understanding the reasons behind the structure of the equation, the students can think more deeply before transforming the equation.

Justification

Furthermore, students will be expected to justify each statement by writing the properties for each subsequent equivalent equation until they become experts with the Properties of Equality.

1. Addition Property of Equality: if $a = b$, then $a + c = b + c$.
2. Multiplication Property of Equality: if $a = b$, then $a(c) = b(c)$.
3. Transitive Property of Equality: if $a =b$, and $b =c$, then $a = c$.

Here, a, b, and c stand for arbitrary numbers.

Because the process is so cumbersome, I did not stress the need to justify each step in the past. Students used the shortcuts, transforming equations without justification, before they became experts with the properties of equality, which lead to confusions about when to use the rules of operations versus properties of equality. In YNI 2017 Math Seminar, Dr. Howe, seminar leader, Professor Emeritus at Yale, and Professor of Mathematics Education at Texas A&M promoted the mantra that “Shortcuts are privileges of the experts”. When shortcuts are shown to students before they have a thorough understanding, they become bewildered and develop misconceptions. Promoting the shortcut before understanding is in place may block future learning. In order to avoid misconceptions, students need to see the logical flow and the process must be transparent to them. They need to understand that just as the Order of Operations is used to simplify numerical expressions and Rules of Arithmetic are used to simplify algebraic expressions, the Properties of Equality are used to manipulate equations.

Unlike the traditional approach, I will use only the three Properties of Equality mentioned above. Most textbooks treat the Addition Property of Equality and the Subtraction Property of Equality as separate entities. However, I will treat them as one property and take time to show students that subtraction is addition of the negative; thus my students will only need the Addition Property of Equality. Similarly, rather than the traditional way of using the Multiplication Property of Equality and Division Property of Equality by themselves, I will treat them as one property. Since division is multiplication by reciprocal, students only need to use the Multiplication Property of Equality. There are a couple of benefits to this approach. First, students will focus on the inherent inverse relationship between addition and subtraction, and multiplication and division. Second, students will have fewer Properties of Equality to learn.

Verification

My students will also verify the solution. According to G. Polya, author of ‘How to Solve It’ (8), verifying a solution by substituting it into the original problem and referring back to the context in the text of the problem is an important and necessary step. A solution needs to be interpreted in the context of the given situation to determine whether the obtained result makes sense in the context of the problem. Since a variable is a placeholder for a number in a given set, the obtained value must be a member of the predetermined set. If the solution is not a member of the set, then the solution would not make sense, and the plan must be reevaluated. My students will practice the cycle of devising a plan, reflecting on the process, and revising the plan until a solution that makes sense in the context of the situation is identified.

For example, in the problem, *"Thesum of two consecutive even numbers is 46. Find the numbers.",* if the variable represents the smallest even integer, then the solution must be an even number and also the smaller of the two numbers. If the solution found is not an even number, then it would be a signal for the student to rethink the problem and revise their plan. If the solution is not the smaller of the two even numbers, then that would also be a signal for the student to revise their thinking. This essential step is often skipped by the student but I will have my students practice it continually until it becomes a habit. The process of stopping to reflect becomes more and more valuable as problems become more complex. Famous mathematicians realize that solutions are often found through much trial and error, but most of my middle school math students come to my class believing "smart" students know the path to the solution right away. Thus when the path is not apparent immediately, they believe they are not smart and cannot do math. One of goals of this unit is to provide many opportunities for students to experience the need to revise their plans and try different approaches. Hopefully, my students will come to realize that the process of solving complex problems is often fraught with revision and that perseverance is a normal part of the process and a practice to be valued.

Additionally, verifying by substitution into the original equation is a great opportunity to emphasize the concept of equivalence present in an algebraic equation, where two algebraic expressions are set to equal each other. An algebraic equation essentially is a question: it asks, of all the values that the variable might take, what particular value can be substituted into the two expressions so that the value on the right side of the equal sign "=" is the same as the value on the left side. This is a good opportunity to discuss the multiple meanings associated with "=" sign. From previous math classes, most students understand equal sign (=) only as signal to perform operations to find an answer. However, in algebra and higher math, it is essential to take the = sign as meaning that the two expressions on the either side of it should have the same value. For example $2x = y$ expresses the relationship between x and y, which can be interpreted as *"the value of y is double the value of x"* or *"twice the value of x is equal to the value of y."* Thus, there is a defined relationship between the values of x and the values y that follow the rule described by $2x = y$. The values of x and y are dependent on each other and each is responsible for how the other value changes: if the value of x changes, then the value of y changes in response to the change in x. This difficult concept of relational equivalency is introduced in 7^{th} grade but needs to be mastered in 8^{th} grade in order for a student to be successful in Algebra 1 and beyond.

