Curriculum Units by Fellows of the National Initiative
2017 Volume V: From Arithmetic to Algebra: Variables, Word Problems, Fractions and the Rules

Sequencing Math DNA: Differences, Nth Terms, and Algebraic Sequences

Curriculum Unit 17.05.11, published September 2017
by Lawrence E. Yee

Introduction

In mathematics, a *sequence* is defined as an ordered list of numbers. Each number in a sequence is called a *term*. The length of a sequence can vary. *Finite sequences*, are sequences that stop after a certain number of terms. *Infinite sequences* do not stop and are assumed to continue indefinitely. For example, the sequence 2,4,6,8,10 is a finite sequence with five terms, while 2,4,6,8,10,... is an infinite sequence. Many potentially infinite sequences are formed according to simple rules, and such sequences are also often referred to as "patterns".

At the high school level, students are introduced to three main special types of sequences: *arithmetic* (linear), *geometric* (exponential), and *quadratic*. This curriculum unit discusses each of these three types in the sections below. When given a few terms in a sequence, students are expected to classify the sequence type, as being potentially one of the three special ones, or "other". They are also asked to identify key characteristics of the sequence, and to use these to write both *recursiveequations* and *explicit equations* for a_n, the *n*th term in the sequence. Note that some texts may use the term *rule* in place of *equation*. Many of my students are able to observe patterns represented in the sequences, but struggle in taking those observations and translate translating them into algebraic representations.

In textbooks, formulas are frequently just given to students based on specific types of sequences. This unit will look at how the *common difference*, *second difference*, or *common ratio* between consecutive terms of a sequence can be used to determine a sequence's recursive and explicit equations. In addition to tabular models, geometric representations of sequences will be used as an alternative approach to represent sequences, leading to algebraic representations. Overall, this unit serves as a supplement to enhance my existing textbook resources, to better support students in any course that works with linear, quadratic, and exponential sequences and functions.

Background and Demographics

William C. Overfelt High School is located in San Jose, California. When people hear of San Jose, they tend to think of Silicon Valley, high tech, and astronomical housing prices. Those of us who live and work in and around the neighborhood of Overfelt, commonly refer to Overfelt as being located in the East Side of San Jose. In a city of over one million people, the East Side is home to predominately Latinos, with African Americans, Vietnamese, Filipinos, Cambodians, and Samoans making up smaller, but still significant ethnic populations. At Overfelt, approximately 80% of our student body consists of Latino/Hispanic students. Although the average cost to rent a house in Overfelt's area is close to $3000 a month, the average annual family income is just over $60,000 a year. Currently, over 80% of students at Overfelt qualify for federal free or reduced lunch, and under 50% of our students' parents are high school graduates. The East Side is its own unique, vibrant, and strong community that is a different world from the rest of San Jose.

During the last three years, the East Side Union High School District (ESUSHD) has transitioned to the Common Core State Standards for Mathematics from the earlier California mathematics standards. This process has included a gradual replacement of the three course math sequence comprising Algebra 1, Geometry, and Algebra 2, by a series of three Integrated Math courses titled Common Core State Standards (CCSS) Mathematics I, II, and III. There was significant debate regarding which curriculum would be adopted for the integrated math course sequence. During the transition period, there were four different pilot curricula spread unevenly across the eleven high schools in our district. Math teachers were able to choose their preferred curriculum to pilot. Though each of the pilots addressed similar content, what they emphasized varied drastically, and the order in which content was taught also varied significantly.

Because of the different curricula used throughout the district, and even within a single school, students were most affected when their schedules changed during the school year, or if they transferred between schools. Students who moved into our school and district struggled, as the ESUHSD no longer offered Algebra 1, Geometry, and Algebra 2 courses. For example, arithmetic and geometric sequences were previously in the second semester of Algebra 2, but in our current curriculum, they appear during the first semester of Math 1. The misalignment between old and new course sequences inadvertently created significant gaps in content knowledge for some students.

An additional point of difficulty for making appropriate course assignments for entering students is the fact that the ESUHSD serves seven feeder elementary school districts. At Overfelt, we receive students from multiple middle schools. Most come from Alum Rock Union Elementary School District, but we also receive a small, yet significant, number of students from the Evergreen Elementary School District. Though it should not matter which mathematics textbook curriculum a school uses, the lack of consistency in background contributed to the prevalence in gaps of math knowledge. Partly because of the gaps in knowledge, the Integrated Math 1 course I taught experienced a high amount of student turnover. In my experience, it is not uncommon to have over fifty different students enrolled in one section of Math 1 over the course of the school year.

Rationale

For the last eight years I have taught every level of high school math, from Algebra 1 through AP Calculus BC and AP Statistics. As a result, I gained a unique perspective on the progression of mathematical content and topics in secondary schools. The students I receive in AP Calculus help make me aware of the gaps in mathematical knowledge that accumulate over years. I observed that students who struggle at the pre-calculus and calculus level were successful in the foundational level courses because they memorized formulas and could repeat procedures that closely resembled example problems. However, these students are often unable to explain their reasoning or demonstrate their understanding of the underlying concepts. By my observations, math is not the favorite subject for many of my students. As a consequence, many students take the requisite courses to graduate and simply want to do enough to pass the class. Thus, for various reasons, many students learn superficially through rote memorization and application of formulas and shortcuts, rather than engaging with concepts to gain understanding and build connections between topics within mathematics.

As our school and district transitioned to the Common Core State Standards (CCSS), mathematics courses shifted their focus away from a skills based approach towards the Standards for Mathematical Practice that emphasize both computational fluency and conceptual proficiency.[1] With all the changes in the structures of mathematics courses at Overfelt and our district, there was a frequent mismatch between the course objectives and the content provided through the curricula. A major objective, common across all curricula in the Integrated Math 1 course during the first semester, includes the analysis of arithmetic/linear sequences, and geometric/exponential sequences of numbers, and modeling these sequences by writing explicit equations. Afterwards, students would be given “real life” scenarios that follow arithmetic and geometric patterns. Depending on a curriculum’s approach, this topic could span multiple units of study and be extended for weeks to months.

Last year, my district adopted the Big Ideas Integrated Math curriculum last year. In stark contrast to our previous curriculum, which languished through multiple modules about sequences, the current curriculum rushes through sequences by including only two sections about them, each only two to three pages in length. The abrupt presentation of the topics expects students to quickly glance at a sequence, such as 5,11,17,23,… and within the recommended 1 hour for the entire lesson, identify and verify the common difference, write an explicit function that models the situation, and apply their findings to compute later terms of the sequence. Likewise, for geometric/exponential patterns, students are expected to complete the same tasks as for a linear pattern, except they would be asked to identify the common ratio rather than the common difference.

Many students are successful with each individual task regarding a sequence in the specific units of study, but find it difficult to respond to the following question: *Is the sequence arithmetic, geometric, or neither? Explain.* Most of my students are able to recognize if a pattern is linear, geometric, or neither but have difficulty explaining how they know. If students do recognize sequences as arithmetic or geometric, they have significant difficulty when asked to write an explicit equation. When the two types of sequence are taught as discrete content units in different chapters, the ability to analyze a sequence and make connections between the sequence and its defining characteristics is inhibited. Some students find success at writing explicit equations for arithmetic and geometric sequences, as the general formulas are readily available in our textbook and accompanying resources. However, I notice that many of those same students who can write the equations from a sequence cannot take an equation and explain the relationship between the constants and the features of an arithmetic and geometric sequence.

Quadratic patterns are no longer prescribed in the Integrated Math 1 standards and occur in Math 2. Ironically, however, one of our own district standards for Math 1 expects students to be able to construct and compare linear, quadratic, and exponential sequences and to use them to solve problems. In the pilot year of our current curriculum, my co-teacher and I completed the entire course with enough time to include additional lessons during the year. Thus, there is time in Math 1 to introduce, explore and analyze quadratic functions. Introducing the concept in Math 1 will also provide valuable preparation for the Integrated Math 2 and Math 3 courses, and all of these topics are major components of the first unit in the Math Analysis course.

