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Place Value Meets Multiplication: Utilizing Place Value to Comprehend Multiplication

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Introduction

Many educators today would agree to some extent that there has been a shift in mathematics education on many levels and across all grade levels. As a student, one must demonstrate understanding of concepts in mathematics often before one's brain is developmentally able to comprehend that mathematics is more than procedures or a set of rules to reach a specific solution. The brain of an elementary school student is still being formed, it is constantly absorbing the information around it. Thus, a child's response to a mathematical inquiry is generally based on his or her current information that can also vary with the activity.

In my experience as a fourth-grade math teacher, the majority of my students have a strong dislike of all things related to multiplication. I am accustomed to the moans and groans as well as the "it's too hard" or "it takes too long" cries from my students. I often wonder, why do my students feel this way? More importantly, how can I help them feel more confident in their ability to solve problems that involve multiplication? What makes it too hard or too long? What does "too long" even mean? Are my students referring to the number of procedural steps needed to arrive at the product? Realistically, some of their feelings regarding these skills can be linked to their overall feelings about mathematics which could have been damaged by negative experiences with a previous teacher, a parent/guardian's view on mathematics or lack of support, or even past performance in math class in earlier grades. There could also be some elements of low self-esteem and lack of confidence wrapped up into these feelings. Fourth graders can be a very peer approval seeking group of impressionable minds.

When I think about my personal experiences in elementary school, I distinctly recall the teacher standing in front of the classroom giving directions, showing us the steps to solve the math problem of the day and then giving the class problems to solve on their own. There was little to no small group instruction, movement around the classroom, or technology usage. There was, however, an adult in the classroom who I truly believed cared about my success in her classroom as well as for my overall growth as a child. I could go on and on about the importance of how a student's perspective regarding how the teacher feels about him or her can affect the student's performance in the classroom, but that would lead into a completely different unit.

I hope that the curriculum unit I have created will lessen the anxieties of my future students as we delve into how the basic principles of place value can help us understand and solve multiplication problems. We will explore the Base 10 Number System, the role of place value within that system, and how a clear and complete understanding of both are needed to conceptualize multiplication parameters before one can thoroughly apply the procedures needed to investigate and solve multi-digit equations, as called for in the state guidelines, by demonstrating to the students that multiplication is an extension of the skills they already possess for addition. I will also demonstrate how numbers can be broken up into more manageable pieces to arrive at the product sought in the problem, which will help my students to better visualize the end result. I believe that a greater understanding of the concepts behind the algorithms will increase their overall confidence, foster a growth mindset, and increase their degree of inquiry.

Once a clear foundation has been laid, I will focus on the Virginia Standards of Learning requirements for third graders including both single and multi-digit multiplication. Although my focus is not the Standards of Learning or Common Core mathematics standards for third graders, I do believe that looking at the objectives of the previous year, as well as more widely used standards could prove beneficial in showing how the objectives shift from year to year and what the students should have been exposed to prior to arriving in fourth grade. Having this information gives me a frame of reference to access the student's prior knowledge as well as gauge their retention of the previous year's learning.

As we move from conceptualizing multiplication to application and computation, I will dig deeper into several multiplication algorithms. I will discuss three methods for multiplying whole numbers: Area Models, the Box Method, and the traditional U.S. multiplication algorithm (which I generally call Old School in my classroom as it pays homage to the way my students' parents and I learned multiplication). Each of the methods will be modeled for the students, with emphasis on the role place value plays and what the digits really mean once the algorithm is put in place and solved.

Because I want my students to truly understand that math connections can be made everywhere, I also plan to incorporate math related picture books into the introduction of each skill. For example, the book, *Two Ways to Count to Ten: A Liberian Folktale*[1] retold by Ruby Dee illustrates to the audience that it is easier and quicker to skip count, i.e. use multiplication, instead of counting by ones. I can use the text to address my students' complaint that "it takes too long" to solve multiplication problems. Many of the inquiry-based questions we will solve in class will focus on everyday occurrences that my students will come across throughout the school year. (Sample questions are listed in the resources section)

Demographics

I am entering my ninth year as a fourth-grade teacher in a Title I school. For several years, like many elementary school teachers across the nation, I taught all core subjects (reading, writing, math, science, and social science/history) to one class of students all year. Three years ago, my school transitioned to a departmentalized model for the fourth-grade students (fifth grade was already following this model). We have three fourth grade teachers and each teacher provides instruction in one tested core subject to the entire grade level, plus science (which is not a state tested subject in this grade level) to each respective homeroom. I recently completed my second year of teaching mathematics to the entire grade level. This past school year, I taught approximately 80 students who were broken up into three classes ranging from 23-30 students, with each class lasting about 80 minutes each day. Although my school is not as affected by transience (students who move from school to school) as some of the other schools in the district we do have notable spurts of transiency at certain times of the school year, which affect the students' ability to learn concepts thoroughly.

Background

I have several reasons for wanting to write this unit on utilizing place value to understand multiplication. First, I want my students to know how to use the basic principles of number sense to assist them in solving more complex problems and to demonstrate how that understanding is a driving force in their overall understanding of mathematics in their future courses. Number sense can be defined as:

A person's ability to use and understand numbers:

- *knowing their relative values,*
- *how to use them to make judgments,*
- *how to use them in flexible ways when adding, subtracting, multiplying or dividing*
- *how to develop useful strategies when counting, measuring or estimating.*[2]

My students have an extremely difficult time with all aspects of number sense. It seems that students move from one grade level to the next without a clear understanding of the basics which only serves to increase their overall deficits in mathematics. These deficits include, but are not limited to, computation with basic addition, subtracting across zeros, and knowing the basic multiplication and division facts. With such deficits in their foundation, how can they progress with success to more challenging or grade level expectations? A student can read a book fluently, memorize the historical contributions to the arrival of Africans in the Jamestown Settlement, and even carry out the steps of the scientific method, but as Paul Halmos[3] expresses it, "the only way to learn mathematics is to do mathematics."

Secondly, it is also of vital importance to me that my students begin to feel that math is indeed something that they can do and that we can have some fun while doing it. In my nine years of teaching, I have had the pleasure of teaching, mentoring, and interacting with hundreds of students yearly on multiple levels inside and outside of the classroom, but I cannot recall one student who has simply said, "I get math, it comes easy to me." I have many students say, "I like math but I simply do not get it." The most common statement is, "Math is too hard so I don't like it." Personally, I receive all three statements as they are and I am grateful that the student would share their thought with me. But in doing so they knew that I would do something with the information. I am 100% confident in knowing that my students trust that I care about them and want them to do well, and that is a great honor that I will always cherish. This level of trust also allows me to take them to a place of discomfort in regard to mathematics.

Third, for far too long students have been given math instruction based on simply arriving at a solution without an understanding as to how they arrived at the solution, and more importantly, what the solution means in terms of the initial problem given. In this unit, my desire is to take these components to bring my students' understanding and application of number sense full circle.

Content Objectives

Base 10 Number System

The Base 10 number system uses the digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9 to form every whole number, and also extends to express decimals and fractions. The zero in this system represents a place holder; for example, 9,099 indicates that there are no hundreds in this number. It is necessary to insert the zero to read the number correctly. Susan Smith[4] states that base-ten is a positional system, one in which the one's place is to the right and the next place represents the base, then the base times the base, with the numbers continuing in this way. Meaning, since the base of our system is ten, there is an increase in size by a factor of ten at each step or position. Another important feature of the number system is that "the value of the whole number is the sum of its parts or the value assigned to each digit.[5] " When the above components are put together we can use the number system to perform all of the basic operations: addition, subtraction, multiplication, and division. The overall structure of the base ten system, the sequence of the digits, the relationships the digits have with each other in each number provides a framework for us to work with numbers in a cohesive manner. Howe and Epp[6] state, "We hope that making more aspects of this structure explicit will increase conceptual understanding and improve computational flexibility, thereby helping to make mathematics instruction more effective." Additionally, understanding the base ten system will enhance the students' capacity to understand place value.