Transitive Property of Equality

Less discussed in the classroom is the equivalence among the transformed equations. Using word problems, my students will understand that the original equation and all the transformed equations are essentially the same. The Transitive Property of Equality (*if $a = b$ and $b = c$, then $a = c$*) is used almost unconsciously in conjunction with Properties of Operations to transform complex equations to simpler equations. My students can have a more robust understanding around equivalent equations if this process is made more transparent.

For example: Sum of two consecutive numbers are 31. Find the numbers.

Another way to explain the arithmetic approach:
31 is the sum of the two consecutive numbers. If 1 is taken away, then the sum is total of two of numbers that have the same value, the lower of the two consecutive numbers.
$31 – 1 =30$
Since 30 is two times the lower number, 15 is the lower number.
$30/2 = 15$
Therefore, the consecutive numbers are 15, 16

Let x represent the smaller of the two numbers. Then next consecutive number is $(x + 1)$.
Therefore, the two consecutive numbers whose sum is 31 is represented by $x + (x + 1) = 31$.
$(x +x) + 1 = 31$ Associative Property.
$2x + 1 = 31$ Combine like terms
$2x + 1 + -1 = 31 +-1$ Addition Property of equality.
$2x = 30$ Combine like terms.
$(1/2) 2x = (1/2) 30$ Multiplication Property of equality
$((1/2)2)x=15$ Associative Property of Multiplication.
$1x = 15$ Inverse Property of Multiplication.
$x =15$ Identify Property of Multiplication.
The smaller of the two numbers is 15.
Therefore, the consecutive numbers are 15, 16.
Verify in the original equation.
$15 + (15 + 1) = 31$
$15 + 16 = 31$
$31 = 31$ verified

The equation $x + x + 1 = 31$ is equivalent to the $2x + 1 = 31$ by the Transitive Property of Equality because $x + (x +1) = (x + x) +1= (1x + 1x) +1 = (1 + 1)x + 1 = 2x +1$.Thus, x + (x +1) is simplified to $2x + 1$ through a sequence of operations, of which the most important is combining like terms. Since $x + x + 1 = 31$ *and* $x +x + 1 = 2x + 1$, *thus* $2x + 1 = 31$, we can conclude that the two equations $x + x + 1 = 31$ and $2x + 1 = 31$ are essentially the same equations asking the same question.

Multiplication Property of Equality

The following one-operation word problems are intentionally sequenced to teach the Multiplication Property of Equality and highlight any misunderstanding of multiplicative relationships, which is taught in previous grades. Students need a deep conceptual understanding of the multiplicative relationship to fully grasp how the Multiplicative Property of Equality is used to transform the equations to uncover the unknown value. When the meaning of multiplication, division, and fractions are explicitly dealt with in a concrete setting, the misconceptions are more easily resolved. Because the computational demands are lower, students can focus on the structure of the equation and how each part functions during the transformation process.

Students will be asked to solve the following simple word problems using computation. They are designed and sequenced in such a way that most students will likely find the solution using computation. The purpose of the following word problems is to make the steps used in computation more transparent, so they can form a generalized rule. The steps in computational approach will be explicitly connected to the steps in algebraic approach. Rather than the conventional view of seeing arithmetic and algebraic as two completely different worlds, hopefully students come to realize that the approaches are closely connected.

Example: Equal Group: Number of group members unknown

Your friend tells you she bought 12 bananas. On this particular day, every bunch had the same number of bananas so she bought 2 bunches. She says to you "How many bananas do you think were in each bunch?