In my Integrated Math 1 courses, I have many math-phobic students who, due to a lack of success in prior courses, feel as though they are scarred by mathematics. I see many students struggle when they confront the intellectual jump from arithmetic computations to algebra, especially, using variables. For many of my students, it is a major obstacle that elicits allergic reactions to mathematics.

Students who struggle in the higher level math courses often have the opposite problem. They can perform computations and write equations by following steps or replicating the procedures shown in example problems. Where these students struggle is when problems do not look like the examples, or if they do not remember the formulas from the textbooks. One reason for this struggle is that the Integrated Math courses give students quite limited exposure to sequences and functions that that define arithmetic, geometric, or quadratic sequences. I had students taking pre-calculus and calculus level courses who struggled to write and graph linear equations.

One goal of this unit is to provide students with a tool to analyze sequences of numbers. This will be accomplished using difference tables, which provide a structure to organize and assist students as they look for patterns to deduce whether sequences are arithmetic, geometric, quadratic, or none of these. Through examples, this unit shows how to use difference tables to analyze arithmetic sequences with the ultimate goal of writing an explicit equation. For geometric sequences, this unit replaces the difference table with an analogous ratio table, again with the goal of writing an explicit equation. Further extension of the difference table will be applied to quadratic cases. With the quadratic case, a computation based procedure will be shown to enable quadratic sequences to be written as explicit equations.

Mathematical Sequences

Each number in a sequence is a *term* and can be identified by its position in a sequence. Given a sequence, students are expected to extend patterns to find subsequent terms, create an algebraic representation of the problem by writing an explicit formula in terms of *n* for the *n*-th term in the sequence, and use their explicit formulas to further analyze the sequence. This unit will discuss arithmetic and geometric sequences using the *sequence notation*, a_n, while for quadratic sequences, we will be using *function notation* f(n). Either representation is acceptable, and I chose to use different notations because the sequence notation appears in the common course exams used at our school for Math 1, while function terminology is introduced after arithmetic and geometric sequences. Note that the terms *arithmetic* and *linear* are often used synonymously, as are *geometric* and *exponential*.

A *difference sequence* is a sequence found by taking the difference between two successive terms of an original sequence. In general, any sequence is determined by its first term and its difference sequence. This follows because the *nth term, a_n*, of the original sequence is the sum of the i) first term of the original sequence and ii) the first n-1 terms of the difference sequence. At the high school level, this fact is often provided without proof or further formal elaboration, as the proof requires induction. Al Cuoco provides a thorough explanation and justification through examples and proof in the text *Mathematical Connections.*[2] Finding the difference sequence is useful because it may be simpler than the original sequence. We can also take the difference sequence of an existing difference sequence, which is called the *second difference.* Using the same argument as seen with the first difference, it would then follow that the original sequence is determined by i) its first term; ii) the first term of its first difference sequence and iii) its second difference sequence. And so forth, for further difference sequences. As demonstrated in the sections below, I plan to discuss with my students these facts and demonstrate how to find difference sequences and writing linear sequences in terms of the first term and their difference sequence through multiple examples.

This unit uses the fact that any sequence is determined by its first term and its difference sequence to highlight the connection that the general result about reconstructing a sequence from its first term and its difference sequence can be followed by study of sequences for which the difference sequence is *constant*, i.e., all terms have the same value. In particular, for linear sequences, the first difference sequence is constant, and vice versa. With my Math 1 students, I would discuss the derivation of the formula for the nth term of a linear sequence with a constant difference, by showing multiple examples of these types of sequences. Below, I will show how I would build an equation for an arithmetic sequence.

For quadratic sequences, the second difference sequence is constant. I would show my students how to verify sequences as quadratic by finding the second differences of a sequence. For my pre-calculus students, I would show to my students through subsequent applications of the same argument, the general theorem for differences which states that a constant nth difference implies a nth degree polynomial will model the sequence. The general theorem is also stated as a fact for high school students to use by Cuoco, with no formal proof provided for students, but a proof is provided for teachers as enrichment in *Mathematical Connections*. For exponential sequences, the difference sequence is also exponential, with the same ratio between terms. Difference sequences are not as useful for studying exponential sequences since the result is a sequence with the same level of difficulty/complication. I would have my students work with sequences without constant differences so they understand that difference sequences are only a tool to help us analyze and classify the type of sequence.

Arithmetic (Linear) Sequences

In the current textbook adopted by our district, students are given the definition of a linear sequence $a_n=a_1+d(n-1)$ to find the nth term of a sequence if they know a_1, the *first term*, and *d*, the *common difference*. The curriculum resources available offer exercises for students to practice writing an algebraic representation by telling students to identify the first term and find the common difference by taking two consecutive terms and subtracting later term by its preceding term using the equation $d = a_{n+1} - a_n$. Once identified, students are expected to substitute the two values into the equation for a_n, and simplify to find the equation that models the sequence.

My students who possess strong number sense skills are often able to recognize arithmetic patterns and determine the common difference using mental arithmetic. Identifying the first term is usually a simple task since it is often already labeled in problems. Despite being able to write an equation for a_n, students are following a procedure and using an equation that is presented to them for specific cases. Students who struggle tend to only find the difference between the first and second terms and fail to compare the differences between subsequent pairs of consecutive terms. This is a consequence of some textbooks grouping content into discrete sections, each devoted to a single topic. Rather than thinking, a student may just assume that a given problem is answered by the formulas of the section. The structure of the table will help students develop the habit of computing many or all of the terms in the difference sequence, as the difference between subsequent terms is recorded in its own column. Also, including sequences that have no special forms to be analyzed throughout the unit will reinforce the necessity to compute many terms of a difference sequence. Discussing a general sequence and its relation to its difference sequence may give students the tool they need to think more flexibly when analyzing and classifying sequences.

Building an Equation from an Arithmetic Sequence

The following exercise demonstrates how to set up a difference table and use the information provided to write an explicit equation for a_n. In my classes, I would first model this procedure with my students using direct instruction.

Exercise: Consider the first terms in the infinite sequence: 5,7,9,11,13,… . Determine if the sequence is arithmetic, geometric or neither.

To set up the linear difference table, we begin by writing the terms in the sequence as the output values. We label the corresponding term number to coincide with each term value.

Input (n)	Output (a_n)	$\Delta_n = a_{n+1} - a_n$
1	5	7-5=2
2	7	9-7=2
3	9	11-9=2
4	11	13-11=2
5	13	

At this point, I would ask my students if they notice anything about our first differences. Since they are all the same number 2, we can say that the first difference is constant, where the constant $\Delta=2$. I would emphasize the fact that since the first common difference is constant, we know that this is an arithmetic sequence. Using this fact, I will express each term of the sequence as a sum of the first term and repeated addition of 2.

$a_1 = 5$

$a_2 = 5+2$

$a_3 = 5+2+2$

$a_4 = 5+2+2+2$

$a_5 = 5+2+2+2+2$

...

By expressing each term as a sum of the initial term and repeated addition of the first common difference, we can make the following observation regarding the number of times we add the common difference:

$$a_1 = 5 + \underbrace{\square}_{0} = 5 + 2(0)$$
$$a_2 = 5 + \underbrace{2}_{1} = 5 + 2(1)$$
$$a_3 = 5 + \underbrace{2+2}_{2} = 5 + 2(2)$$
$$a_4 = 5 + \underbrace{2+2+2}_{3} = 5 + 2(3)$$
$$a_5 = 5 + \underbrace{2+2+2+2}_{4} = 5 + 2(4)$$

...