Place Value

Place value can be described as prescribing the value of a digit based on its position in a number. In other words, place value allows us to reuse a single scheme that assigns ten digits in different positions in a number to tell how many of each of the groups there are. For example, the 3 in 369 has a value of 300 whereas the 3 in 39 has a value of 30. Additionally, prior to going deeper with the mathematics concepts, my students must be able to reliably understand that when one asks for the value of the digit 6 in the number 369 it is not 6, nor is it the tens place, but it is 60 or 6 tens. Stein et al. [7] reiterates that "reading and writing numerals accurately, along with a strong conceptual understanding of place value concepts prepares students for more advanced computation and problem solving.

Howe-Reiter[8] presents the five stages of place value understanding as a focus to aid the elementary school teacher through a comprehensive guide to how teaching students the basics in mathematics leads to a more in depth conceptual understanding of number sense, which allows one to build on more rigorous standards later.

The Five Stages of Place Value according to Howe and Reiter are as follows:

1. Write the numbers-

369.

2. Recognize that the number is a sum of "place value pieces" or building blocks that explicitly show the place value of each digit-

369 = 300 + 60 + 9.

3. Each place value piece is a product of the digit times a base ten unit-

369 = (3 × 100) + (6 × 10) + (9 × 1).

4. Each base ten unit is a product of some copies of the base (base being 10):

369 = 3 × (10 × 10) + 6 × 10 + 9 × 1.

Each base ten unit is a result of multiplying the previous one by 10. Thus, the base ten units are 1, 10, 10 × 10 = 100, 10 × 10 × 10 = 1,000, and so forth.

5. The repeated products of 10, which are often called *powers* of 10, can be conveniently written using exponents:

$369 = 3 \times 10^2 + 6 \times 10^1 + 9 \times 10^0$

What does the Five Stages of Place Value look like in the classroom after solving a multiplication problem?

If I purchased a 4-piece and 6-piece nugget meal from Chick-fil-A for each student in fourth grade, how many nuggets would we have in all?

*# of students = 75 (for demonstration purposes only)

1. 4 piece nugget and 6 piece nugget = 10 piece nugget; 75 students=

75 = 70 + 5 = 7 × 10 + 5 × 1, so

75 × 10 = (7 × 10 + 5 × 1) × 10 =

(70 × 10) × 10 + (5 × 1) × 10 =

7 × (10 × 10) + 5 × 10 =

700 + 50 =

750

The product in place value form would look like the remaining stages below:

2. 750 = 700 + 50 + 0

3. 750 = (7 × 100) + (5 × 10) + (0 × 1)

4. 750 = 7 × (10 × 10) + 5 × 10 + 0 × 1

5. $750 = 7 \times 10^2 + 5 \times 10^1 + 0 \times 10^0$

In fourth grade, we typically review and expand upon stage 1 as the students are required to read, write, and identify the value of numbers up to nine digits (which is an increase from the third-grade requirement of six digits). That lesson then leads into stage two of writing numbers in expanded form and in most classrooms not much is done with the stages beyond that point.

There have been many times in my classroom in which a student reads the number 66 as "sixty-six" but when asked to write the number in expanded form, they write "6 + 6." Another error I have encountered over the years is that the student does not demonstrate understanding that the position of a digit determines its value. This means the student writes 6009 when asked to write six hundred nine or 60029 when asked to write six hundred twenty nine. As we move into larger numbers, when asked the value of the digit 4 in 430,516,802,703, the student may respond with "4" or "one hundred billion" as opposed to "four hundred billion." These simple yet critical errors serve as an alarm to me. As the teacher, I know that I need to backtrack and spend more time with that student to improve their understanding of the individual place value pieces that make up the numbers listed above. Not taking the time in the elementary years to solidify this understanding causes the student to further miss out on the connections between place value and computation.

Simply put, place value is what governs all other arithmetic operations. An understanding of the properties of operations also leads to better understanding of place value and computation. My students need to know why we are combining these numbers and then breaking them down into pieces and how it allows us to perform multiplication. They need to understand that the base ten units have different sizes and the number of digits tells that size, called the *order of magnitude* of the unit.[9] For example,

$9 = 9 \times 1$

$30 = 3 \times 10$

$300 = 3 \times 100$

$3{,}000 = 3 \times 1000$.

From the information above we know that 3,000 means 3 thousand because there are three places to the right, filled by zeros. The number 300 means 3 hundred because there are two places to the right filled with zeros, 30 means 3 tens or thirty, as there is one place or one zero to the right, and the number 9 means 9 ones with no remaining whole numbers to the right of the digit.[8]

As you look at the five stages of place value above, it is fairly clear that each stage builds upon the previous and connects one's understanding of place value to all arithmetic operations.

Properties of Operations

Addition

In elementary schools across the United States teachers are constantly telling the students that we must line up our digits to add correctly. Why is it important to do this? Simply put, we must add like units. The ones should be added with the ones, tens with tens, hundreds with the hundreds etc. until all digit units have been summed together. On the lower elementary level and for visual learners it is best to use concrete models (Base-10 blocks etc.) to demonstrate this before moving the students to conceptual understanding. Lining up the digits ensures that we do add like units.

Prior to moving into multiplication, I plan to briefly review the properties of addition as they connect rather well with the properties of multiplication.

The Commutative Property of Addition says that changing the order of the addends does not change the sum.

$33 + 66 = 66 + 33$

The Associative Property of Addition states that the way in which addends are grouped does not change the sum.

$33 + (66 + 99) = (33 + 66) + 99$

The Identity Property of Addition states that the sum of any given number and zero is equivalent to that number.

$99 + 0 = 99$

Keeping these basic properties in mind, let’s shift our focus to the star of the unit.

Multiplication

Multiplication has properties parallel to the properties for addition.

The Commutative Property of Multiplication says that one can multiply the factors in any order and the product will not be affected.

$369 \times 33 = 33 \times 369$

The commutative property can be illustrated using a rectangular array. Using a rectangular array allows the students to see how each grid space represents one digit and is the product of multiplying the rows by the columns. Meaning 6 rows times 7 columns is equivalent to 42 grid spaces. In the figures below (a) represents 7 × 6 and (b) represents 6 × 7. Figure (b) is the result of rotating (a) 90 degrees, and yet both illustrations have the same number of grid spaces, therefore 6 × 7 is equivalent to 7 × 6.