Arithmetic approach: 12 bananas are equally shared between the two bunches. Therefore, each bunch has 6 bananas. 12/2=6	Algebraic approach: *Let x represent the number of bananas in one bunch.* *2x=12* (1/2)2x=(1/2)12 Multiplication Property of Equality. Since (1/2)2x=((1/2)2)x= (1)x=x, and (1/2)12=6, we can conclude by the Transitivity Property of Equality that x=6 There are 6 bananas in each bunch Verify by substituting into the original equation: 2(6) = 12 12 = 12 (verified)

The most important part is defining the variable: *let x represent the number of bananas in one bunch.* Once the variable is clearly defined, students can focus on the meaning of *2x*, which means two times the number of bananas in each bunch since *the friend bought two bunches*. Thus the multiplicative relationship represented in the algebraic equation and in the text of the word problem can be explicitly connected. Furthermore, the connection between *dividing by two* in the arithmetic approach and the *multiplying by half* in the algebraic approach will be brought to the attention of the students to make the process transparent. Through questioning and class discussions, students would see that the computational step used in the arithmetic approach is also present in the algebraic approach.

For the multiplication property of equality, I will once again refer to the situation described in the word problem. Since *2x* is two times the number of bananas in one bunch, *half of 2x* is the number of bananas in one bunch. Since *12* represents the twelve bananas from two bunches, *half of 12* is the number of bananas from one bunch. Therefore, students can clearly understand and concretely see that *the half multiplied* to the left side of the equation is also multiplied to the right side of the equation to identify the number of bananas in one bunch. Thus, the rationale for the Properties of Equality used to maintain the balance between the two sides of the equation when transforming equations is shown explicitly.

Example: Comparison: Larger number unknown

Your teacher and your mom are talking at the registration when the teacher suddenly turns to you and asks how old your grandmother is. You forgot how old your grandmother is but you remember that your youngest aunt Kristie is 30 years old. Furthermore, your aunt Kristie is half the age of your grandmother. How old is your grandmother?

One possible approach: Kristie is 30 years old. Since 30 years is half of grandmother's age, grandmother's age is twice 30, or 60. 2(30)=60	*Let p represent grandmother's age.* 30= 1/2 p . (2)30=(2) 1/2 p Multiplication Property of Equality. (2) 30=((2) 1/2)p Associative Property of Equality. 60 = 1p Inverse Property of Multiplication. 60=p Identify Property of Multiplication. Since(2) 1/2 p=((2) 1/2)p=(1)p=p, and 2(30) = 60, we can conclude that p = 60 by the Transitive Property of Equality. Grandmother is 60 years old. Verify by substituting into the original equation. 30 = ½(60) 30 = 30 (verified).

The purpose of this example is to show students that the two above examples have multiplicative relationships represented in slightly different ways. Both equations actually have the same structure, Thus, the same property, Multiplication Property of Equality, is used to simplify an equation in the form of *ax=b* to the form *x = c (where c = b/a).*

Addition Property of Equality

Two operation word problems are arranged to teach the Addition Property of Equality and to address any misconceptions and gaps regarding prerequisite concepts of operations with negative signs, unlike terms, and the inverse property of addition. I will still write the equation that models the word problem since translating words to equations was covered by Rachelle Soroten, 2017 NYI fellow and author of the first curricular unit of the four complementary units. The problems are still simple enough that the arithmetic approach would be more efficient than using algebraic procedure. In fact, using arithmetic would avoid the usual mistakes I see around the operation with negative numbers and other prerequisite concepts. However, as students continue to compare the two approaches, they may begin to see the benefits of using an algebraic method. As problems become increasingly complex, more students may switch their approach from arithmetic to algebraic in the middle of the process when they experience difficulties. Thus, students are constructing their own knowledge around the limitations and benefits of both approaches through their personal experience rather than being told which method should be applied to which types of problem.

Example: Change (+): Array Group Size Unknown

There are 26 desks in your math class. Because the classroom is not a perfect rectangle, some desks are arranged in equal rows and other desks are arranged in a group of 6 desks. You are sitting in the group of 6 desks. As you look around, you realize there are 5 rows. You started to wonder how many desks are in a row.

Arithmetic approach: If the 6 desks in the group are taken away from the total, then 20 desks are arranged in rows. 26-6=20 Since 20 desks are arranged in 5 rows, 4 desks are in each row. 20/5=4 Therefore, 4 desks are in each row.	Algebraic approach: *Let x represent the number desks in each row.* 5x+6=26. 5x+6+(-6)=26+(-6) Addition Property of Equality. 5x=20 Combine like terms. (1/5)5x=(1/5)20 Multiplication Property of Equality. ((1/5)5)x=4 Associative Property of Multiplication. 1x = 4 Identity Property of Multiplication. x = 4 4 desks are in each row. Verify by substituting into the original equation: 5(4) + 6 = 26 20 + 6 = 26 26 = 26 verified