I would ask my students to consider the relationship between the number of times we add 2 to the first term and the term number. Here, I would highlight the fact that the number of times we add 2 is exactly one less than the term number. Next, I would ask my students to conjecture what would happen if we extend the pattern and try to make the generalization for the nth term in the sequence, which will give us our explicit equation. This is an area where I expect some students to struggle. To assist students, I would how many times we are adding the first constant difference to get a_n. This unknown quantity is indicated by "?" in the equation.

$$a_n = 5 + \underbrace{2+2+\ldots+2+2}_{?} = 5 + 2(?)$$

I may write the additional computations representing the term number directly to the number of times we multiply 2, as shown below. Also, this is a good opportunity for students to encounter the idea of *variable*, a symbol that stands for a number in some set of numbers. Here the set is the positive integers.

$$n=1;\quad a_1=5+\underbrace{\square}_{0} \quad =5+2(0)=5+2(1\text{-}1)$$

$$n=2;\quad a_2=5+\underbrace{2}_{1} \quad =5+2(1)=5+2(2\text{-}1)$$

$$n=3;\quad a_3=5+\underbrace{2+2}_{2} \quad =5+2(2)=5+2(3\text{-}1)$$

$$n=4;\quad a_4=5+\underbrace{2+2+2}_{3} \quad =5+2(3)=5+2(4\text{-}1)$$

$$n=5;\quad a_5=5+\underbrace{2+2+2+2}_{4} \quad =5+2(4)=5+2(5\text{-}1)$$

$$\ldots$$

Applying the observation for the *n*th term gives us the expansion:

$$\text{nth term; } a_n=5+\underbrace{2+2+\ldots+2+2}_{n-1}=5+2\big(n\text{-}1\big)=5+2(n\text{-}1).$$

Thus, the sequence 5,7,9,11,13,… can be represented algebraically in terms of n, the term number, as the equation a_n = 5+2(n-1). We can also simplify this equation to a_n = 2n+3. I expect many of my students to be able to follow along with the procedure, and I would probably model at least two or three examples for the whole class. For students that understand quickly, I would place students into groups of three or four to work on analyzing different sequences. Either my co-teacher or I would also be able to work with small groups of students who needed more guided practice or more examples to show how to write the explicit equations.

The General Linear Sequence

The following exercise demonstrates one way to construct a general explicit equation of a generic arithmetic sequence. The exercise can be viewed as a proof through an example, allowing students the opportunity to develop a conceptual understanding of the components of the general equation of a linear sequence. I would show this example to my students because it is one way of demonstrating where the equation in their textbook comes from. In the past, students who were given a formula could find an equation, but often could not explain what information an equation tells us about a sequence.

Exercise: Consider a sequence with first term, a_1. Subsequent terms are computed by adding the real number, Δ, to the previous term in the sequence. Write an equation in terms of *n*, the term number for a_n, the nth term in the sequence.

To examine this sequence, we can start by writing out the first few terms.

a_1=first term

$a_2=a_1+\Delta$

$a_3=a_1+\Delta+\Delta$

$a_4=a_1+\Delta+\Delta+\Delta$

$a_5=a_1+\Delta+\Delta+\Delta+\Delta$

…

$a_n=a_1+\Delta+\Delta+\cdots+\Delta+\Delta$

By writing out each term in terms of a_1 and Δ, we can use the structure of this list to help us make a generalization for the nth term in the sequence. Starting with the first five terms of the sequence, we count the number of times we add the first common difference to the initial term. Using an underbrace, we record that number for each term. Since we are performing repeated additions of the first common difference, we can express the repeated addition of Δ as a product of Δ and the number of times it is added in each term.

$$\begin{aligned}
a_1 &= a_1 + \underbrace{\square}_{0} &&= a_1 + \Delta(0)\\
a_2 &= a_1 + \underbrace{\Delta}_{1} &&= a_1 + \Delta(1)\\
a_3 &= a_1 + \underbrace{\Delta+\Delta}_{2} &&= a_1 + \Delta(2)\\
a_4 &= a_1 + \underbrace{\Delta+\Delta+\Delta}_{3} &&= a_1 + \Delta(3)\\
a_5 &= a_1 + \underbrace{\Delta+\Delta+\Delta+\Delta}_{4} &&= a_1 + \Delta(4)
\end{aligned}$$

Observe that, just as in the special example discussed above, the number of Δs we added in each term is exactly one less than the term number. To highlight this connection, we add one more column of computations where we rewrite the quantities of Δ. Through extension of the observable patterns, we can make a generalization for the general term, a_n.

$$
\begin{aligned}
a_1 &= b + \underbrace{\square}_{0} &&= b + \Delta(0) = a_1 + \Delta(1-1) \\
a_2 &= a_1 + \underbrace{\Delta}_{1} &&= a_1 + \Delta(1) = a_1 + \Delta(2-1) \\
a_3 &= a_1 + \underbrace{\Delta + \Delta}_{2} &&= a_1 + \Delta(2) = a_1 + \Delta(3-1) \\
a_4 &= a_1 + \underbrace{\Delta + \Delta + \Delta}_{3} &&= a_1 + \Delta(3) = a_1 + \Delta(4-1) \\
a_5 &= a_1 + \underbrace{\Delta + \Delta + \Delta + \Delta}_{4} &&= a_1 + \Delta(4) = a_1 + \Delta(5-1) \\
&\dots \\
a_n &= a_1 + \underbrace{\Delta + \Delta + \dots + \Delta + \Delta}_{n-1} &&= a_1 + \Delta(n-1) = a_1 + \Delta(n-1)
\end{aligned}
$$

This gives us the equation $a_n=a_1+\Delta(n-1)$. As mentioned earlier, this is the definition of a linear sequence, as given in the Larson textbook. I would make the connection between the general case and the various special examples that my students would have practiced. The 'only' difference is, the difference term here is a variable, Δ, rather than a specific number.

Recursive Equations for Arithmetic Sequences

Using the previous structure we can also deduce a recursive form of the equation. Students in Math 1 should also be able to represent arithmetic and geometric sequences recursively. Students in the higher level math classes also work with recursive sequences. In general, we have the equation $a_n = a_{n-1} + \Delta_{n-1,}$ by the definition of a difference sequence, and constant differences means that $\Delta_{n-1} = \Delta$ for any n. I would discuss this fact with my students to show that recursive equations can be written for any sequence if we know a term and its difference sequence and that the recursive sequence is a special case of the general equation.

$$a_1 = a_1 \textit{ the first term}$$

$$a_2 = a_1 + \Delta \qquad = a_1 + \Delta$$

$$a_3 = \underbrace{a_1 + \Delta}_{a_2} + \Delta \qquad = a_2 + \Delta$$

$$a_4 = \underbrace{a_1 + \Delta + \Delta}_{a_3} + \Delta \qquad = a_3 + \Delta$$

$$a_5 = \underbrace{a_1 + \Delta + \Delta + \Delta}_{a_4} + \Delta \qquad = a_4 + \Delta$$

$$\ldots$$

$$a_n = \underbrace{a_1 + \Delta + \Delta + \Delta}_{a_{n-1}} + \Delta \qquad = a_{n-1} + \Delta$$

The result is the recursive equation of an arithmetic sequence with first term a_1 and nth term a_n=a_{n-1} Δ, where Δ is the common difference.

Geometric (Exponential) Sequences

A geometric sequence is characterized by the common (constant) ratio, r. Each term is found by multiplying the previous term by the common ratio. The structure of the explicit equation of a geometric sequence can be analyzed using a table in a way similar to the first difference table for arithmetic sequences. To start analyzing a sequence, the ratio between two consecutive terms is calculated using the definition $r_n = a_{n+1}/a_n$, and then checking to see if the ratios are constant. Only if the ratios are constant, do we have a geometric sequence. In my classes, I plan to have students try analyzing examples that are geometric sequences while we study linear sequences to see what observations and conjectures they make. I would then show them how to find the ratio between consecutive terms and build an equation as outlined in the following exercise.

