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# When Your Plan to Multiply Polynomials is FOILED

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> *"To a scholar, mathematics is music."*
>
> *- Amit Kalantri, Wealth of Words*

## Introduction

Many rising 8th grade scholars around the country will be given the opportunity to take Algebra I; some skipping 8th grade Math all together. That was my experience until last year. I advocated for scholars to receive a double block in Math to ensure those valuable concepts in 8th grade math was introduced. My administration team agreed and we were able to adjust their schedules to assure they didn't miss any valuable standards. But for many teachers around the county, that isn't the case. Many find themselves teaching 2 years' worth of math topics in 180 academic school days, and not surprisingly, there are just some students who can't handle the work load. In her book *Mathematical Mindsets* Jo Boaler states,

> Students were failing algebra not because algebra is so difficult, but because students don't have number sense, which is the foundation for algebra.[1]

This statement alone paints a picture of what numerous of Algebra teachers across the country experience time and time again. Moving from numerical concepts to algebraic ideas often opens the door for frustration and disconnection for many students. They simply do not understand or see how variables belong in math nor do they find much similarity between variables and numerical values. I intend on designing a curriculum unit that will highlight the relevance of applying some arithmetic concepts, especially rules for multiplication, to multiply polynomials, using area models and box models. I find that polynomials are the building blocks for algebraic thinking from the 8th grade and beyond. Being able to effectively and accurately manipulate polynomials will allow students to represent a variety of situations/scenarios algebraically. I plan on making significant connections between my previously written curriculum, *Closing Deficits Exponentially: Addressing*

*Base Ten & Small Numbers Using Exponents* and the rules for multiplication, in such a way that students begin to transfer their elementary math understandings to more complex algebraic ideas.

The goal is to take these elementary structures and use them as stepping stones to introduce operations with polynomials. In a lot of curriculum materials, they miss the mark on making these basic associations when using the same or similar process in a more advanced manner. I am hoping to use this unit to illustrate the similarities and ease in using strategies learned in elementary grades to serve as a link to Algebra concepts. As we move into the less traditional and conventional ways of doing Math, I will demonstrate various visual models that will provide my students with concrete examples. These examples will serve as a bridge for understanding how basic mathematical ideas can be utilized to conquer problems with a higher level of difficulty. I will use an array model, often known as the box method, to allow students to have a visual representation of how polynomials can be manipulated. By using the box method, I've witnessed fewer errors, and increased ability to keep track of the transformation each polynomial makes when an operation is applied.

By teaching my students to understand the process of using the box method for multiplying multi-digit numbers, and then having them advance to working with monomials, binomials, and trinomials, I hope to strengthen their understanding of not just polynomial operations but, the laws and properties of exponents, the behavior of the degree of polynomials under multiplication, area models, and certain geometric notions. Also, I hope they will come to appreciate the value of the Rules of Arithmetic (aka, Properties of the Operations). We will again explore the basis for the *5 Stages of Place Value*[2]to set up this practice, and use those general ideas to master the procedure.

Lastly, aside from the idea of doing the actual math, I hope to shift the mindsets of my scholars as they enter the last year of their middle school careers. I hope to get them excited and engaged about diving into algebraic thinking.

## Demographics

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This year I will remain at Johnson Middle School (JMS) as the 8th grade Math/Algebra I Teacher. With a current population of just under 300 scholars, 100% of the school population is eligible for Free and Reduced Lunch (FARMS) and live in crime infested neighborhoods. Johnson is located in Ward 8 of the southeast quadrant of the Nation's Capital, Washington, D.C. The school is located in a community that is ridden by violence, an overwhelmingly amount of poverty, and students who experience daily traumas. Of the rising 8th grade class of 105 scholars, 25 of them have Individualized Educational Plans (IEP) and specific academic accommodations outlined in each. We have a truancy rate of barely under 50%, and 45% of students are classified as chronically absent. Our scholars carry a backpack of trauma and often times are depleted upon arriving to school. Recently, there have been tremendous shifts in the academic culture at JMS, and despite all of these obstacles, scholars have continuously risen to the occasion, with increasing the academic gains each year. With various incentives and an administrative team that has made student achievement its number one priority, JMS has continued to strive.

## Objectives

This unit is intended to be a one to two-week (5 -10 instructional days) unit based on scheduling, pacing calendars etc. I will not start this unit until my scholars show a solid understanding with the 5 stages of place value and the law of exponents. I would like to increase my students' fluency, precision and solid understanding of polynomial operations using area models for multiplication. I would like them to demonstrate multiplying polynomials using the box method as a way to classify the various terms and degrees of polynomial (magnitude of exponent). My hope is that, by reintroducing the associative and commutative rules of multiplication, I will enable my students to successfully find area and surface area of geometric figures.