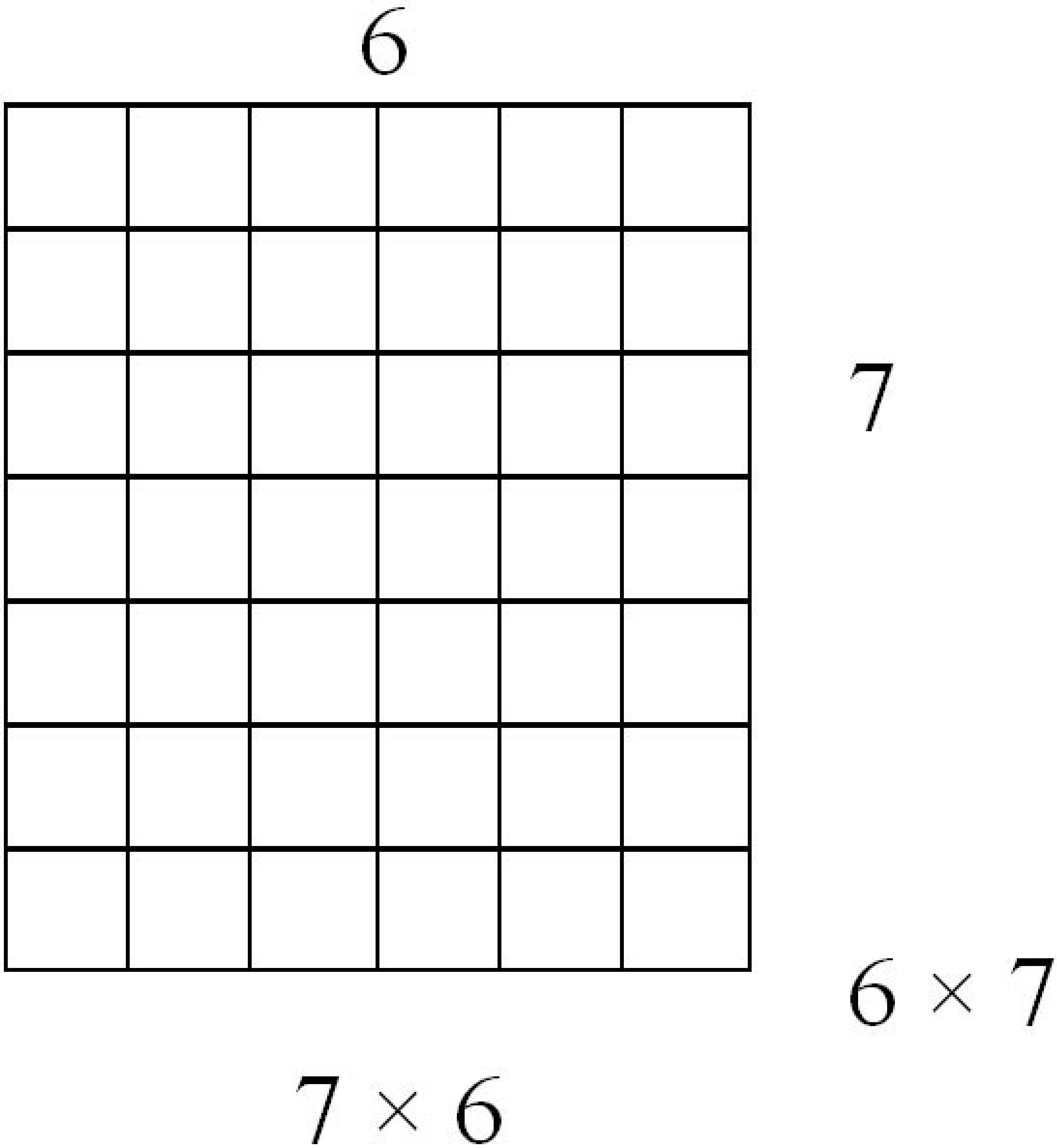


Figure (a)

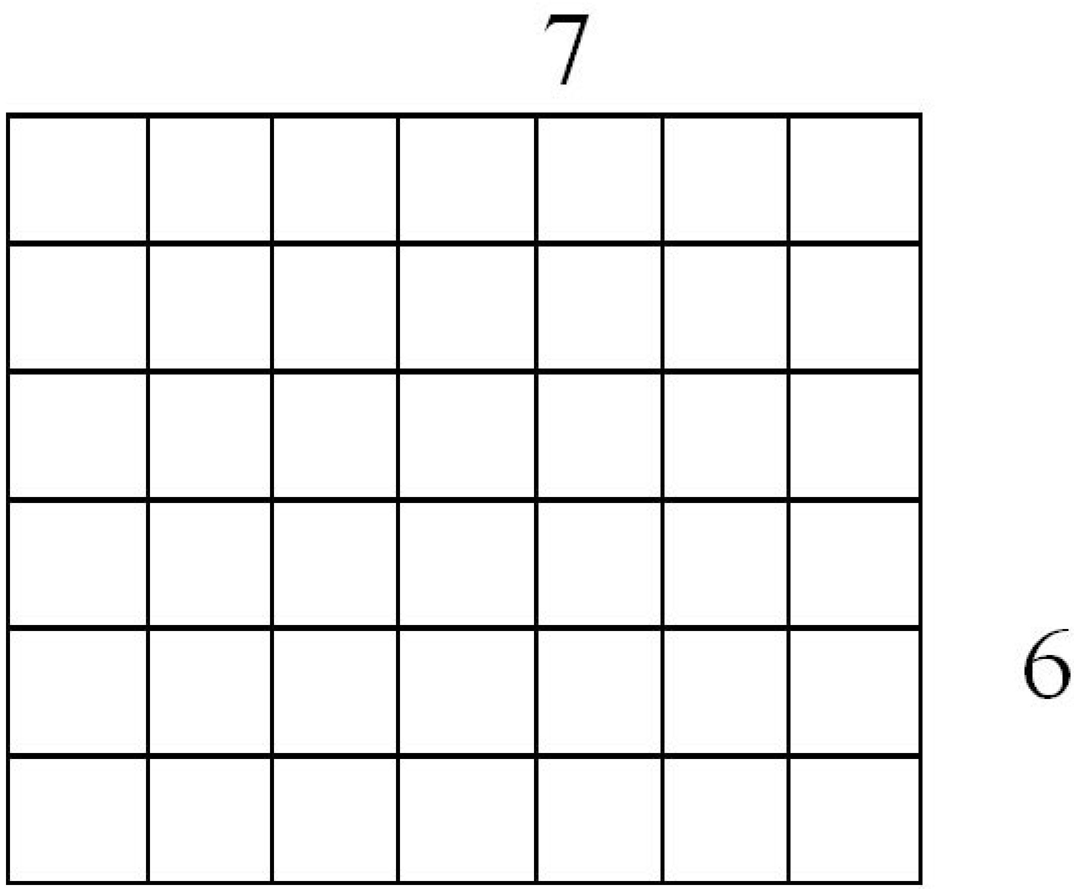


Figure (b)

Bennett et al[10] emphasizes that, it is equally important to note that understanding the commutative property practically cuts the number of basic multiplication facts that students must learn in half. Giving the students, this foundational information will indeed assist them in solving more complex multiplication problems. Knowing the basic multiplication facts makes it easier for the student to multiply by the powers of ten.

More Multiplication Properties of Importance

The Associative Property of Multiplication says that, when three or more numbers are multiplied, the product is always the same regardless of their grouping. Thus, for example

(5 × 14) × 19 = 70 × 19 = 1330, and   5 × (14 × 19) = 5 × (266) = 1,330.

The Identity Property of Multiplication states that any number multiplied by one is that number.