Exercise: Consider the sequence 3,6,12,24,48,…. Determine if the sequence is arithmetic, geometric or neither.

We begin by analyzing the sequence. We can observe that the sequence is not arithmetic because the difference between two consecutive terms is not constant. For example, Δ_1=6-3=3, but Δ_2=12-6=6. Also Δ_3=24-12=12. The differences seem to be getting bigger and bigger. At any rate, they are certainly not constant.

To check if the sequence might be geometric, we compute the ratio between two consecutive terms to find r_1, r_2, r_3, and r_4. The key fact that I would point out to my students is that the ratio is the same value, and that this feature implies that the sequence is indeed geometric.

Since the ratio is constant, the sequence is geometric. To build an explicit equation, we can set up a ratio table for the geometric sequence that is analogous to the difference table used to analyze the arithmetic sequence.

n	a_n	r_n
1	3	r1=6/3=2
2	6	r2=12/6=2
3	12	r3=24/12=2
4	24	r4=48/24=2
5	48	…
…	…	

Using a procedure similar to arithmetic sequences, each term can be written as the product of the first term and some number of copies of the constant ratio, That is, a_n is a product of a_1 and a power of the common ratio *r*.

a_1=3

a_2=3•2

a_3=3•2•2

a_4=3•2•2•2

a_5=3•2•2•2•2

…

At this step, I would ask my students to count the number of times we multiply the first term, 3, by the common ratio, 2. I would then show that we can rewrite each term as a product of a_1 with some power of the common ratio, by using the product property of exponents with the common ratio as the base. This would also be an opportunity to discussing with your students that the product property of exponents is a consequence of (a generalized version of) the Associative Rule for multiplication. I would then ask my students to, conjecture about the number of times we multiply by the common ratio to obtain the a_n term. Once we know this quantity, we can write the explicit equation.

$$a_1 = 3\ \underbrace{\square}_{0} = 3(2)^0$$

$$a_2 = 3 \cdot \underbrace{2}_{1} = 3(2)^1$$

$$a_3 = 3 \cdot \underbrace{2 \cdot 2}_{2} = 3(2)^2$$

$$a_4 = 3 \cdot \underbrace{2 \cdot 2 \cdot 2}_{3} = 3(2)^3$$

$$a_5 = 3 \cdot \underbrace{2 \cdot 2 \cdot 2 \cdot 2}_{4} = 3(2)^4$$

$$\ldots$$

$$a_n = 3 \cdot \underbrace{2 \cdot 2 \cdots 2 \cdot 2}_{?} = 3(2)^{?}$$

For any term in the sequence, notice the exponent of the common ratio is one less than the term number. This connection between the term number and the exponent of the common ratio can be made more obvious by rewriting the exponents asthe term number, minus 1. Note that this process parallels the structure for arithmetic sequences.

$$a_1 = 3\ \underbrace{\square}_{0} = 3(2)^0 = 3(2)^{(1-1)}$$

$$a_2 = 3 \cdot \underbrace{2}_{1} = 3(2)^1 = 3(2)^{(2-1)}$$

$$a_3 = 3 \cdot \underbrace{2 \cdot 2}_{2} = 3(2)^2 = 3(2)^{(3-1)}$$

$$a_4 = 3 \cdot \underbrace{2 \cdot 2 \cdot 2}_{3} = 3(2)^3 = 3(2)^{(4-1)}$$

$$a_5 = 3 \cdot \underbrace{2 \cdot 2 \cdot 2 \cdot 2}_{4} = 3(2)^4 = 3(2)^{(5-1)}$$

$$\ldots$$

$$a_n = 3 \cdot \underbrace{2 \cdot 2 \cdots 2 \cdot 2}_{n-1} = 3(2)^{n-1} = 3(2)^{(n-1)}$$

Extending the observations for the first five terms in the sequence, we can generalize the results to the *n*th term gives us the equation $a_n=3(2)^{n-1}$ representing our geometric sequence. After this example, I may have some students work with another geometric sequence with the goal of writing an equation. I anticipate some students will be ready to work independently, while smaller groups of students would work with either myself or my co-teacher as we go through additional examples.

The General Geometric Sequence

The structure of the process showing how to find an algebraic representation for geometric sequences follows a structure similar to the method for finding an arithmetic equation. The variable choice in the exercise is purposely meant to lead to the equation, $a_n=a_1(r)^{(n-1)}$, which is found in the Big Ideas Integrated curriculum, as well as other math textbooks. I would demonstrate this example to my students to show where the formula in their textbook comes from.

Exercise: Suppose we know a sequence is geometric with first term a_1 and a constant ratio r. Can we find a formula for a_n?

In my classes, I would use the example in the section above, and point out that we can replace the 3 with a_1 and the 2 with r. I would make sure to point out that we are able to write the equation because there is a constant ratio. This would give us the following:

$$
\begin{aligned}
a_1 &= a_1 \underbrace{\square}_{0} &&= a_1(r)^0 = a_1(r)^{(1-1)} \\
a_2 &= a_1 \cdot \underbrace{r}_{1} &&= a_1(r)^1 = a_1(r)^{(2-1)} \\
a_3 &= a_1 \cdot \underbrace{r \cdot r}_{2} &&= a_1(r)^2 = a_1(r)^{(3-1)} \\
a_4 &= a_1 \cdot \underbrace{r \cdot r \cdot r}_{3} &&= a_1(r)^3 = a_1(r)^{(4-1)} \\
a_5 &= a_1 \cdot \underbrace{r \cdot r \cdot r \cdot r}_{4} &&= a_1(r)^4 = a_1(r)^{(5-1)} \\
&\ldots \\
a_n &= a_1 \cdot \underbrace{r \cdot r \cdots r \cdot r}_{n} &&= a_1(r)^{n-1} = a_1(r)^{n-1}
\end{aligned}
$$

From this procedure, we have $a_n=a_1(r)^{n-1}$.

This is the same formula, which is the definition of a geometric sequence, found in the Larson textbook used by my students.

Recursive Equations for Geometric Sequences

We can use the structure to write the recursive form of a geometric sequence. This recursive formula follows directly from the definition of successive ratio and the condition that it be a fixed number r, if the sequence is geometric, i.e., the definition of geometric sequence. The main job here is to get students to understand that this is an immediate consequence of the definitions.

$$a_1 = a_1$$

$$a_2 = \underbrace{a_1}_{a_1} \cdot r \qquad = a_1 \cdot r$$

$$a_3 = \underbrace{a_1 \cdot r}_{a_2} \cdot r \qquad = a_2 \cdot r$$

$$a_4 = \underbrace{a_1 \cdot r \cdot r}_{a_3} \cdot r \qquad = a_3 \cdot r$$

$$a_5 = \underbrace{a_1 \cdot r \cdot r \cdot r}_{a_4} \cdot r \qquad = a_4 \cdot r$$

$$\ldots$$

$$a_n = \underbrace{a_1 \cdot r \cdot r \cdots r \cdot r}_{a_{n-1}} \cdot r = a_{n-1} \cdot r$$

From this procedure, we have a recursive form with first term a_1 and nth term $a_n = a_{n-1} \bullet r$.

Neither Arithmetic nor Geometric Sequences

When analyzing sequences in Math 1, students are often asked to determine if a sequence is arithmetic, geometric, or neither. The use of the word neither is meant to say that a sequence is not arithmetic, nor geometric. However, I often have many of my students interpret the word “neither” to mean none, such that they believe patterns that are not arithmetic, nor geometric may have no pattern or algebraic representation. In response to this issue, I would discuss and show my students examples of a variety of sequences that behave in different ways. For example, the Fibonacci sequence is not arithmetic or geometric, but is defined by a simple recursive equation. I also know that some students work with this sequence in our introductory computer science class.