I will be introducing this unit in parallel with Algebra I Module 1 Topic B: Structure of Expressions, in the curriculum as presented in the Eureka/EngageNY curriculum instructional materials. By focusing on multiplying polynomials using the box method and justified by both the associative and commutative rules, scholars will gain a strong grasp on polynomial operations, and solving problems involving them, specifically area models. All standards in my unit are compiled of grade level Common Core State Standards (CCSS) as noted in the Appendices, and will follow the pacing calendar as provided by the District of Columbia Public Schools. This unit comprises three topics:

i. the area model for multiplication,
ii. the box method using exponential notation (*5$^{th}$ stage of the five stages of place value*),
iii. geometric illustrations of the associative and commutative rules for multiplication.

There are 4 overall objectives that I would like my scholars to master, they are as follows:

1. Justify how the associative and commutative rules prove their calculations of products of polynomials to be correct.
2. Demonstrate multiplying polynomials using area models and the box method.
3. Find the area and perimeter of a rectangle with polynomial side lengths.
4. Create an area model using algebra tiles and create an equation involving the appropriate polynomial operations.

## Rationale

Over the years our students have been conditioned to look for short-cuts when trying to find a solution or solutions to math problems. There was one "shortcut" that my students attempted to teach me when they had to multiply multi-digit numbers. This shortcut was in fact the *lattice method*, sometimes known as the *gelosia* method of multiplication or the box method. Originating in India, the gelosia strategy appeared in various Hindu works, where they called the method *quadrilateral*. As this method became popular in other places, it's taken on other names such as: *the method of the sieve, method of the net, shabakh, and/or venetian squares.*

The name was given in Italy as it resembled a grid or lattices that were added to the windows of popular women to protect them from public view. By the time Fibonacci and Europeans were introduced and familiar with the method, it was not practiced as much due its difficulty in being printed. [3]

The diagonals in the lattice method represent the various digit place values. Unlike the lattice method the box method offers a very streamlined way of identifying the place value of each unique part. In fact, the lattice method comes from the box method but is very rarely explained.

In the calculation illustrated below, you will see the common schemes in both methods.

$$
\begin{array}{lllllll}
\mathbf{(345 \times 12)} & = & \underset{(3000)}{300 \times 10} & + & \underset{(400)}{40 \times 10} & + & \underset{(50)}{5 \times 10} \\
 & + & & & & & \\
 & & \underset{(600)}{300 \times 2} & + & \underset{(80)}{40 \times 2} & + & \underset{(10)}{5 \times 2} \\
\hline
\end{array}
$$

$$
= 3000 + (400 + 600) + (50 + 10 + 80) \; = \; 3000 + 1000 + 140 \; = \; 4000 + 140 \; = \; \mathbf{4140}
$$

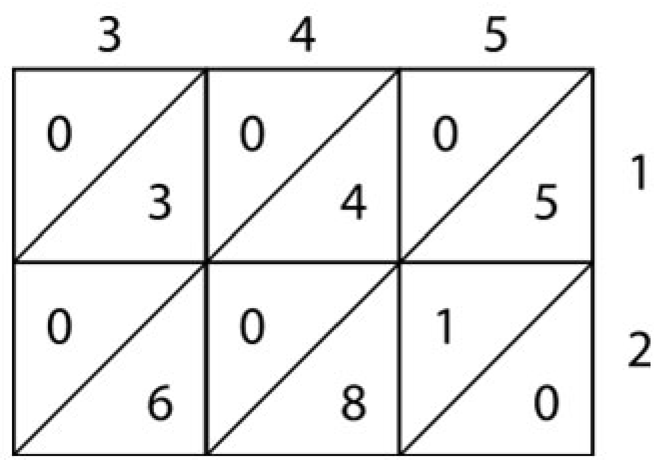


Figure 1: The Lattice (Gelosia Method) Box of Multiplication

Here is the basic part of the lattice method for 345´12. The antidiagonals, from the lower right to the upper left, represent the place values, from ones to ten thousand. To complete the calculation, sum along each antidiagonal, and carry the tens part of the sum to the next place.

| 300 | 40 | 5 | |
|---|---|---|---|
| 3000 | 400 | 50 | 10 |
| 600 | 80 | 10 | 2 |

Figure 2: The Traditional *Box Method*

Here you can see that each factor was decomposed into its place value parts (300, 40 and 5 for 345, and 10 and 2 for 12), and each place value part of one factor was multiplied by each place value part of the other factor. This is justified by way of an extended version of the distributive rule of multiplication. I will discuss this more general version of the distributive rule with my students, as it forms a key part of the justification of the box method. It is a natural consequence of the relationship between multiplication and area.