As noted above the equation *5x + 6 = 31*, which models the word problem, will be given to the students. The purpose of the above example is for the students to notice the structure of the equation and connect each feature to the text. This is the “reverse engineering” discussed above. Thus, students understand that + *6* represents the six desks that are not part of the array, *5x* represents five rows times some the unknown number of desks in each row, and *31* represents the total number of desks in the classroom. The aim is for students to see that each part of the equation has a purpose and a concrete meaning. Once students understand the role of each feature, they can more easily understand why *5x* cannot be combined with *6*. My hope is that by spending time connecting each feature of the equation to the context in the word problem, misconceptions such as transforming 5x + 6 = 31 to 11x = 31 can be avoided.

To teach the Addition Property of Equality, I would bring students' attention to the fact that the same operations are performed in the arithmetic approach and the algebraic approach. In the arithmetic method, *31* is decreased by *6* to take away the extra desks that were not part of the array. The same procedure is present in the algebraic approach. Since *6*, the six desks, is not part of the array, *-6* is added to 5x + 6 so that the expression can be simplified 5x: five rows times the number of desks in a row. Since 31 represents the total desks in the classroom, 31 would also have to decrease by 6: the six desks eliminated in 5x + 6. Thus students can observe how the Addition Property of Equality is used to simplify the equation. Students can understand the rationale for needing to simplify the equation. They would realize that if they could focus only on the arrays, then finding the number of desks in a row is much simpler.

Once the students simplify the equation to 5x = 25, they can use the Multiplication Property of Equality, the concept already addressed in discussing the one-step equation. By comparing the arithmetic approach and the algebraic approach, students will notice that the same operations are performed in both approaches. In computation, students took away 6 from 31, then divided the result, 25, by 5 to obtain 5, five desks in a row. In algebraic approach, 6 is taken away from the both sides of equal sign (=), then divide by 5 both sides of the equal sign. The same operations are used for both approaches.

Example: Change (-): Compare – Bigger Unknown

Your grandmother is telling you a story of why she only has 6 hats even though she used to have a lot them. On her 40th birthday, there was a fire because not all the birthday candles were put out. She was able to salvage only one-third of her hats. Then she lost 2 more hats most recently. How many hats did she have before the fire?

Arithmetic approach	Algebraic approach
Arithmetic approach: Right after the fire, grandmother had 8 hats since she currently has 6 hats and lost 2 hats after the fire. (6+2)=8 Since 8 hats is one-third of the total hats, she had three times as many hats before the fire. 8×3=24 Therefore, grandmother had 24 hats before the fire.	Algebraic approach: *Let t represent the number of hats grandmother had before the fire.* 6= 1/3 t-2 6+2=(1/3 t-2)+2 Addition Property of Equality. 8 = 1/3 t+(-2+2) Associative Property of Addition. 8= 1/3 t + 0 Inverse Rule of Addition. 8= 1/3 t Identify Property of Addition. (3)8=(3) 1/3 t Multiplication Property of Equality. 24 =((3) 1/3)t Associative Property of Multiplication. 24 = 1t Inverse Property of Multiplication. 24 = t Identity Property of Multiplication. Grandmother had 24 hats before the fire. Verify by substituting into the original equation: 6= 1/3 (24)-2 6 = 8 – 2 6 = 6 verified

The aim of the above example is to show that the Addition Property of Equality is used for equations that are written slightly differently but in effect, have the same structure. This is an opportunity to reinforce the concept that subtraction is the same as adding the opposite. In other words, $(1/3)t-2$ is effectively the same as $(1/3)t + -2$. Therefore structurally, the two above examples are the same. Therefore, both of the equations can use the Addition Property of Equality to simplify the equation $ax +b = c$ to $ax = d$ (with $d = c – b$). Then, the Multiplication Property of Equality can be used to simplify the equation $ax =d$ to the equation $x = e$.

Deriving and Solving Multi-step Equations

For the following multi-step word problems, reverse engineering will not be used because the previous curricular unit written by Rachelle Soroten stopped at deriving algebraic equations with simple two-step problems. Thus, with the introduction of the problem involving multi-step operations, my students will be expected to derive algebraic equations. However, they will also be expected to solve using both arithmetic and algebraic approaches. My hope is that, since we spent a significant amount of time connecting the algebraic equation to the texts in the word problems, my students will have developed the habit of reading word problems more closely. I hope they will come to the realization that word problems require patience and perseverance but are very much approachable and not impenetrable. In fact, word problems help with understanding the rules that govern procedural algebra.