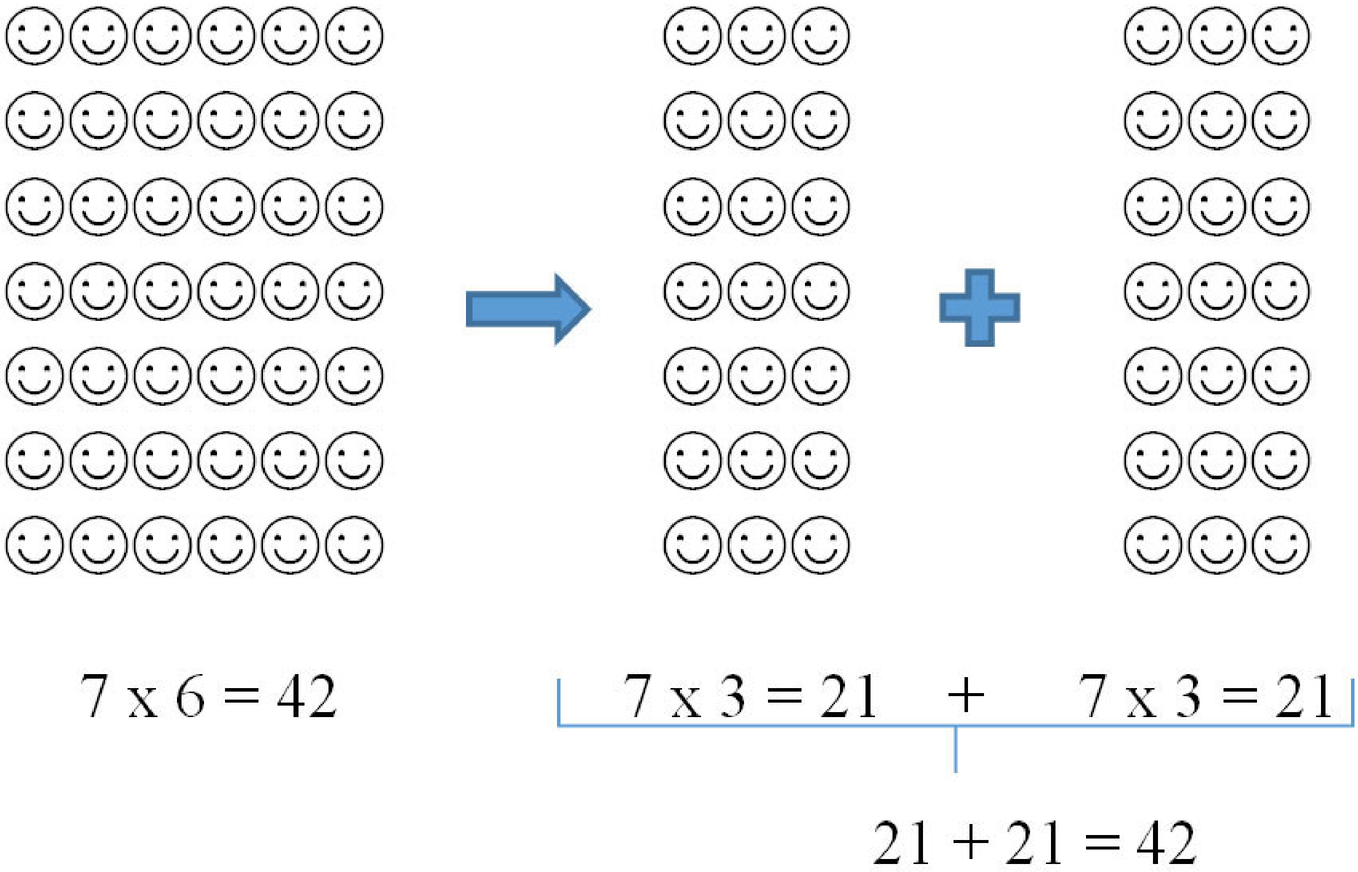
369 × 1 = 369

The Distributive Property is the crucial property of arithmetic that connects multiplication and addition. It states that a multiplication fact can be broken up into the sum of two multiplication facts. We can decompose or "break one of them into parts, multiply each of the parts by the other factor and add the results.[11] " Let's model the distributive property using the basic multiplication problem we used above in figures (a) and (b).

7 × 6 can also be calculated as 7 × 3 + 7 × 3 since 3 + 3 = 6.

The end result supports 7 groups of 6 which equates to 42. The 6 can be broken up in other ways as well such as (5 and 1) or (4 and 2). Due to the distributive property the product will continue to be equivalent no matter how the factors are broken up.

The distributive property can be modeled using an array. I will use 7 × 6 again for demonstration purposes.



The Extended Distributive Rule per Howe and Epp [12] states, “If A and B are sums of several numbers, then the product AB may be combined by multiplying each addend of B by each addend of A, and adding all the resulting products.”

For example, if A = a + b and B = c+ d + e, then

AB = (a + b) (c + d + e) =

ac + ad + ae + bc + bd +be

Let’s look at this rule using the example: 45 × 239

(40 + 5)(200 + 30 +9) =

(40 × 200) + (40 × 30) + (40 × 9) +

(5 × 200) + (5 × 30) + (5 × 9) =

8,000 + 1,200 + 360 +

1,000 + 150 + 45 =

9,560 + 1,195

= 10,755

The example above allows us to link multiplication and addition through the Extended Distributive Rule while multiplying in base 10 form. Each row of the grid is the result of multiplying each single place piece of one factor by each place value piece of the other factor, and then adding all the results.

The Distributive Property, and more general versions of it, as in the example above, is the key to performing multi-digit multiplication.

The Zero Property of Multiplication states that any number multiplied by zero is zero. This rule is a consequence of the Identity Rule for addition and the Distributive Rule.

369 x 0 = 0

Zero is a digit that is often presented to students as having no value but instead occupies the role of a place holder. This can cause confusion for students when completing more complex multiplication computation such as multiplying 2 digit by 2 digit numbers. Zero in this context simply means that there is nothing in that place however; we will take in consideration how easily this misconception can alter the product in a later example.

Each of the properties above provides rules to help solve multiplication problems that will always apply to those specific equation types. Being able to identify the situations in which a property applies would be beneficial for a student who is just learning the basics of multiplication.

Multiplication Basics

> "When students are learning base-ten concepts, they are combining multiplicative understanding (each place is ten times the value of the place to the right) with a positional system (each place has a value) - something hard to do prior to learning about multiplication."[13]

It is equally challenging for students to jump directly into solving multi-digit multiplication problems as required by the fourth grade standards in Virginia. So, before delving into multi-digit multiplication, let's take a moment to review multiplying single place numbers. When thinking of a one digit number and any place value piece we can easily write: 3, 30, 300, 3,000, and 30,000 etc. which is equivalent to:

3 × 1 = 3

3 × 10 = 30

3 × 100 = 300

3 × 1,000 = 3,000

3 × 10,000 = 30,000

These same principles can be used to multiply one digit numbers and multiples of ten. For example:

30	4	
6 × 30	6 × 4	
= 180	= 24	6

180 + 24 = 204

6 ×34 = 204

In the example above we can see how the distributive property is used to multiply the 6 with each of the place value pieces of the 34 which was written in expanded form.

As the students move from single digits to two digit multiplication it is good practice to expose them to multiples of 10 and 100 beforehand. In Virginia this is generally introduced in third grade. Van de Walle[14] articulates that, this focus supports the importance of place value and an emphasis on the number rather than the separate digits. By looking at the number, the student will be able to see the individual place pieces more clearly and thus decompose the number accordingly when working with larger numbers provided the student is able to keep track of place value.

For example:

19 × 80 = 19 × (8 × 10) = (19 × 8) × 10 = 152 × 10 = 1,520

Similarly, we can use place value to guide the computation of multiplication involving higher powers of ten as well:

19 × 800 = 19 × (8 ×100) = (19 × 8) ×100 = 152 × 100 = 15,200

Again, the distributive rule comes into play as the 19 is multiplied with each place value piece to arrive at the product for the given equation. When the student is able to visualize the place value pieces of powers of ten numbers the computation tends to be easier for the student to grasp and apply the concept to more real-life situations.