When I asked my students what the lattice method was, they could only show me an example of how it was used. The funny thing is, they could use the strategy to apply the rules of multi-digit multiplication; however, they could not explain the make-up or characteristics of the box used or answer why the method worked. “When we do not ask students to think visually about the growth of the shape, they do not have access to important understandings about functional growth.”[4] Having seen my kids using the lattice method and almost always multiplying multi digits with accuracy, I decided to capitalize on my opportunity to use similar methods with multiplying polynomials; this time justifying the procedures using basic area models.

## Unit Content

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*Polynomial:* Is an algebraic expression consisting of a sum of two or more monomials. A monomial is a product of a “coefficient” or “constant”, and a power of a variable. The coefficient may be a specific number, or be denoted by a letter from the beginning of the alphabet: a, b, c, etc. Even when the coefficient is designated by a letter, it is considered a “constant”, which means it is treated as being known for purposes of the calculation to be done. A variable is a number that is not known, at least at the start of the problem, but which can be taken from some collection of numbers (whole numbers, integers, rational numbers, real numbers, etc., etc.). A variable is usually denoted by a letter from the end of the alphabet: x, y, z, etc. A power of a variable is the product of a certain number of copies of the variable, with the number of copies indicated by an exponent. Thus $x^2 = x ´ x$, and $x^3 = x ´ x ´ x$, and so forth. The fact that these products do not depend on how the copies of the variable are multiplied together, only on the total number of copies, is fundamental, and is a consequence of the Associative Rule for Multiplication. This will be discussed in further detail below. Polynomials can be combined using the basic mathematical operations, addition, subtraction, and multiplication. Dividing one polynomial by another usually results in a more general type of expression, often called a rational function. For this reason, we will avoid division of polynomials in this unit.

A few of the commonly used polynomials are:

*Monomial:* By monomial, we mean a product of $\boldsymbol{ax^n}$, where a indicates *a* constant (any known number), *x* is a variable, and $x^n$ is a power of *x,* and where n =  0 is a non-negative whole number.

For example, the monomial $2x^3$ , is interpreted as 2 copies of *x* to the 3rd power.

*Binomial:* polynomial consisting of two monomial terms

*Trinomial:* polynomial consisting of 3 monomial terms

The exponent of $x$ in a monomial $ax^n$ is called a *degree*. The largest degree of the monomials in a polynomial is called the *order* of the polynomial. Order allows you to identify the type of parent function or polynomial you are working with. In the 8th grade we are only required to work with polynomials of the third order or less in a variety of ways. While there are higher order polynomials, they are often very difficult to work with and they are often unable to be factored effectively once they reach the fifth order ($x^5$).

$3x^2 + 4x - 2$

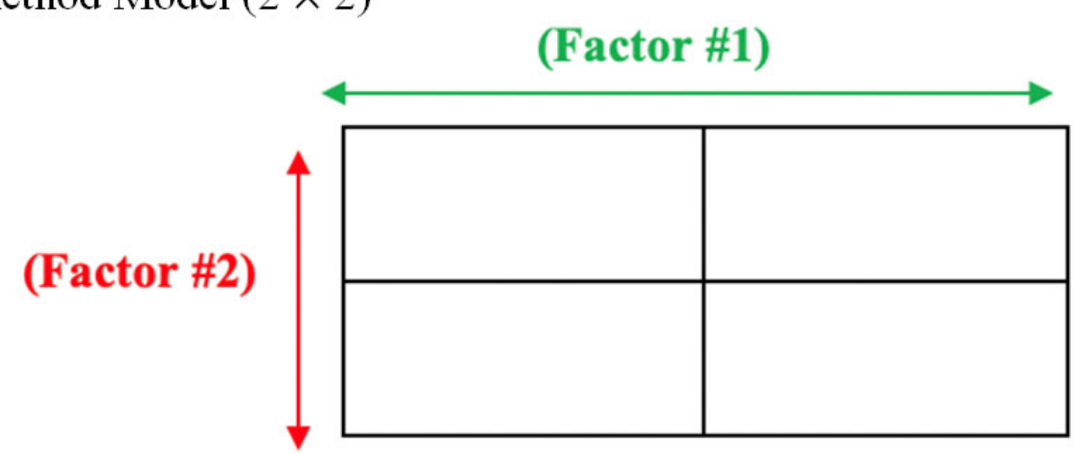
The above polynomial is an example of a trinomial; containing 3 terms of degrees 0, 1, and 2. The highest degree is 2, so this is an order 2 polynomial, often also called a quadratic polynomial. There are a few other specific conditions for defining a polynomial that are not important in this unit but can be mentioned, for example: all exponents must be non-negative whole numbers (including zero).