Example: Consecutive Odd Numbers: All Numbers Unknown

The sum of the three consecutive odd numbers is 99. What are the numbers?.

Arithmetic approach: The average of three numbers whose sum is 99 is 33. 99/33=3. However, if the numbers are consecutive odd numbers, then their average is equal to the middle number. Since 33 is the middle number, the smaller consecutive odd number is 2 smaller (33 – 2), and the bigger consecutive odd number is 2 greater (33 + 2). Therefore, the consecutive odd numbers are 31, 33, and 35.	Algebraic approach: *Let s represent the smallest of the three odd numbers. Since the numbers are odd numbers, the other numbers are (s +2) and (s + 4).* However, variable s can represent the middle or the highest number. If *s* represents the middle number, then the other numbers would be (s – 2) and (s + 2). If *s* represents the highest number, then other numbers would be (s – 2) and (s – 4). s+(s+2)+(s+4)=99. 3s+6=99 Commutative Property of Addition and Combine like terms. 3s+6+ -6=99+ -6 Addition Property of Equality. 3s=93 Combine like terms. (1/3)3s=(1/3)93 Multiplication Property of Equality. ((1/3)(3))s= 31 Associative Property of Multiplication. 1s = 31 Inverse Property of Multiplication. s = 31 Identify Property of Multiplication. 31 is the smallest consecutive odd number. The consecutive odd numbers are 31, 33, and 35. Verify by substituting into the original equation: 31+31+2+31+4=99 31 + 33 + 35 = 99 99 = 99 verified

After students have found the three consecutive numbers using arithmetic, they will work on deriving an algebraic equation that models the situation. One of the most important aspects of writing an algebraic equation is defining the variable including the unit. For the above example, a solution of *32*, which is an even number, may not signal to the students that the solution does not make sense if they did not clearly state the variable must represent an odd number. Thus, in this case, the students most likely did not pay close attention to the fact the numbers are odd numbers and are not just consecutive numbers. Without clearly defining the variable, students would struggle in making sense of the solution in the context of the problem. If students are not taking time to make sense of the solution, they are not taking time to reflect and revise their approach during the problem solving process. When students take time to reflect on their work, they are learning that the process involves revision, attention to detail, and perseverance. These are many of the mathematical habits described in Common Core Mathematical Standards. (See Appendix 1)

Once the variable is clearly defined, students can refer to their computation to help set up the algebraic equation. If the variable is defined as the smallest odd integer, *31*, then the subsequent odd number would be *33*, which is *31 + 2*. The largest odd number would be *35*, which is *31 + 4*. Structurally the numerical equation *(31 + 31 +2 + 31 + 4 = 99)* is the same as the algebraic equation (*s + s +2 +s + 4 = 99)*. If the students defined the variable as the largest odd number, *35*, then the preceding consecutive odd numbers would be *33,* which is *35 – 2,* and *31*, which is *35 – 4*. Thus, the algebraic equation would change to s + s – 2 + s – 4 = 99. Showing different equations that model the same situation emphasizes the importance of defining a variable and the impact the definition can have on writing algebraic equations.

Simplifying the Multi-step Equations to ax + b =c

Another reason for the above example is to teach students to simplify complex equations to simple two – step equations. Multi-step equations with variables on both sides of the equation are addressed later in this unit. Students will use rules of operations, such as combining like terms and distributive property, to rewrite the equation in *ax + b = c* form. In the above example, students will rewrite s+s+2+s+4=99 to the simpler form 3s+6=99. After the equation is simplified to *ax + b =c*, students will transform the equation using the Addition Property of Equality and the Multiplication Property of Equality. In the past, some of my students did not have a clear understanding of the purpose of the Rules of Arithmetic and the Properties of Equality, nor did they understand the differences between them. By explicitly showing that the Rules of Arithmetic are used to rewrite complex equations to the simpler *ax + b =c* form, and the Properties of Equality are used to transform *ax + b =c* to the form *x = d*, essentially finding the value of the variable, the roles of the two sets of the rules should become much clearer to the student.