When looking at two digit by two digit multiplication and the role of zero as in the example below, the importance of understanding each place value piece increases in computing a product. This example models a frequent misconception amongst students who do not understand the role of the zero for the second row of partial products as well as the value of the “1” in the “13.”

$$\begin{array}{r} 44 \\ \times 13 \\ \hline 132 \\ +44 \\ \hline 176 \end{array} = \begin{array}{r} 44 \\ \times\ 3 \\ \hline 132 \end{array} + \begin{array}{r} 44 \\ \times\ 1 \\ \hline 44 \end{array}$$

Figure (c)

Compared to the correct algorithm:

$$\begin{array}{r} 44 \\ \times\ 13 \\ \hline 132 \\ +\ 440 \\ \hline 572 \end{array} = \begin{array}{r} 44 \\ \times\ 3 \\ \hline 132 \end{array} + \begin{array}{r} 44 \\ \times\ 10 \\ \hline 440 \end{array}$$

Figure (d)

In figure (c), one can clearly see that understanding that the “1” in the multiplier represents one ten which produces an increased value of the total product of the algorithm as evidenced in figure (d). Choosing not to identify the "1" as one tens caused the student to multiply it as another ones digit instead. Ignoring place value completely in this example is a common misconception amongst students in this age group. Figure (d) shows how the student was able to insert a "0" in the second partial product to indicate that the ones digit of the multiplier had already been multiplied with each number of the multiplicand.

Multiplication Situations

There are several types of one-step multiplication/division problems, according to three kinds of situations in which multiplication gets used.

Equal Groups

In multiplication, one factor represents the number of equal groups while the other factor represents the number in or the size of each group. The result is the product of the total of all groups. This model allows the students to use concrete items to visualize multiplication as an extension of addition; multiplication here amounts to adding a number to itself several times.

Arrays

An array consists of row and columns and reinforces an understanding of the commutative property as referenced above (e.g. 7 rows and 6 columns produces a 7 x 6 array).

Multiplicative Comparisons

Multiplicative comparison compares two quantities by saying that one of them is some number of times larger (or smaller) than the other one.

The chart below provides examples of each multiplication problem type in word problem form emphasizing the whole unknown as these problem types require multiplication to solve them and does not include the multiplication types that require the usage of other operations. The word problem examples will be used in conjunction with the teaching strategies for demonstration purposes.

Multiplication Problem Type	Examples
Equal Groups Whole Unknown	There are 12 boxes of cupcakes on the table. Each box holds 24 cupcakes. How many cupcakes are there in all?
Array Whole Unknown	There are 18 basketball teams competing in a national basketball tournament. Each team has 12 basketball players stand for the National Anthem. How many basketball players are standing all together?
Multiplicative Comparison-Result Unknown	Khalil was in the weight room for 55 minutes last week. Gerald was in the weight room fourteen times as long as Khalil during the same week. How long was Gerald in the weight room?
Multiplicative Comparison-Smaller Amount Unknown	Essence ran 120 miles. She ran three times as many miles as Erynn. How many miles did Erynn run?
Multiplicative Comparison-Comparison Factor Unknown	Taiya drove 420 miles to a dance competition. Monay drove 70 miles. How many times farther did Taiya drive than Monay?

Area Model

Using an area model to solve a multiplication problem can be useful to students when computing 2 digit by 2 digit multiplication, as it is based on the place value components since each number is broken up into tens and ones. By drawing the boxes as close as possible to scale the student can see why we multiply the factors in the way we do. Van de Walle[15] reiterates a benefit of the area model by stating, "The area model uses a row and column structure to automatically organize equal groups and offer a visual demonstration of the commutative and distributive properties."

Grid paper is a great tool to use to illustrate this concept as the students can count each box to represent the factors as each box is of equal size and it would be easy to outline an area model that represents for example, 12 × 24. Once the boxes are outlined, the student can then use Base-10 blocks to cover the respective piece. Each Base-10 piece results in the partial products that will then be added together to compute the product. When modeling this example for my students, it will be important to reiterate that each flat is composed of 10 rods, which represents 10x10 = 100.

Using the equal groups, whole unknown word problem from the chart above, I will model the steps of using an area model to solve a 2-digit by 2-digit multiplication problem. As with addition, the issues of computation are largely independent of the problem types. Students need to understand the context in which multiplication can be used as well as how to execute it computationally.

Step 1: Write the problem.

12 (columns) × 24 (rows)

Step 2: Draw the grid to scale and write each factor in expanded form.

20	4	
10 × 20	10 × 4	10
2 × 20	2 × 4	2

In this step, the two digit numbers are decomposed into their place value components, and each component of one is multiplied by each component of the other, according to the Extended Distributive Rule. Each product of place value pieces is easy to compute, following the discussion above. The multiplicand of 12 is broken down into 10 and 2 and each factor is multiplied with each piece of the 24. This step also shows a visual representation of the Extended Distributive Rule that was previously explained. Note, drawing the boxes to scale is the key feature of the Area Model.

Step 3: Write the numbers in expanded notation form.

(10 + 2) × (20 +4) =

(10 × 20) + (10 × 4) + (2 × 20) + (2 × 4) =

It is important to note that this step illustrates the symbolic version of the Extended Distributive Rule, in that each place value piece of one factor is distributed amongst all of the place value pieces of the other factor.

Step 4: Perform all the indicated multiplications in step 3, and add the partial products.

200 + 40 + 40 + 8 = 288

Box Method

The Box Method is an abstract version of the Area Model. The students are no longer drawing proportional boxes to compute a product but equal size boxes which is more conducive to logic when multiplying with larger numbers. It would not be practical to draw the boxes to scale for three digit numbers or larger. Further, it has been my experience that the students lose valuable time trying to draw the boxes to scale when working with larger numbers as opposed to drawing equal size boxes.

Let’s look at an example of the multiplicative comparison result unknown word problem from the chart above.

Khalil was in the weight room for 55 minutes last week. Gerald was in the weight room fourteen times as long as Khalil during the same week. How long was Gerald in the weight room?

Because the Area Model and the Box Method have some similarities the steps may parallel a bit. This small progression will add to the students’ overall confidence as they are will be expanding on their previous understanding of computing 2 digit by 2 digit multiplication. This progression will also support the principles behind fostering a growth mindset by allowing the students to see that growth is a continuous process. Additionally, using the Box Method allows the students to continue to see the decomposition of the numbers into their respective place value pieces.

Step 1: Write the factors to be multiplied in expanded form.

$$\begin{array}{r} 55 \\ \times\ 14 \\ \hline \end{array} \quad \Rightarrow \quad \begin{array}{l} 50 + 5 \\ 10 + 4 \end{array}$$

Step 2: Draw a box with equal grid lines that correspond with the number of addends produced from writing the factors in expanded form. Place the expanded form of the multiplicand horizontally across each box and the expanded form of the multiplier vertically beside each box.

50 5

10

4

Step 3: Using the grid, multiply each number of the multiplicand on the top of the box with each number of the multiplier on the side of the box.

50 5

10 × 50 10 × 5 10
500 50
4 × 50 4 × 5
200 20 4

Step 4: Add each partial product from the boxes to obtain the total product.