*Area Model:* An area model represents a multiplication by the area of a rectangle, subdivided into sub-rectangles whose areas are the products of the place value pieces of the factors. The area model provides an effective method for helping students understand multiplication of two-digit numbers, but beyond that, it becomes rather unwieldy and not really practical. However, a symbolic version of the area model is very useful for understanding the extended distributive rule, and I will discuss this symbolic version with my students.

*The Box Method:* A modification of the area model often known as *the box method,* is a rectangular model used in math for multiplication. In the box method, one makes an array, with one column for each place value part of the first factor, and one row for each place value part of the second factor. Then in the box corresponding to a given pair of place value parts, one from each factor, one puts the product of the two parts. The whole product is found by summing all these products. The lattice method can be seen as a slightly refined and more automated, but less transparent, version of the box method.

Figure 3: Box Method Model (2 × 2)

Figure 3: Box Method Model (2 × 2)




We will give examples of the Box Method below.

## Area Models with Whole Numbers

I will introduce polynomial multiplication with my students by first getting them acclimated with multiplying whole numbers using the box method. We will study multiplication of two-digit numbers using the area model, then transition to the box method for multiplication when one (or both) factor has three or more digits. The area model is a great way of illustrating to students how quickly the value represented by each digit increases when moving to the left, place by place, and to help them visualize how each part of the number contributes to the overall product. I will emphasize to my students to make notes and observations as the powers of ten increase. I would explicitly like my students to observe the relationships within the antidiagonals & increasing powers of ten using the box method. As they transition to using this method with variables, I want them to notice how those general ideas around the powers of ten are justified the same way, based on the rules of arithmetic.

These models will be used in rich discussions and activities as the unit progresses, there are a few explicit examples below.

One by Two (1 × 2) Digit Multiplication

In the examples below, the area model is set up by having one row and two columns. This model is consistent with the number of digits/powers of ten that are available in the two factors being multiplied. Each factor is broken into its respective place value parts, which are and then multiplied by the place value parts of the other factor, one place value part at a time. This process is repeated until all parts have been multiplied. The final product is then determined by the sum of all parts. In the first three examples, the left-hand factor is a single digit, so has only one place value part.

Example 1: $(5 \times 12) = 10 + 2$

| | 10 | 2 | |
|---|---|---|---|
| 5 | $(5 \times 10)$ **50** | $(5 \times 2)$ **10** | $= 50 + 10$ $=$ **60** |

Example 2: $(6 \times 13) = 10 + 3$

| | 10 | 3 | |
|---|---|---|---|
| 6 | $(6 \times 10)$ **60** | $(6 \times 3)$ **18** | $= 60 + (10 + 8)$ $= (60 + 10) + 8 =$ **78** |

Example 3: $(7 \times 132) = 100 + 30 + 2$

| | 100 | 30 | 2 | |
|---|---|---|---|---|
| 7 | $(7 \times 100)$ **700** | $(7 \times 30)$ **210** | $(7 \times 2)$ **14** | $= 700 + 210 + 14$ $= 910 + 14 =$ **924** |

In the above examples you can see how the product of the factors in the ones place, play a significant role in the final result. Students will be able to visualize how quickly and easily the powers of tens can increase using this box method.

Two by Two (2 × 2) Digit Multiplication

In order to introduce the box method of multi-digit numbers, I will use the basic area model in detail for my students. Using the area model in parallel with the box method will allow students to see how the box method is the shorthand way of multiplying multi-digit numbers. The following examples show a similar process with factors that both have more than one digit. This is visually represented by adding an additional row to allow for the additional digit in the first (red) factor.

Example 4: $(13 \times 22) = 20 + 2$

| | 20 | 2 |
|---|---|---|
| 10 | $(10 \times 20)$ **200** | $(10 \times 2)$ **20** |
| + 3 | $(3 \times 20)$ **60** | $(3 \times 2)$ **6** |

$= 200 + 60 + 20 + 6$
$= 200 + 80 + 6$
$=$ **286**

Example 5: $(14 \times 62) = 60 + 2$

| | 60 | 2 |
|---|---|---|
| 10 | **600** | **20** |
| + 4 | **240** | **8** |

$= 600 + 240 + 20 + 8$
$= 600 + 260 + 8$
**= 868**

I will ask students what observations did they make as they moved to multiplying two-digit factors? How did the area model/box change? What do you notice about the products inside each box? I’m looking for students to make connections with the powers of ten increasing as you move up the columns, or move to the left in the rows. I hope that they will also observe that the same powers of ten appear along the antidiagonals from each other (as illustrated in the example below).