The Tipping Point

Due to the gradual increase in complexity, each student will reach a tipping point, which will be different for different students, where they come to rely more heavily on the algebraic procedures to solve problems. As the computations get more complicated, students will see the need for symbols to represent the situation. Their reliance only on computational strategy will gradually shift to include procedural algebraic strategies. Therefore, the challenge for more fluent algebraic procedural problem solvers would be to solve the problems using the arithmetic approach and show the operational connections between the two approaches. Interestingly, the process of switching to the arithmetic approach from the algebraic approach may be extremely difficult for the students. However, this process will deepen their understanding of number relationships. Furthermore, students may have a new found appreciation for the beauty of Algebra.

Example: Different Prices: Both Prices unknown: multi–step equation with distributive property a(x + b) = c.

Movie tickets cost: $14 for adults and $8 for children. The movie theater sold 40 tickets for a total of $416. How many adult tickets were sold?

Arithmetic approach:
If only children tickets were sold then the total cost would be \$320.
8(40) = \$320
The difference in the two totals is \$96.
\$416 - \$320 = \$96
The difference between adult tickets and children tickets is \$6.
\$14 - \$8 =\$6
16 adult tickets were sold.
96/6 = 16.

Algebraic Approach*:*
Let x represent the number of adult tickets sold.

14x+8(40-x)=416
14x+320-8x=416 Distributive Property
6x+320=416 Combine like terms
6x+320-320=416-320 Addition Property of Equality
6x=96 Combine like terms
(1/6)6x=(1/6)96 Multiplication Property of Equality
x = 16
16 adult tickets were sold.
Verify by substituting into the original equation:
14(16)+8(40-16)=416
224 + 320 – 128 = 416
544 – 128 = 416
416 = 416 verified

The above problem is a good example of the tipping point. Many students find the algebraic approach easier to understand compared with the arithmetic approach. The concepts used to solve the problem using the algebraic approach are similar to the ones used in previous problems, thus the difficulty level does not significantly increase. Because the arithmetic approach is much more difficult than the previous problems, I expect most students to solve the above problem successfully using the algebraic approach but struggle with the arithmetic approach. By the way, the arithmetic solution given above uses what was traditionally known as the *method of false position*, which was a well-recognized technique in the era before symbolic algebra (roughly, before 1600), when algebra word problems had to solved by direct reasoning, without the use of symbols.

I will continue to emphasize that all complex equations can be simplified to ax + b = c by combining like terms and using the distributive property. The Properties of Equality can then be used to find the solution.

Example: Catch up problem: Time unknown: multi-step equations with variables in both sides of the (=) ax + b =cx + d

Larry has \$250 in a bank. He takes out \$20 every week. Jessica has \$40 in her savings account and she deposits \$15 each week. When will they both have the same amount in the savings account?

Arithmetic approach:
Difference in the initial amount is \$210.
\$250 - \$40 = \$210 in favor of Larry
Weekly difference in deposits is \$35.
\$15 – (-\$20) = \$35 in favor of Jessica
It will take 6 weeks for Jessica to catch up to Larry since \$210/\$35 = 6

Algebraic approach:
Let k represent the number of weeks until the accounts are equal.
250-20k=40+15k
250-20k+20k=40+15k+20k Addition Property of Equality.
250=40+35k Combine like terms
250-40=40-40+35k Addition Property of Equality
210=35k Combine like terms
(1/35)210=(1/35)35k Multiplication Property of Equality
6 = k
It will take 6 weeks for Larry and Jessica to have the same amount.
Verify in the original equation
250-20(6)=40+15(6)
250-120=40+90
130=130 verified

The purpose of the above problem is to show students that if the two sides of the equation are simplified to ax +b = cx + d, then one possible efficient step would be to use the Properties of Equality to transform it to ex = f. This will be the first problem where students are using the Addition Property of Equality on variable expressions. In all previous examples the Addition Property of Equality was used only on numerical expressions. Students would need to be reminded that variable expressions and numerical expressions both represent values thus the same property can be used in similar ways to transform the equation.

Teaching Strategies

Partner – Group – Whole Class Discussion

Incorporating cultural differences improves engagement and learning for not only the students of the color but for all students. North emphasizes that learning styles are different for different groups (9). She further states that traditional mathematical pedagogies of the teacher lecturing while students take notes and student working independently may be a good fit when individualism is stressed. However, many of our students come from cultures where community plays a prominent role and the collective need of the community is valued more than individual needs. They are from an environment all members work together to solve problems; everyone is expected to be part of the solution. Through her research, North found these students may learn better in collaborative learning settings. Another cultural difference is in communication. European culture is "often very direct, avoids elaborations, has linear and logical flow" (10). Other cultures are less direct and much more elaborate - very much like telling a story. Numerous back and forth exchanges between the listeners and the speakers are expected and valued. Therefore, I will provide frequent opportunities for students to participate orally in the classroom. Partner work and group work followed by a whole class discussion is beneficial not only for the students from other countries, but also for students raised in the United States. Partner work and group work will be structured to hold students accountable and give equal voice to all students. Sentence frames will be used to support students to express their thinking more clearly in written and oral responses.