It is not until the Math 2 course that students are shown that there are also sequences that can be modeled using quadratic expressions. However, in Math 2, the discussion is brief and relies on telling students that a sequence is quadratic if the second difference is constant. Textbooks show how to find the second difference by drawing a diagram,[3] and then ask students to write an equation to represent the terms in the quadratic sequence. Yet, the examples given rely on other properties of quadratics, such as knowing both intercepts, to find equations. Moreover, the discussion and examples in the text do not provide students with any methods to find equations for the general quadratic sequence.

Quadratic Sequences

Quadratic sequences are characterized by having a constant second difference. This is often stated as fact in textbooks. In the curriculum my school uses, there are no explanations as to why a second constant difference implies we have a quadratic equation. By searching through multiple high school curriculum resources, I have found that the second common difference is usually presented through examples, and students are told that if they find a constant second difference, they know they have a quadratic sequence. Though students are expected to write the equation of a quadratic sequence, many exercises rely on knowing key features of quadratics, such as the vertex or both intercepts.

Al Cuoco outlines one method to find the equation of a quadratic function in *Mathematical Connections.*[4]The procedures involve taking a quadratic sequence and creating a table that includes an analysis of the first difference, Δ, and the second difference, ΔΔ which is found by taking the differences of the first differences. Using this structure as a starting point, and using the skill of creating a first difference table for arithmetic sequences, an equation can be constructed for any quadratic sequence.

Quadratics are studied in Integrated Math 2 and subsequent math courses. In Math 1, students will have practiced using both the sequence notation, a_n, and the function notation, f(n), to represent sequences. For the purpose of the following example and its place in the mathematics curriculum currently used in my school, function notation will be used. The following exercises will show that a unique quadratic equation can be written for a given sequence with constant second differences. In particular, the exercises will demonstrate:

1. A Quadratic function implies constant second difference.
2. How to reconstruct a quadratic function from its second difference.
3. A constant second difference implies we have a quadratic. This is meant to be an algebraic exploration of quadratic functions that will assist students with understanding how to analyze quadratic sequences and make generalizations.

The General Quadratic Sequence

There are three ways of connecting a quadratic sequence with its first and second difference sequences:

i) You can compute the coefficients d and e of the first difference sequence dn + e, and the value d of the second difference sequence, in terms of the coefficients $ax^2 + bx + c$ of the orginal sequence, and solve for *a, b, c* in terms of *d* and *e,* and the first value a+b+c of the sequence.

ii) You can sum the first n (or n-1) terms of the first difference sequence (and add the first value of the sequence to that sum). This can be done in two ways: iia) geometrically or iib) by formal algebraic computation.

In this section, approach (i) will be used to express a quadratic function in terms of its first and second differences. The next section will show by example how both methods (i) and (iia) can produce the same quadratic equation. Cuoco utilizes formal algebraic computation in *Mathematical Connections*, and I will not use method (iib) as it requires use of combinatorics, which is beyond the scope of the courses for which this unit is intended.

General Quadratic Equation

The following exercise outlines one process that can be used to analyze a sequence that is quadratic in order to write an equation in the form ax^2 +bx+c. Though this example uses the general quadratic equation as a starting point, I would consider starting by examining quadratic number sequences, such as 1, 4, 9, 16, 25, … before showing this process to my students.

We begin by considering the standard form of a quadratic equation, f(x)=ax^2 +bx+c, where a,b, and c are real numbers with a≠0.

First, we evaluate the function for the integers from 1 to *n*.

$f(x)=ax^2+bx+c$

$f(1)=a(1)^2+b(1)+c=a+b+c$

$f(2)=a(2)^2+b(2)+c=4a+2b+c$

$f(3)=a(3)^2+b(3)+c=9a+3b+c$

$f(4)=a(4)^2+b(4)+c=16a+4b+c$

$f(5)=a(5)^2+b(5)+c=25a+5b+c$

…

$f(n-1)=a(n-1)^2+b(n-1)+c=a(n-1)^2+b(n-1)+c$

$f(n)=a(n)^2+b(n)+c=an^2+bn+c$

As a sequence: $(a+b+c),(4a+2b+c),(9a+3b+c),\ldots,(an^2+bn+c)$

As a table:

x	$f(x)=ax^2+bx+c$
1	$a+b+c$
2	$4a+2b+c$
3	$9a+3b+c$
4	$16a+4b+c$
5	$25a+5b+c$
…	…
n-1	$a(n-1)^2+b(n-1)+c$

n	$an^2 + bn + c$

The next step is to take the first difference, $\Delta_x = f(x+1) - f(x)$. Note that the first differences in this example use function notation, whereas the linear example above used the sequence notation $\Delta_n = a_{n+1} - a_n$.

For $x=1, \Delta_1 = f(2) - f(1)$

$\Delta_1 = (4a+2b+c)-(a+b+c)=3a+b$

For $x=2, \Delta_2 = f(3) - f(2)$

$\Delta_2 = (9a+3b+c)-(4a+2b+c)=5a+b$

...

For $x=n-1, \Delta_{n-1} = f(n) - f(n-1)$

$\Delta_{n-1} = f(n) - f(n-1)$

$=(an^2 + bn+c)-(a(n-1)^2 + b(n-1)+c)$

$=an^2 + bn+c-a(n-1)^2 - b(n-1)-c$

$=a(n^2 - (n-1)^2)+b(n-(n-1))+(c-c)$

$=a(n^2 - (n^2 - 2n+1))+b(n-n+1))+(c-c)$

$=a(2n-1)+b(1))+(0)$

$=(2n-1)a+b.$

Adding a column to our table, we record the first differences.

x	f(x)	$\Delta_x = f(x+1)-f(x)$
1	$a+b+c$	$\Delta_1 = 3a+b$
2	$4a+2b+c$	$\Delta_2 = 5a+b$
3	$9a+3b+c$	$\Delta_3 = 7a+b$
4	$16a+4b+c$	$\Delta_4 = 9a+b$
5	$25a+5b+c$	$\Delta_5 = 11a+b$
...	...	...
n-2	$a(n-2)^2 + b(n-2)+c$	$\Delta_{n-2} = (2n-3)a+b$
n-1	$a(n-1)^2 + b(n-1)+c$	$\Delta_{n-1} = (2n-1)a+b$
n	$an^2 + bn+c$	

All of the steps up to this point were used to determine if a sequence is arithmetic. We notice that the first difference is not constant. I will ask my students to look at the first differences carefully, and I hope that they will be sufficiently comfortable with arithmetic sequences to recognize that this difference sequence is arithmetic.

Writing the first difference sequence, we have:

3a+b, 5a+b, 7a+b, 9a+b, …, a(2n-3)+b, a(2n-1)+b.

Since we know how to analyze an arithmetic sequence, we use the same process to analyze the first differences. We will accomplish this by taking the second difference, $\Delta\Delta_x = \Delta_{x+1} - \Delta_x$.