Three by Three (3 × 3) Digit Multiplication

As with the previous examples of box models, as the number of digits/place value parts in one of the factors increases, so does the number of rows or columns of the box. This model also offers a very helpful view of numbers with the same order of magnitude being directly diagonal to one another. I would use this example as an opportunity to really have a deep discussion as to how this way of multiplication allows for such organization.

Can you recall a time where you identified something similar? I’d point out to my students that in fact looking at the basic multiplication chart offers such a pattern also. Due to the associative and commutative rules, the products of the place value parts are just the products of the base ten parts times the products of the digits.

Example 6: (112 × 222) = 200 + 20 + 2

| | 200 + | 20 + | 2 |
|---|---|---|---|
| 100 + | **20000** | **2000** | **200** |
| 10 + | **2000** | **200** | **20** |
| 2 | **400** | **40** | **4** |

= 20000 + 2000 + 2000 + 400 + 200 + 200 + 40 + 20 + 4
= 20000 + 4000 + 800 + 60 + 4
**= 24864**

I would like to take a moment to point out that with this example in particular you can really begin to introduce the concepts of polynomial multiplication by way the of the fourth stage of the *5 Stages of Place Value*. Below I will take a moment to demonstrate with my students the possibility to connect the process of multiplying whole numbers with multiplying in polynomial form.

Example 6a. (112 × 222) = 200 + 20 + 2

| | 200 + | 20 + | 2 |
|---|---|---|---|
| 100 + | **(1×2) × 10000** | **(1×2) × 1000** | **(1×2) × 100** |
| 10 + | **(1×2) × 1000** | **(1×2) × 100** | **(1×2) × 10** |
| 2 | **(2×2) × 100** | **(2×2) × 10** | **(2×2) × 1** |

= $(2 \times 10000) + (2 \times 1000) + (2 \times 1000) + (4 \times 100) + (2 \times 100) + (2 \times 100) + (4 \times 10) + (2 \times 10) + (4 \times 1)$

= $(2 \times 10000) + (4 \times 1000) + (8 \times 100) + (6 \times 10) + (4 \times 1)$

= $20000 + 4000 + 800 + 60 + 4$

= **24864**

Example 6b. (112 × 222) = $((1 \times 10^2 + 1 \times 10^1 + 2 \times 10^0) \times (2 \times 10^2 + 2 \times 10^1 + 2 \times 10^0))$ =

| | $2\times10^2$ + | $2\times10^1$ + | $2\times10^0$ |
|---|---|---|---|
| $1\times10^2$ + | $(2 \times 10^4)$ | $(2 \times 10^3)$ | $(2 \times 10^2)$ |
| $1\times10^1$ + | $(2 \times 10^3)$ | $(2 \times 10^2)$ | $(2 \times 10^1)$ |
| $2\times10^0$ | $(4 \times 10^2)$ | $(4 \times 10^1)$ | $(4 \times 10^0)$ |

= $(2 \times 10^4) + (2 \times 10^3) + (2 \times 10^3) + (4 \times 10^2) + (2 \times 10^2) + (2 \times 10^2) + (4 \times 10^1) + (2 \times 10^1) + (4 \times 10^0)$

= $(2 \times 10^4) + (4 \times 10^3) + (8 \times 10^2) + (6 \times 10^1) + (4 \times 10^0)$

= $\mathbf{(2.4864 \times 10^4) = 24864}$

## Basic Arithmetic Rules of Addition & Multiplication

### Basic Rules for Addition

The process of using the array method turns out quite nicely due to some of the basic arithmetic rules for addition and multiplication. There are 4 basic rules of each operation (addition or multiplication) in the system of real numbers, explicitly: the commutative, associative, identity and inverse rules for addition or of multiplication in the set of real numbers. In addition, the distributive rule connects the two operations. In this unit we will look closely at how the distributive (or its more general version, the extended distributive rule), commutative and associative rules justify the use of area models/box method in both numerical and polynomial form.

Before we move into the rules of multiplication, it is necessary to briefly discuss the rules of addition in order to solidify the ideas of multiplication in this unit.

**The Commutative Rule of Addition:** the sum of two addends will be the same no matter the order of the addends.

$a + b = b + a$

$37 + 25 = 25 + 37$

$\mathbf{62 = 62}$

**The Associative Rule of Addition:** the sum of three non-negative integers, does not depend on the combination of numbers.