Multiple Readings of the Problem

To ensure everyone understands all the layers in word problem, students read multiple times with different purpose for each time they read before solving the problem. Students read the word problem the first time with the intent to understand the situation without worrying about specific numbers. The goal of the first reading is to comprehend the text. Students need to have a total clarification of what is being described in the situation before trying to understand the relationship between the quantities. This is especially important for the English learners. Students read the situation the second time with the goal of analyzing the language used to present the mathematical structure and create models of the given information. In other words, students read the problem again with the focus on the quantities and the relationship between the quantities so that they can map out and organize data. Students read the situation the final time with focus on the question(s) being asked. After students fully understand the situation, and have organized the data including the quantities and units, they can focus on the question being asked. They can use the information to solve the problem.

Word problems

Teaching the concepts of simplifying expressions using number properties, understanding the roles of variables, translating words into symbolic language, and solving algebraic equations under the umbrella of word problems gives students one storyline to follow. This method shows how all the concepts are related and provides reasons for learning these skills. Many of my students have a "global orientation for learning" (11), which means they learn better when they can understand how all the parts are related and used to solve problems.

Writing Word Problems

In the beginning of the unit, more elaborate word problems are written to tell a story so that students can see themselves in them. A student shared that "because the instruction used life situations that she saw on the daily basis at home, she paid attention to what was going on in class, and therefore she learned more and understood mathematics better" (12).

Word Problem Sets

Problem sets follow a deliberate progression with careful consideration for the complexity of the problems, computational challenge posed in the problems, and exposure to different problem types. See Classroom Activities section for sample problem sets.

One Problem at a Time

At the beginning of the lesson, I will present word problems one at a time. Students have a paper with only one problem so that they are fully engaged in the discussion. The lesson focuses on making connections, explaining their thinking, explaining other student's thinking, and being part of the learning community. Also if there is more than one problem, some students are likely to disengage with the discussion for several reasons. Some students may feel they already know the answer to the problem so why engage in the discussions. Others may get anxious about finishing other problems and their anxiety may interfere with full engagement.

Classroom Activity

Activity 1 One Step Equation: Focus on Multiplication Property of Equality

1. Your friend tells you she bought 12 bananas. On this particular day, every bunch had the same number of bananas so she bought 2 bunches. She says to you "How many bananas do you think were on each bunch?"
2. Your teacher and your mom are talking at the registration when your teacher suddenly turns to you and asks your grandmother's age. You forgot how old your grandmother is but you remember that your youngest aunt Kristie is 30 years old and half the age of your grandmother. How old is your grandmother?
3. Your little brother is having a big birthday party. Your mom invited all 24 students in his class and they all came, how unusual. Your brother wants to form groups with 4 kids to play in a game at his party. He

wants to know how many groups he would have so the right amount of the material is prepared for the game.

4. You are playing a guessing game with your friend Mary. She says that she has $25 in her bank account. This is ¼ the amount her sister Emily has in her bank account. Mary asks you how much Emily has in her bank account
5. Solve the following one-step equations and justify each step.
    a. 8y = 16
    b. x/8=16,
    c. a/5=10
    d. 10 = 5a
    e. 10 = z + 7
6. Write a word problem that can be modeled by 3x = 18. Use at least 3 complete sentences to describe the situation with detail. Solve the problem using arithmetic and algebraic approach.
7. Write a situation in which the unknown value is 2/3. In other words when you solve your algebraic equation the solution would have a value of 2/3. Use at least 3 complete sentences to describe the situation. You may use any of the four operational symbols (+, -, /, x).