For x=1, we have,

$\Delta\Delta_1 = (5a+b)-(3a+b)=5a+b-3a-b$

$=5a-3a+b-b$

$=2a.$

For x=2,

$\Delta\Delta_2 = (7a+b)-(5a+b)=7a+b-5a-b$

$=7a-5a+b-b$

$=2a.$

And so on…

Taking x=n-2

$\Delta\Delta_{n-2} = (2n-1)a+b-((2n-3)a+b)=2an-a+b-2an+3a-b$

$=2an-2an-a+3a+b-b$

$=(2an-2an)+(-a+3a)+(b-b)$

$=(0)+(2a)+(0)$

$=2a.$

Expanding our table by adding a fourth column, the table is a second difference table for the quadratic.

x	f(x)	Δ_x	ΔΔ_x
1	a+b+c	$\Delta_1 = 3a+b$	2a
2	4a+2b+c	$\Delta_2 = 5a+b$	2a
3	9a+3b+c	$\Delta_3 = 7a+b$	2a
4	16a+4b+c	$\Delta_4 = 9a+b$	2a
5	25a+5b+c	$\Delta_5 = 11a+b$	2a
…	…	…	…

n-2	$a(n-2)^2 + b(n-2)+c$	$\Delta_{n-2}= a(2n-3)+b$	2a
n-1	$a(n-1)^2 + b(n-1)+c$	$\Delta_{n-1}= a(2n-1)+b$	
n	$an^2 + bn+c$		

Observe that $\Delta\Delta_x = 2a$ for $1 \le x \le n$. The second difference table shows that a general quadratic, $f(x)=ax^2 + bx+c$ has a constant second difference. This demonstrates condition (1).

Next, we want to show that a quadratic can be reconstructed from the second difference. Though this procedure is not shown in the math textbook we use, I definitely plan to show this to my pre-calculus students, and to my Math 1 students if there is additional time, because it connects the constant second difference and first difference to the equation of a quadratic. To accomplish this feat, we use the general theorem that a sequence is determined by its first term and its difference sequence. Applying this twice, we will get our quadratic.

Using the information in the second difference table, notice that the second difference is constant. We can express this relationship in the equation, $2a=\Delta\Delta$. Solving for *a*, we have

$2a=\Delta\Delta$

$2a/2 = \Delta\Delta/2$

$a= \Delta\Delta/2$.

The result shows us that for a quadratic sequence, the coefficient of the x^2 term is equal to half the second common difference. I would point out to my students that the second difference of a quadratic effects the vertical stretch of a quadratic because it is twice the leading coefficient.

Since we can find the value for a, we need to find the b and c values. The method is to remove the quadratic component of the sequence by subtracting the ax^2 from each term. Let $g(x)=f(x)-ax^2$. This gives us

$g(x)=f(x)-ax^2 =(ax^2 +bx+c)-ax^2 =bx+c$.

By doing so, we have a linear equation $g(x)=bx+c$. Evaluating this expression for values $1 \le x \le n$, we can make a new table, which we will analyze.

x	$g(x)=f(x)-ax^2 =bx+c$
1	1b+c
2	2b+c
3	3b+c
4	4b+c
5	5b+c
…	…
n-2	b(n-2)"+c
n-1	b(n-1)"+c

From the table above, we will attempt to analyze the values obtained by finding Δ_x' , the first difference of g(x).

x	g(x)	Δ_x'
1	1b+c	(2b+c)-(1b+c)=b
2	2b+c	(3b+c)-(2b+c)=b
3	3b+c	(4b+c)-(3b+c)=b
4	4b+c	(5b+c)-(4b+c)=b
5	5b+c	(6b+c)-(5b+c)=b
…	…	
n-2	b(n-2)+c	((n-1)b+c)-((n-2)b+c)=b

Computing the first difference shows that Δ_x' =b for all of the first differences. Thus, the coefficient of the x term in the quadratic is equal to the first difference of f(x)-ax^2 . Since the first difference is constant, we showed that f(x)-ax^2 is linear. At this point, I would expect my students to notice that they have a linear sequence and would be able to write the general expression c+Δ_x' .

This shows that f(x)-ax^2 =c+Δ'x, which we can then solve for *c*, by subtracting Δ'x from both sides of the equation. Substituting b for Δ' and ax^2 +bx+c for f(x), we have

c=f(x)-ax^2 -Δ$'$ x

=ax^2 +bx+c-ax^2 -bx

=c, for all x.

I would point out that a quadratic equation in the form f(x)=ax^2 +bx+c can be written for a sequence with constant second common difference ΔΔ if we follow the three steps:

Step 1: To find a, take a= ΔΔ/2.

Step 2: b is the the first common difference, Δ', of the sequence formed by subtracting (ΔΔ/2)x^2 from each term in the original sequence.

Step 3: We can find the constant value c, by subtracting ((ΔΔ/2)x^2 +Δ$'$ x) from any term in

the original sequence.

Though the computations can be tedious, the process is an extension of the difference tables used to analyze arithmetic/linear sequences. One valuable outcome in going through this process is to emphasize the fact that quadratic functions are characterized as having a second common difference, which lays the foundation for calculus and the study of rates of change and derivatives. In calculus, one could characterize quadratic functions as those functions with constant second derivative. Another outcome is that there is a way to write an explicit equation for quadratic sequences, though it does require some work.

Example of a Quadratic Sequence

The Handshake Problem: There is a party. Each person at the party shakes the hand of every other person in the party. People are unable to shake hands with themselves, and you are not counting multiple handshakes with the same person. Starting with two people, create a sequence with five terms to show the relationship between the number of people at the party and the number of handshakes. Then write a function to determine the number of handshakes if there are n people at the party.

To start this problem, one approach is to make a table with columns indicated the number of people at the party and number of handshakes. With two people, persons A and B, at the party, there is only one handshake possible. If a third person, C, arrives, there will be two additional handshakes as C must shake hands with A and B, for a cumulative total of 3 handshakes. If a fourth person, D, arrives, the only new handshakes are between D and each of the three other guests, for a total of 6. If a fifth person E arrives, there are four more handshakes for a total of 10. Continuing the trend, with each new arrival, the person must shake hands with those already present. So when a sixth person arrives, there are five more handshakes for a total of 15 and so on.

Note that this constructing of the handshake function uses an argument based on what its first difference sequence must be, and adding the differences. This argument shows that the difference sequence is linear which will allow us to conclude that the original sequence is quadratic because a linear first differences implies constant second differences.

Number of people	*# of handshakes*
1	0
2	1
3	3
4	6
5	10
6	15
…	…

As a starting point, we can compute the first difference using Δ_n.

n people	*# of handshakes*	Δ_n
1	0	1-0=1
2	1	3-1=2
3	3	6-3=3
4	6	10-6=4
5	10	15-10=5
6	15	…
…	…	

We notice that the first differences are not constant, but appear linear. This sequence could be quadratic, so we should check to see if there if the second difference is constant. Computing $\Delta\Delta_n$, we extend our table and observe that the second common difference is constant. Therefore, we will be able to write a quadratic equation in the form $an^2 + bn + c$ for the handshake problem.

Number of people	*# of handshakes*	Δ_n	$\Delta\Delta_n$
1	0	1-0=1	2-1=1
2	1	3-1=2	3-2=1

3	3	6-3=3	4-3=1
4	6	10-6=4	5-4=1
5	10	15-10=5	…
6	15	…	
…	…		

With $\Delta\Delta_n$ =1, we find *a* by taking a = $\Delta\Delta_n/2$ = ½.

To find the value of *b* and *c*, we take our original values for the number of handshakes and subtract $(\Delta\Delta_n/2)\ n^2$ = $½n^2$ from each term.

Number of people	# of handshakes - $(\Delta\Delta_n/2)\ n^2$
1	$0- ½(1)^2 =- ½$
2	$1- ½(2)^2 =-1$
3	$3- ½(3)^2 =- 3/2$
4	$6- ½(4)^2 =-2$
5	$10- ½(5)^2 =- 5/2$
6	$15- ½(6)^2 =-3$
…	…

From these values, we will now compute the first differences, Δ'_n.