$(a + b) + c = a + (b + c)$

$(42 + 37) + 25 = 42 + (37 + 25)$

$79 + 25 = 42 + 62$

$\mathbf{104 = 104}$

**The Any Which Way Rule**

By repeated application of the associative rule and the commutative rule, we can justify a much more general and flexible rule that is usable in a wide variety of practical calculations. We will refer to this rule as The Any Which Way Rule. This rule states that, when you are adding a list of numbers, no matter how you combine the numbers or the order that you add the numbers you will yield the same sum.

The Any Which Way Rule:

$22 + 14 + 37 + 53 = (22 + 37) + (14 + 53) = (22 + 53) + (14 + 37) = \mathbf{126}$

The Any Which Way Rule also works for multiplication, as it can be justified by the associative and commutative rules for multiplication, which will be displayed below.

**The Distributive Rule**

The distributive rule is a rule that students are familiar with as it tells us to multiply a number by the sum of two numbers. It is the *only* rule of the operations that involves both addition and multiplication.

**The Distributive Rule:** For non-negative integers, $a$, $b$, and $c$

$a \times (b + c) = (a \times b) + (a \times c)$

The distributive rule can be represented by using an array model or an area model. We can demonstrate this when a nonnegative integer is represented by a horizontal row and through multiplication, we reproduce the row a specific number of times.

Example #7:

**The Distributive Rule:**

5 × (3 + 4)

let *a, b, c* be defined as *a* = 5, *b* = 3, *c* = 4

(5 × 3)+(5 × 4) = (15) + (20) = 35

In the above example I demonstrate how the distributive rule is applied using single digit numbers. It is then demonstrated using an area model below.

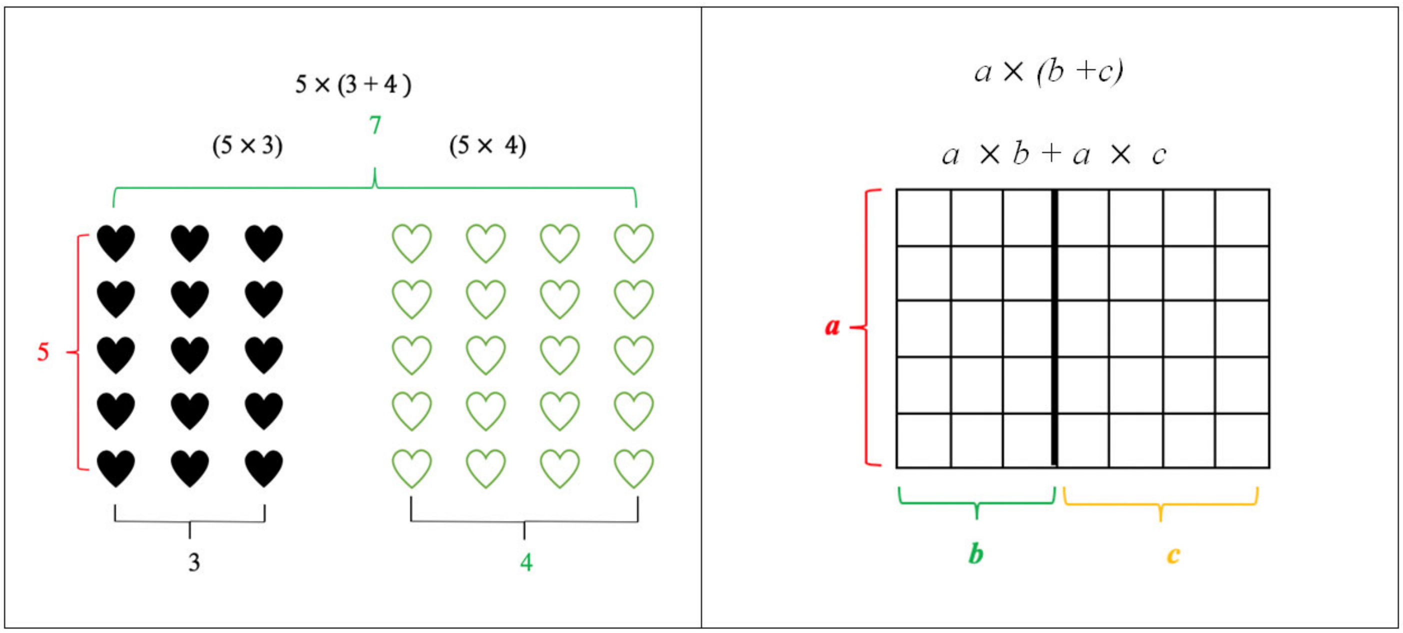



Figure 4: The Distributive Rule as an Array Model & Schematic Model

The above array model shows that this model is very similar to the area model. However, the area model offers a more schematic way of representing the multiplication of products. Using the array model to demonstrate the distributive rule is a great way for students to see how multiplying one number by two partial products will yield the same result no matter how they numbers are multiplied.