500

200

50

+ 20

770

The Box Method example above further illustrates the Extended Distributive Rule that was previously discussed as each factor has been combined by multiplying each addend. Additionally, when looking at the Box Method, the sums along the rows of the partial products produces the addends of the Standard Algorithm as evidenced by the example below. While summing along the columns produces the addends of the Standard Algorithm when the order of the factors are reversed as 14 × 55 (illustrated below).

Standard U.S. Algorithm

The standard U.S. algorithm for solving multiplication may be the most widely used strategy and one could also suspect that it is the least understood by students in classrooms all over the world. Many teachers tend to introduce this strategy very early when teaching students how to multiply, a time in which they are still trying to grasp why they are regrouping and inserting a “0” each time they arrive at a new row of partial products.

The example provided below is the same algorithm used for the box method above to aid as a visual to show how the pieces or the resulting partial products in each strategy are equivalent.

Step 1: Multiply the ones digit in the multiplicand (the top number) by the ones digit in the multiplier (the bottom number). Put the ones in the ones place below the equal bar and regroup the tens above the tens place in the multiplicand.

2

5**5**

× 1**4**

0

2

1**4**

× 5**5**

0

It is important to note that the regrouping is a result of: 5 ones × 4 ones = 20 ones and 20 ones = 2 tens and 0 ones. Be sure to emphasize the base-ten language as it points out the number of tens and why the regrouping is necessary.

Step 2: Multiply the ones digit of the multiplier by the tens digit of the multiplicand and add the regrouped tens from step 1. Write the total beside the "0" in the ones place.

2

5**5**

× 1**4**

220

2

1**4**

× 5**5**

70

Notice that the result, 220, is the sum of the two amounts in the lower row of the grid, while 70, is the sum of the partial products of the columns in the Box Method for this product.

It is vital that students multiply the digits before regrouping as multiplying creates a new tens number that will then need to be added to the previously created tens number from step 1. However, many students make the mistake of adding the 2 to the 5 before multiplying by 4. Meaning, 5 sets of 4 equals 20; 20 plus 2 tens equals 22 tens or 2 hundreds and 2 tens. Students who regroup first, generally do so by imitating the procedure for addition by adding the regrouped digit prior to adding the remaining digits in that place value column.

From my teaching experience, many students have a difficult time keeping the digits in line when using this method which can easily skew their final product. To limit this, I have the students use grid paper in which each number corresponds with a box on the paper or the students will turn their notebook paper vertically so that the numbers are already aligned in rows.

Step 3: Insert “0” in the second row of partial products as we will now be multiplying by the tens piece of the multiplier.

2

5**5**

× 1**4**

220

0

2

1**4**

× 5**5**

70

0

Step 4: Multiply the tens digit of the multiplier by the ones digit of the multiplicand and write the digit beside the “0”.

2

5**5**

× 1**4**

220

50

2

2

1**4**

× 5**5**

70

00

Step 5: Multiply the tens digit of the multiplier by the tens digit of the multiplicand.

2

5**5**

× 1**4**

220

+550

2

2

1**4**

× 5**5**

70

+700

At this point in the process I would show the students that the second addend, 550, is the sum of the two amounts in the upper row of the grid, and 700, is the sum of the partial products in the left column for the Box Method.

Step 6: Compute the sums of the partial products to obtain the total product.

2

5**5**

× 1**4**

220

+550

770

2

2

1**4**

× 5**5**

70

+700

770

Teaching Strategies

Whole Group/Small Group Instruction

One strategy that I will use in my class is whole group/small group instruction. During this time, I will provide direct instruction on the standard for the day using multiple modalities (interactive notes, picture books, inquiry questions, create foldables, video demonstrations, etc.) to frame the lesson. This is a time for the students to learn

explicit ways to solve mathematical problems, content vocabulary, and stretch their thinking beyond what they were previously taught through exploration in a comfortable environment. The students will begin to see that the solution to a mathematical equation can be computed in multiple ways. It is vital that students learn in a place of comfort and understand that mistakes are a part of the learning process. My whole group instruction, like my whole class, focuses on a growth mindset, so the students create the class norms that they agree to follow to help propel us to math success. This growth mindset is modeled in whole group instruction.

Think-Pair-Share

This strategy affords the students the opportunity to think individually about the task and then pair up with a neighboring peer to discuss their thinking before sharing with whole group. The students really enjoy this strategy as it gives them time to bounce their ideas off of someone else, minimizing the intimidation factor, before sharing their thinking or answer with the whole class. This strategy also fosters a good deal of math talk as students converse using content specific vocabulary and mathematical thinking to arrive at a mutually agreed upon solution. As a teacher, I know that students at this age get distracted easily and will at times get off topic in their conversations so the students are given a specific time frame to talk to help minimize this.

Student as the Teacher

Another strategy that I use in my class is to have the student serve as the teacher. I do not use this strategy at the very beginning of the school year since the students are still trying to get acclimated to their new environment and roles as fourth graders. However, when I do implement the strategy, even the most reluctant student wants to participate. As the students get more comfortable with the flow of our class I slowly release the role of teacher to a student for a small task. During this role reversal, the student essentially becomes the teacher and I become one of the students. They must conduct themselves as the knowledgeable one in the classroom and guide the students to understanding and completing the given tasks. I take mental notes on which students are able to help guide the instruction, and I also play the role of the fearful student who simply does not understand what to do in the situation. As my questions progress in complexity, other students in the classroom join forces with the student teacher to help me understand the concept. The teamwork used to help me understand the concept is another benefit of the strategy because it is important to me that my students know that we are a family and that we are all here to learn and grow together.

Classroom Activities

To reinforce the content objectives presented, the students will complete several interactive activities in class. For me, these activities can serve as both informal and formal assessments. The activities can also be completed individually or in a small group. Based on my previous experience, I will have my students complete the first activity listed below in a small group format. I expect this to lend itself to a great deal of math talk which further promotes comprehension of the skill.

Toys R Math Activity

In this activity, the students will have the opportunity to shop from a teacher created toy store advertisement. The advertisement will be a combination of the most popular and not so popular toys that students in this age group enjoy. Students will use multiplication to finalize their purchases i.e. 2 Xbox One Controllers = $60 x 2 = $120 or 17 Zuru Fidget Cubes = 17 × $12 = $204. This activity can be differentiated to fit the abilities of the students. By specifying exactly how many of each toy a student must buy, the student can be supported with scaffolding. I will further allow the students to take a more open inquiry-based approach by giving them a specific budget and instructing them to see how close they can get without going over that budget. This approach provides enrichment for the students who need more of a challenge. This option forces the students to also review subtraction as they make their purchases. Once the students have finalized their purchases they must then write a check to cover the cost of their items. This seemingly small task allows the students to practice this realistic skill in a non-intimidating manner while simultaneously reviewing how to read, write, and identify numbers in both standard and word form. Writing the check also allows the students the opportunity to practice writing their signature, which seems to be a forgotten skill in this era. Once all groups have finalized their transactions, each group will then share with the whole class what they purchased and the method they used to generate a total.