Number of people	# of handshakes - $(\Delta\Delta_n/2)\ n^2$	Δ'_n
1	$0- ½(1)^2 =- ½$	-1-(-1/2)=-1/2
2	$1- ½(2)^2 =-1$	- 3/2 -(-1)=-1/2
3	$3- ½(3)^2 =- 3/2$	-2-(-3/2)=-1/2
4	$6- ½(4)^2 =-2$	- 5/2 -(-2)=-1/2
5	$10- ½(5)^2 =- 5/2$	-3-(-5/2)=-1/2
6	$15- ½(6)^2 =-3$	…

Since the first difference Δ'_n is constant, we know from the explanation above that $b=\Delta'_n=-½$. We then can find *c* using the formula $c=f(n)-(\Delta\Delta_n/2)n^2-\Delta'_n$. For n=1, we have

$$c=f(1)-\frac{\Delta\Delta_1}{2}1^2-\Delta_1'$$

$$c=0-\frac{1}{2}1^2-\left(-\frac{1}{2}\right)$$

$$c=-\frac{1}{2}+\frac{1}{2}$$

$$c=0$$

With $a=\frac{1}{2}, b=-\frac{1}{2}$ and $c=0$, our equation is

$$f(n)=an^2+bn+c$$

$$f(n)=\frac{1}{2}n^2-\frac{1}{2}n+0$$

$$f(n)=\frac{1}{2}n^2-\frac{1}{2}n+0$$

$$f(n)=\frac{1}{2}n^2-\frac{1}{2}n.$$

We can rewrite this equation in the form f(n)= ½(n)(n-1).

This is one way that we can find an algebraic representation for the handshake problem, based on the structure outlined in the prior example.

Geometric Representation of Handshake Problem

An alternative approach to write an equation that models the handshake problem is represent it geometrically using a graph. See Figure 1.

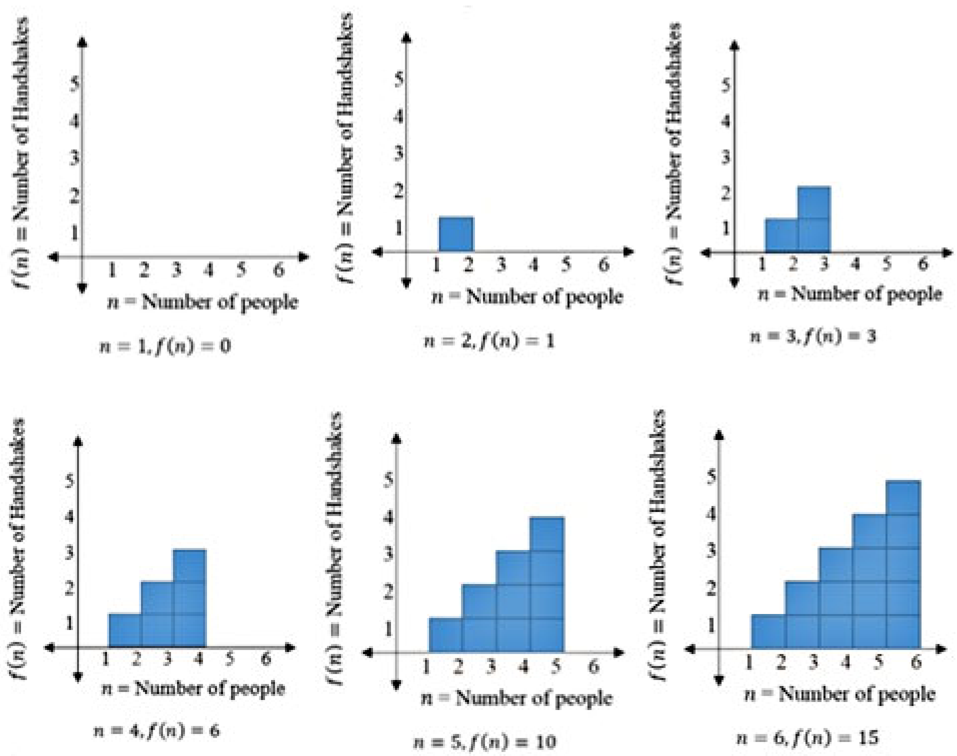


Figure 1: Geometric Representation of Handshake Problem

Examining the shape of the geometric function, observe that if we were to copy the exact figure, rotate the figure 180°, then we could combine the rotated figure and the original one to make a rectangle.

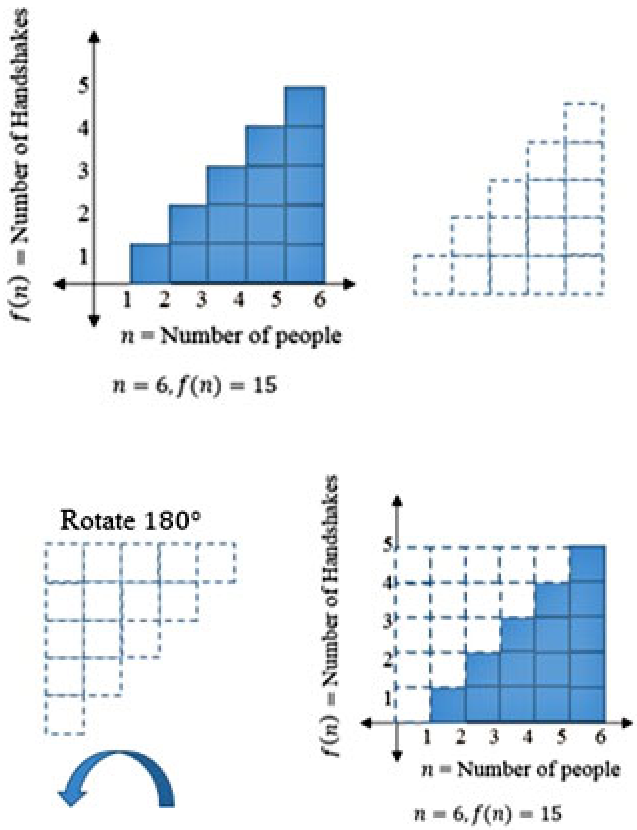


Figure 2: Geometric Approach to Finding an Explicit Quadratic

Observe the resulting rectangle's horizontal and vertical lengths and their relation to *n.* See Figure 3. The horizontal length is equal to n units, while the vertical length is n-1 units. Thus, its area is given by n(n-1). The total area of the rectangle represents double the number of handshakes.

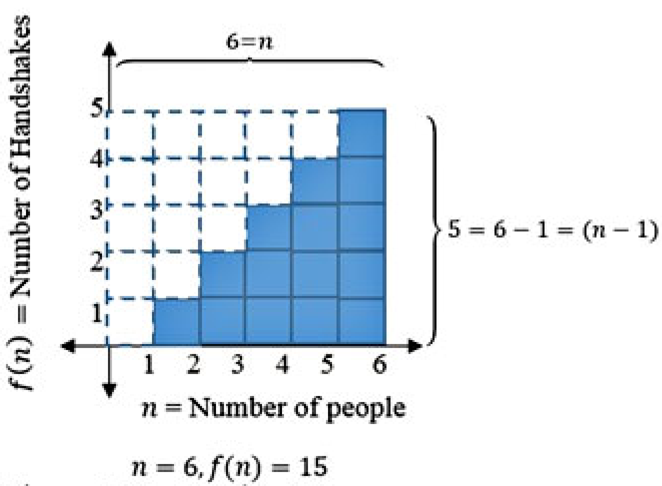


Figure 3: Rectangular Area

To get the area of the shaded region, we can take half of the area of the rectangle. The resulting equation can be written as $f(n)= \frac{1}{2} (n)(n-1)$, which is equivalent to our equation found using the first term, first difference and second difference sequences. From the handshake problem example, both a computational algebraic approach and a geometric visual representation produced equivalent equations that modeled the situation. Interpreting problems in multiple representations will provide opportunities for students to see the connections between the first and second differences and quadratic equations.