Activity 2 Two-Step Equations: Focus on Addition Property of Equality & Multiplication Property of Equality

1. There are 31 desks in your math class. Because the classroom is not a prefect rectangle, some desks are arranged in rows and other desks are arranged in a group of 6 desks. You are sitting in a group of 4 desks. As you look around you realize there are 5 rows. You started to wonder how many desks are in a row.
2. Your grandmother is telling you a story of why she only has 6 hats even though she used to have a lot them. On her 40th birthday, there was a fire because not all the birthday candles were put out. She was able to salvage only one-third of her hats. Then she lost 2 more hats most recently. How many hats did she have before the fire?
3. Your friend tells you that she is grounded for next four weeks. She will not receive any allowances. Fortunately, she had $100 dollars saved up babysitting for $5 an hour. If she spends the same amount each week and has $60 left at the end of 4 weeks, how much did she spend each week?
4. A martial arts school is offering a special where students can enroll for two-fifths of the regular price, after a $3 application fee. Find the regular price of the enrollment if you paid $32.50.
5. Write a situation in which the unknown value is 1 2/3. In other words when you solve your two-step algebraic equation the solution would have a value of 1 2/3. Use at least 3 complete sentences to describe the situation with detail. You may use any of the four operational symbols (+, -, /, x). Solve the problem.
6. Solve following two-step equations using Properties of Equality. Verify your solution.
    a. 7 = 8x + 31
    b. x/8 -4=16
    c. 103 = −9x – 13
    d. 8 + b/-4 = 5
    e. (4/5)x+2= 2x

Classroom Activity 3 Multi – Step Equations Problem Sets

Combine like terms

1. Jacqueline bought four cases of vitamin water drinks for an upcoming meet. She bought three more cases and, in addition spent $6.95 on snacks. If she spent $134.35, how much did she pay for each case of vitamin drinks?
2. Mary's sister is three years younger than twice Mary's age. The sum of their age is 33. How old is Mary?
3. Solve
   a. $2a + 3 + 3a = 18$
   b. $24 = 5x – 2x + 3$
   c. $9x – 5 – 6x – 1.3 = 2.1$
   d. $5v – 0.9v + 2.1 = 4.5$

Distributive Property

1. Jack and Jim collect Pokemon cards. Jim has 12 more than Jack has, and together they want to triple their collection for a total of 66 Pokemon cards. How many Pokémon cards does Jack currently own?
2. Maria is selling tickets to her school's talent show. Adult tickets cost $5 and children's tickets cost $3. Maris sells a total of 50 for $214. How many adult tickets and children's tickets did she sell?
3. Jeremy bought post cards on a trip. He sent 15 cards to his friends. He sent two-third of the remaining cards to his family. Then he sent one of the remaining cards to his teacher. He was left with 3 cards. How many post cards had Jeremy bought?
4. Solve
   a. $32 = 8(x + 2)$
   b. $5(1 – 2w) + 8w = 15$
   c. $3 = 3(x – 2) – 5(2x + 1)$
   d. $(-3+y)/4=103$

Variables on both sides

1. Four times Nia's age, decreased by three is equal to three times Nia's age, increased by seven. How old is Nia?
2. Find three consecutive integers such that twice the greatest integer is two less than three times the least integer.
3. Mike shared a package of lined paper with three of his classmates. He gave 1/4 of the pack to Aaron, 1/3 of what was left was given to Emily. Then Kyle took 1/6 of what was left in the package. Mike kept the remaining 30 sheets. How many sheets of paper were in the original package?
4. Solve
   a. $7a – 17 = 4a + 1$
   b. $3m – 10 = 2(4m – 5)$
   c. $5 + 2(n – 4) = 1 – 3(n + 2)$
   d. $1/3(x+1)=(2/9)x+ 7/9$

Appendix 1 Implementing District Standards

California Common Core State Standards for Math (CA CCSS) (13)

8. EE7b Solve linear equations with rational number coefficients, including equations whose solution requires expanding expressions using the distributive property and collecting like term.

Mathematical Practices (14)

1. Make sense of problems and persevere in solving them.

Students solve real world word problems.

2. Reason abstractly and quantitatively

Students verify the solution of the algebraic equation, and check that it makes sense in the context of the word problem.

3. Construct viable arguments and critique the reasoning of others.

Students analyze and compare the context in the word problem with the features in the equation and are able to explain the connection.

4. Model with mathematics.

Students construct algebraic equations to model word problems.

5. Use appropriate tools strategically

Students use different approaches to solve the word problems including guess and check, computation, illustrations, direct reasoning, defining variables, and writing equations, and more

6. Attend to precision.

Students define variables carefully and completely, including specifying the units.

7. Look for and make use of structure.

Students look for structure in the equations, to use proper rules to simplify and solve the equations.

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