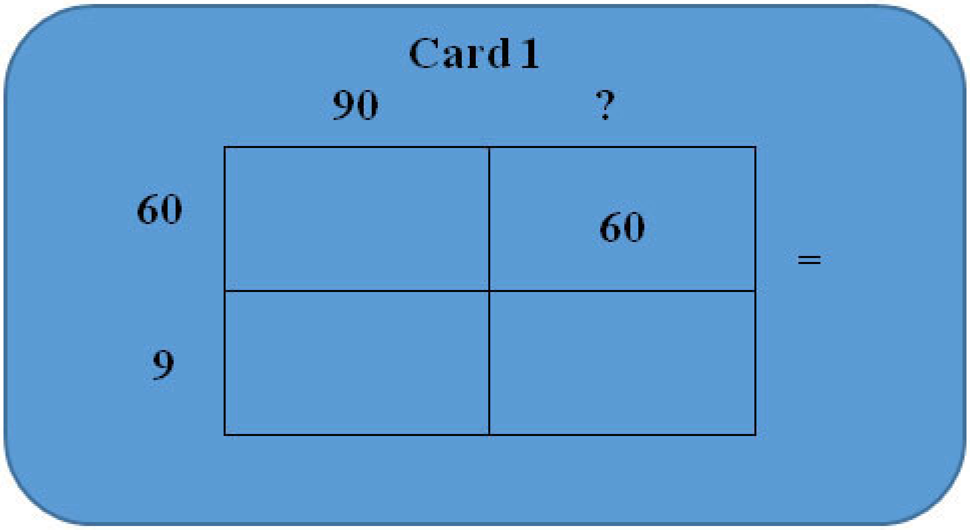
If I want to extend this activity to include review standards, I could ask the students to order the total amount spent by different groups from least to greatest or vice versa. The students could also add the totals that each group obtained to comprise a class total which could then be compared amongst the three classes that I teach.

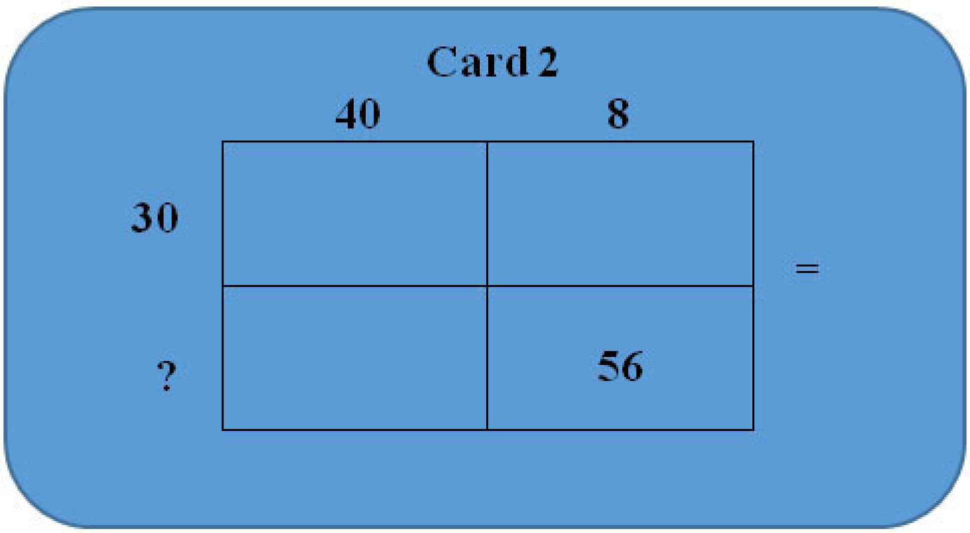
What's Missing?

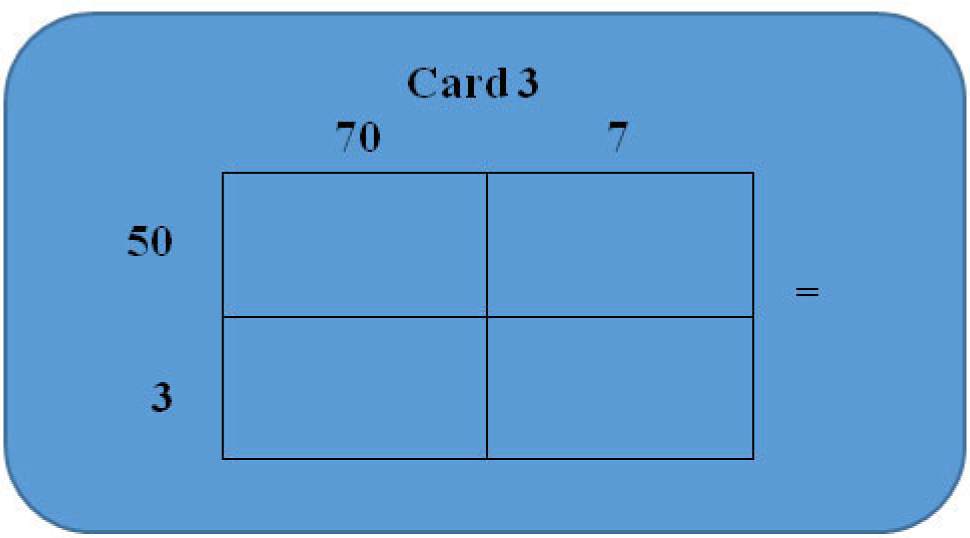
This activity is a scoot game. Scoot games are played with the whole class simultaneously in which the teacher will place a question card on each student's desk. To play, students move around the room, from desk to desk, solving the math problems. As the students rotate around the class they will have to either complete what is missing from the grid model and then solve the problem or find the missing partial product that completes the Box Method. (Examples of game cards are shown below.) Since the students are moving around the classroom this will be an individual activity. To allow the students the opportunity for more math talk (Think-Pair-Share Strategy), they can work in pairs or small groups depending on spacing which will limit the number of scoot cards that are distributed on each desk.

This activity, like the previous one, can be differentiated. When looking at the three types of cards below one can easily see that there are different aspects of each algorithm missing or the card requires you to carry out all of the steps of the Box Method to complete the card. The cards could be sorted in a way that a specific group or individual students only solve one type of problem consistently until confidence and understanding is achieved before implementing the other card types.

Sample Scoot Cards







Math Stories in a Box

In this activity the students will fold a blank sheet of paper into four equal boxes just as if they were drawing the Box Method for binomials in their interactive notebooks. In each box, the student will illustrate an aspect of a self-created multiplicative comparison word problem. The students will then swap story boxes within their small group and repeat the activity until each student has had an opportunity to share their story. To ensure accuracy and further make the connection that the Box Method and the standard algorithm are related, the students will use the Box Method to solve the word problem and the standard algorithm to check their work. The students will need to understand the row sum to partial product connection as illustrated in the example below. Upon completion, each group will choose one story to share with the whole class. The group must collectively teach the class (Students as the Teacher Strategy) how to solve the word problem using either the Box Method or standard algorithm.

9 × 4, 353 = 9 × 4,000 + 9 × 300 + 9 × 50 + 9 × 3 =

Box Method Example:

4,000	300	50	3	
4,000 × 9 = 36,000	300 × 9 = 2,700	50 × 9 = 450	3 × 9 = 27	9

36,000 + 2,700 + 450 + 27 =

30,000 + 2,000 + 400 + 20

+ 6,000 + 700 +50 + 7 =

30,000 + 8,000 + 1100 + 70 + 7 =

30,000 + 9,000 + 100 + 70 + 7 = 39,177

U.S. Standard Algorithm Example:

3 4 2

4353

× 9

39,177

The key observation here is that the product of a digit and a place value piece consists of at most two place value pieces, one of the same magnitude as the original piece, and one of one magnitude larger. The numbers in the first row are the partial products from the grids of the Box Method. In the second row, each number has been decomposed into a place value piece of the same magnitude as the part of the second factor it comes from and a place value piece of the next higher magnitude. The larger magnitude piece is above and the smaller one is below. It is as if the boxes have been partitioned in two sub-boxes, and the larger magnitude piece put on top, and the smaller one on the bottom. Next, to get the 3rd row, the place value pieces of the same order of magnitude are added. This is like adding the contents of the boxes along diagonals in the more finely partitioned box. As indicated in the 4th row, more regrouping took place before writing the final answer (on the right in the 4th row).