Throughout this unit, difference sequences will be used to analyze original sequences. Noticing key features will help students to identify arithmetic, geometric and quadratic sequences. Though they are the three special cases that are given most of the attention at the high school level, there are many more types of sequences that do not receive as much attention, but can be explored by analyzing difference sequences. Starting in the Integrated Math 1 classes, students will develop the tools that can be applied to linear, exponential, and quadratic functions. At the Math Analysis level, repeated applications of the general theorem for difference sequences is a tool to analyze polynomial functions. By taking time to understand the roles of the differences, students will see an alternative approach to write algebraic equations that model numerical patterns.

Additional Problems

The following are example problems that can be modeled by each of the special cases of problems, or other types of sequences.

Arithmetic/Linear

1. A bridge railing is to be formed by placing equilateral triangles in a row. Starting with the first triangle, each subsequent triangle is placed so that it shares one side with the triangle before it. It takes three rods to build the first triangle, 5 rods to build two triangles, and 7 rods to build three triangles. The contractor wants to find the number of rods represented by t_n, the nth term of the sequence $\{3,5,7,\ldots, t_n\}$.

2. Some types of bamboo plants can grow 4cm every hour. If one of these bamboo plants is 18cm tall, write an expression for the height (in cm) of the bamboo after h hours?

3. A decorative railing is to be constructed from sections in the shape of a regular hexagon placed such that two hexagons share 1 side. Create a sequence for t_1 to t_{10} representing the number of metal rods needed to build from one to ten sections. Write a rule for t_n, the number of metal rods needed to build n sections of railing.

Geometric/Exponential

4. A rubber ball bounces up half the previous height it fell. If a rubber ball is dropped from a height of 20 feet, what is the height the ball bounces on the nth bounce?

5. *Chain Letter Problem*. You send out 7 letters to different people on the first day. On the second day, each of the 7 people send out 7 more letters to new people. The process continues for several days. How many letters will be sent on the d-th day?

6. A bacteria cell can divide into two identical cells after 1 minute. Each of the new cells can also divide after 1 more minute. If each cell can divide every minute, how many cells are there after one hour? Write an expression for the number of bacteria cells after t minutes.

7. A tennis tournament begins with 128 players. After the first round 64 teams remain, After the second found, 32 teams remain. How many teams remain after the third, fourth, and fifth rounds? How many rounds will it

take before there is a winner? Write a finite sequence that represents the number of teams that have been eliminated after r rounds of the tennis tournament. What is the domain of r?

Quadratic

8. Diagonals are formed on regular polygons, starting with a three-sided regular polygon or equilateral triangle. The variable n represents the number of sides in the polygon. Determine the number of diagonals in a regular polygon with 20 sides. Then determine the equation for the number of diagonals in a regular polygon of n sides.

9. *Round-Robin Tournament* (Variation of Handshake Problem). In a round-robin tournament, each team plays each other team only once. Starting with two teams, create a sequence with five terms to show the relationship between the number of teams in a round robin tournament and the number of games that need to be scheduled. Write a function to determine the number of games if there are t teams.

10. *Double Round-Robin Tournament*. In a double round-robin tournament, each team plays each other team twice. Starting with two teams, create a sequence with five terms to show the relationship between the number of teams in a round robin tournament and the number of games that need to be scheduled. Write a function to determine the number of games if there are t teams.

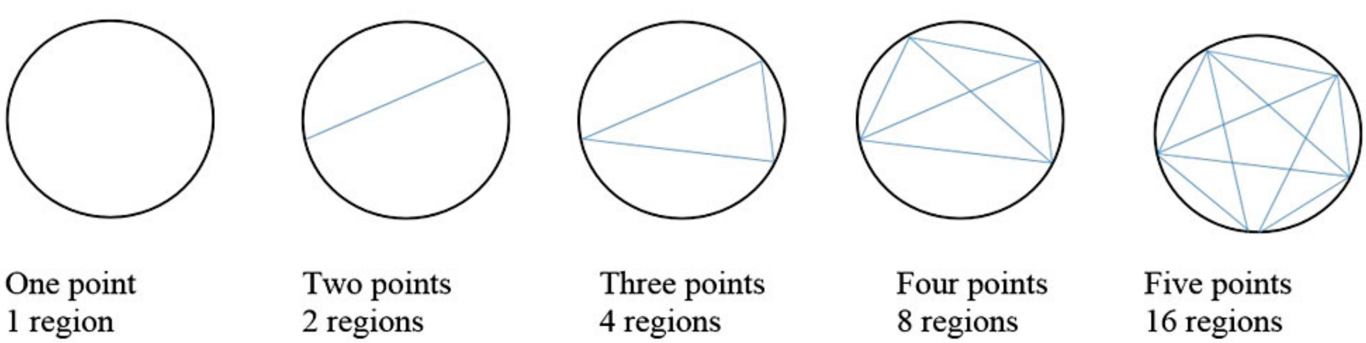
11. *Triple Round-Robin Tournament*. In a triple round-robin tournament, each team plays each other team three times. Starting with two teams, create a sequence with five terms to show the relationship between the number of teams in a round robin tournament and the number of games that need to be scheduled. Write a function to determine the number of games if there are t teams.

12. How many regions are there in the plane cut out by *n* lines in general position (meaning, no two parallel, no three concurrent)

13. Find the sum of the first *n* odd positive integers.

Other

14. *Partitioning of a Circle Problem*. Given a circle, begin with one point on the circle. Add a point on the circle and draw a chord connecting all points. Count the number of partitioned regions in the circle. Add another point and draw chords connecting the new point to all existing points on the circle. Again, count the number of regions partitioned in the circle. Continue the pattern and record the number of partitioned regions.



15. Fibonacci Sequence {1,1,2,3,5,8,....}

16. Cubed Numbers. {1,8,27,64,…}

Appendix

District Academic Standards

The unit will address the following focus standards and supporting standards for Math 1, as well as standards from the Integrated Math 2 course.

ESUHSD Curriculum Unit Focus Standards

Focus standards for this lesson include the following:

F-LEA: Construct and compare linear, quadratic, and exponential models and solve problems

ESUHSD Curriculum Unit Supporting Standards

F-IF-A: Understand the concept of a function and use function notation

F-IF-B: Interpret functions that arise in applications in terms of the context

F-IF-C: Analyze functions using different representations

F-BD-A: Build a function that models a relationship between two quantities (linear & exponential)

Students will be able to understand the use of variables, inequalities, linear equations, and linear inequalities given a story context and use units as a tool for understanding problems.

I also see the content of this unit being applied to my Math Analysis course, as the first three chapters are devoted to developing the idea of a function and study of polynomial and exponential functions. With a stronger understanding of what is a function and how to analyze polynomials, the students will be better prepared for AP Calculus and AP Statistics.

Bibliography

Barry, Maurice. *Mathematical modeling, Book 4*. Toronto: Nelson Canada, 2002.

Bluman, Allan G. *Math word problems demystified*. New York: McGraw-Hill, 2011.

Cuoco, Al. *Mathematical Connections: A Companion for Teachers and Others*. Washington, DC: Mathematical Association of America, 2005.

"Handshake Problem." From Wolfram MathWorld. Accessed June 26, 2017. http://mathworld.wolfram.com/HandshakeProblem.html.

KS3 Maths - Number patterns - Revision 1

http://www.bbc.co.uk/education/guides/zdkwxnb/revision

Larson, Ron, and Laurie Boswell. *Big Ideas Math: Integrated Mathematics I*. Erie, PA: Big Ideas Learning, 2016.

Larson, Ron, and Laurie Boswell. *Big Ideas Math: Integrated Mathematics II*. Erie, PA: Big Ideas Learning, 2016.

Larson, Ron, and Laurie Boswell. *Big Ideas Math: Integrated Mathematics III*. Erie, PA: Big Ideas Learning, 2016.

Endnotes

1. http://www.corestandards.org/Math/
2. Cuoco, A.
3. http://www.bbc.co.uk/schools/gcsebitesize/maths/algebra/sequencesquadrev1.shtml
4. Cuoco, A.