If you compare this with the standard method, you will notice that the amounts in the top boxes are the "carries", the numbers that get written as superscripts in the standard method, to be added to the product at the next place. Doing things like this and comparing might help students realize why, at each place, they should multiply first, and then add the carry to the product, rather than the other way around.

Since I will be sharing place value and multiplication picture books (tiered list of books provided in the resources section) with the students throughout this unit, having them create these mini stories will add to their ability to apply mathematics to connect both reading and writing. It also allows the students an opportunity to be creative with their illustrations, which at times can be a rarity for this grade level, especially in mathematics.

All of the activities above correspond with 2 digit by 2 digit multiplication as that is what the standards require for my students. However, each example can also be expanded on a larger scale with more digits later in the year for enrichment purposes.

Key Vocabulary

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algorithm: a step-by-step way to solve a problem

array: a way of displaying objects in equal groups

decompose: separating a number into 2 or more parts

factors: the numbers being multiplied to get a product

multiple: any number multiplied by a counting number

multiplicand: the number that gets multiplied

multiplication: the operation that gives the total number when joining equal groups

multiplicative comparison: compares amounts by asking or telling how many times one amount is than another

multiplier: the number the multiplicand is multiplied by

order of magnitude: number assigned to the ratio of two quantities, generally expressed in powers of ten

partial product: the product you get when you multiply by one digit at a time that makes up the parts of the total

product: the answer in a multiplication problem

Resources for Teachers and Students

Relevant Math Picture Books

Tier 1 Books: Illustrate the math concept only and does not have a direct story line

Building Blocks of Multiplication, Joseph Midthun and Samuel Hiti

The Best of Times, Greg Tang

The Grapes of Math, Greg Tang

Zero Is the Leaves on the Tree, Betsy Franco

How Much Is A Million? David Schwartz

A Million Dots, Mike Reed Andrew Clements

Zero, Is It Something? Is It Nothing? Claudia Zaslavsky

Big Numbers, Edward Packard

Now For My Next Number, Margaret Park

Tier 2 Books: Have an engaging plot that weaves in the math connection

Breakfast at Danny's Diner, Judith Stamper

7 x 9 = Trouble, Claudia Mills

Anno's Mysterious Multiplying Jar, Masaichiro and Mitsumasa Anno

Too Many Kangaroo Things to Do! Stuart Murphy

How Do You Count A Dozen Ducklings? Seon Chae

Amanda Bean's Amazing Dream, Cindy Neuchwander

Ten Times Better, Richard Michelson

The Multiplication Monster, Kimberly Gross

Zero the Hero, Joan Holub

A Place for Zero, Angeline Sparagna LoPresti

On Beyond a Million: An Amazing Math Journey, David Schwartz

Trouble With Monkeys, Lucy Ravish

The King's Commissioners, Cindy Neuchwander

Tier 3 Books: Not specifically math but can be used to make connections

City by Numbers, Stephen Johnson

Two Ways to Count to Ten, Ruby Dee

Sample Questions to Investigate

1. How many days old are you?
2. How many roles of toilet paper do we use in a school year?
3. How many cartons of milk does the school drink in a month?
4. How many miles would your school bus have traveled by Winter Break?
5. How many seconds old are you?
6. How long will the morning announcements have run by the end of the school year?
7. If we have recess for 15 minutes each day, how much outside time will you have enjoyed by Spring Break?
8. How many students will it take to stretch the length of the long hall?
9. How many pairs of shoes would it take to outline the perimeter of the classroom?
10. How many laptops would it take to reach the ceiling of the classroom?
11. If we all of the students in grades 2-5 played Minecraft for 60 minutes a week, how many minutes will the students play in all in one week? In six weeks?

Appendix: Implementing District Standards

This curriculum unit is designed to provide a guide for teaching 2 digit by 2 digit multiplication for fourth grade students. It follows the newly revised 2016 standards of learning presented in the curriculum framework that are specific for the state of Virginia.

Virginia Standards of Learning (2016)

4.4 The student will

1. demonstrate fluency with multiplication facts through 12 x 12, and the corresponding division facts;
2. estimate and determine sums, differences, and products of whole numbers;
3. create and solve single-step and multistep practical problems involving addition, subtraction, and multiplication, and single-step practical problems involving division with whole numbers.

The 2016 standards of learning were phased in during the 2017- 2018 school year, thus the third grade students were exposed to the following objectives as a foundation of learning for fourth grade.

3.4 The student will

1. represent multiplication and division through 10 × 10, using a variety of approaches and models;
2. create and solve single-step practical problems that involve multiplication and division through 10 x 10;

and
3. demonstrate fluency with multiplication facts of 0, 1, 2, 5, and 10; and
4. solve single-step practical problems involving multiplication of whole numbers, where one factor is 99 or less and the second factor is 5 or less.

It is important to know the level of exposure the students have prior to arriving in fourth grade as it helps to guide instruction.

Common Core Standards

Virginia is not specifically a common core state but the standards below can also be applied to this unit.

CCSS.MATH.CONTENT.3.OA.A.1 Interpret products of whole numbers, e.g., interpret 5 × 7 as the total number of objects in 5 groups of 7 objects each. For example, describe a context in which a total number of objects can be expressed as 5 × 7.

CCSS.MATH.CONTENT.3.OA.A.3 Use multiplication and division within 100 to solve word problems in situations involving equal groups, arrays, and measurement quantities, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem.

CCSS.MATH.CONTENT.3.OA.A.5 Apply properties of operations as strategies to multiply and divide.2 Examples: If 6 × 4 = 24 is known, then 4 × 6 = 24 is also known. (Commutative property of multiplication.) 3 × 5 × 2 can be found by 3 × 5 = 15, then 15 × 2 = 30, or by 5 × 2 = 10, then 3 × 10 = 30. (Associative property of multiplication.) Knowing that 8 × 5 = 40 and 8 × 2 = 16, one can find 8 × 7 as 8 × (5 + 2) = (8 × 5) + (8 × 2) = 40 + 16 = 56. (Distributive property.)

CCSS.MATH.CONTENT.3.OA.A.7 Fluently multiply and divide within 100, using strategies such as the relationship between multiplication and division (e.g., knowing that 8 × 5 = 40, one knows 40 ÷ 5 = 8) or properties of operations. By the end of Grade 3, know from memory all products of two one-digit numbers.

CCSS.MATH.CONTENT.4.OA.A.1 Interpret a multiplication equation as a comparison, e.g., interpret 35 = 5 × 7 as a statement that 35 is 5 times as many as 7 and 7 times as many as 5. Represent verbal statements of multiplicative comparisons as multiplication equations.

CCSS.MATH.CONTENT.4.OA.A.2 Multiply or divide to solve word problems involving multiplicative comparison, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem, distinguishing multiplicative comparison from additive comparison.

CCSS.MATH.CONTENT.4.OA.A.3 Solve multistep word problems posed with whole numbers and having whole-number answers using the four operations, including problems in which remainders must be interpreted. Represent these problems using equations with a letter standing for the unknown quantity. Assess the reasonableness of answers using mental computation and estimation strategies including rounding.

CCSS.MATH.CONTENT.4.OA.A.5 Multiply a whole number of up to four digits by a one-digit whole number, and multiply two two-digit numbers, using strategies based on place value and the properties of operations. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.

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