## The Associative and Commutative Rules for Multiplication

**The Associative Rule for Multiplication**

Both addition and multiplication satisfy the associative rule. It states that, when adding or when multiplying, you can arrive at the correct sum or product no matter how you group your numbers. To effectively apply this rule, parentheses are used to group the numbers to demonstrate which numbers are added/multiplied first. The associative rule for multiplication can be justified by considering the volume of boxes (i.e. rectangular parallelepipeds).

**The Associative Rule for Multiplication:** For non-negative integers, *a*, *b*, and *c*

$a \times (b \times c) = (a \times b) \times c$

Example #8:

**The Associative Rule for Multiplication:**
$(5 \times 7 \times 4)$
let *a, b, c* be defined as $a = 5$, $b = 7$, $c = 4$
$5 \times (7 \times 4) = 5 \times (28) = 140$ $(5 \times 7) \times 4 = (35) \times 4 = 140$

**TheLaw of Exponents**: for any whole numbers a and b, we have the equality

$x^a \bullet x^b = x^{a+b}$

The argument for this is similar to example 9b, although more complicated.

Example #9a:

**The Associative Rule for Multiplication:**

$(x \bullet x^2)$

$(x \bullet x^2) = x\,(x \bullet x) = (x \bullet x)\,x = x^2\,x = x^3$

Example #9b:

$(x \bullet x^3)$

$(x \bullet x^3) = x(x \bullet x \bullet x) = (x \bullet x) \bullet (x \bullet x) = x^2\,x^2 = x^2 \bullet(x \bullet x)$

$= (x^2 \bullet x) \bullet x = x^3 \bullet x = x^4$

## The Commutative Rule of Multiplication

The commutative rule is a rule of addition, and also of multiplication. For multiplication, it states that you can multiply 2 factors in either order without affecting the product.

Probably the best way to develop an intuitive sense for the rule is through the Array Model. This model is also important because it helps prepare students develop intuition of the concept of area.[5]

The best way to visualize the commutative rule is thinking of taking an array model where two factors are being multiplied, and reflecting the array over its diagonal (or rotating it by $90^\circ$). Then the size of the array, which equals the product of both factors, remains the same.  The same can be done when representing multiplication by the area of a rectangle.

**The Commutative Rule of Multiplication:** For non-negative integers, *a and b*

$a \times b = b \times a$

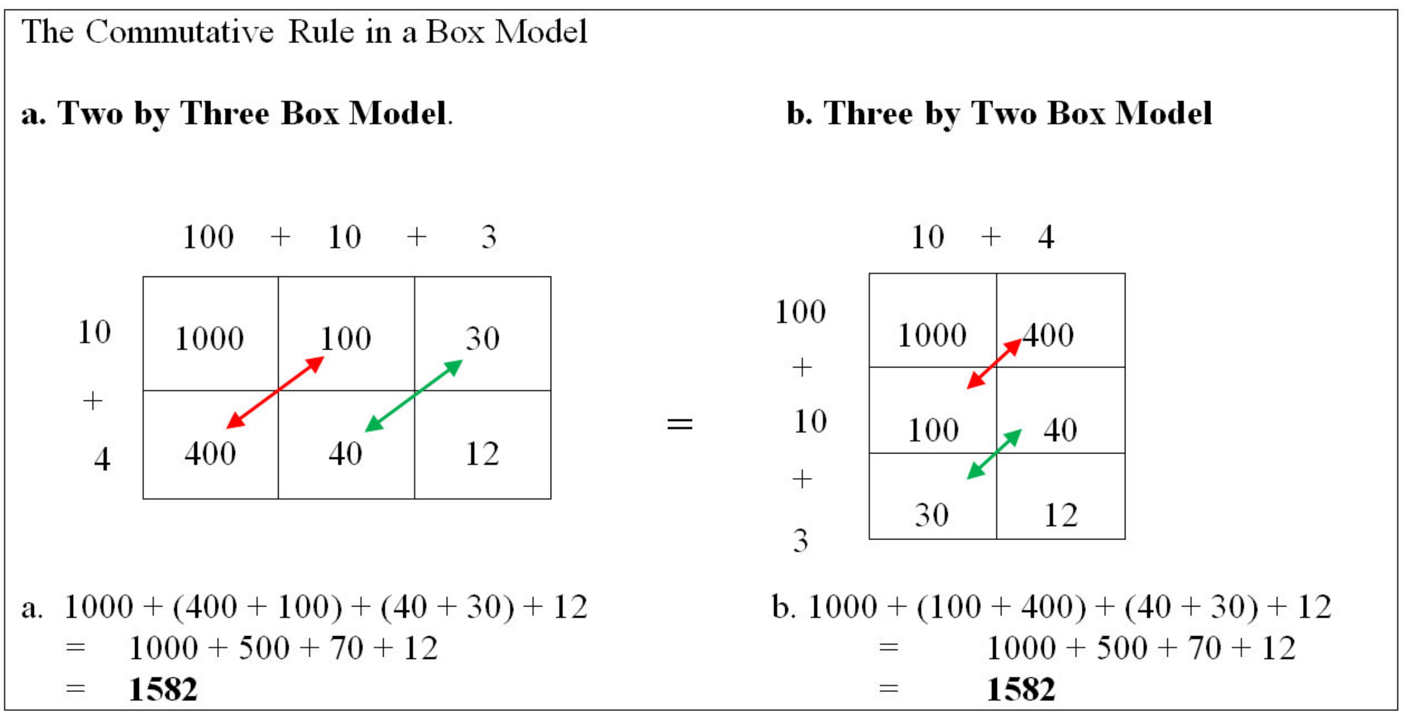
Example #10:

**The Commutative Rule of Multiplication:** For non-negative integers, *a and b*

14 × 113

let *a, and b* be defined as *a* = 14, *b* = 113

14  × 113 = 1582 113× 14 = 1582




Above, observe that I am using the extended distributive rule to evaluate each product. It should be emphasized that the extended distributive rule is consistent with the commutative rules of multiplication and addition. It illustrates that by applying the arithmetic rules of multiplying using the commutative rule, you will obtain the same product regardless of the order of the numbers being multiplied. The same sums of products can be seen in both calculations a and b above.

## The Extended Distributive Rule

The extended distributive rule is derived from the basic distributive rule (for either factor), plus multiple uses of the commutative and associative rules for addition. In discussing the extended distributive rule, the symbols for multiplication (× and •) will be removed; the numbers/quantities that are to be multiplied are placed side-by-side.

**The Extended Distributive Rule**

A and B = sums of some numbers

AB = sum of (any addend of A) ×(any addend of B)

The final product will be determined by adding all the resulting products of AB

We can call this rule *Each with Each (EwE)*: you multiply **each** addend of A with **each** addend of B, and sum of all products.

*Let A and B be defined as,*

*A = (a + b) and B = (c + d)*

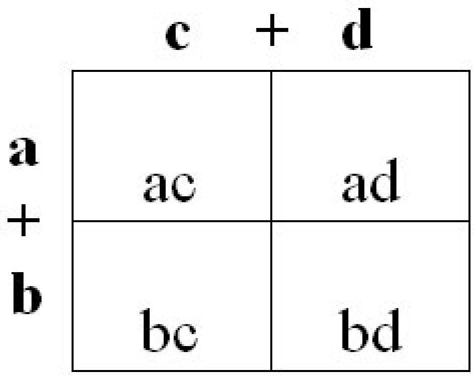
| | |
|---|---|
| **AB** = (a + b) (c + d), | substitution |
| = (ac) + (ad) | first a is multiplied by c and d |
| + (bc) + (bd) | then b is multiplied by c and d |
| = **ac + ad + bc + bd** | the sums of all products are added together |

Note that this gives exactly the same sum as FOIL. However, it is valid for sums with any number of addends, while FOIL applies only to products of binomials.

Example #11:

*As an Area Model: Multiplying a sum of 2 variables by a sum of 2 variables*

*A = (a + b) and B = (c + d)*



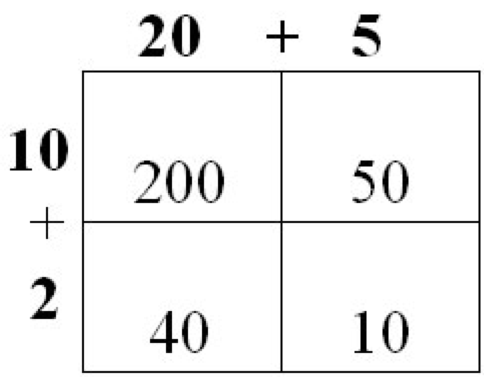

= **(**ac) + (ad) + (bc) + (bd) = **ac + ad + bc + b**

Example #12:

*Area Model with Numerical Values:  Multiplying a sum of 2 numbers by of 2 numbers*

***(12)(25)***

*12 = (10 + 2) and 25 = (20 + 5)*




= 200 + (50 + 40) + 10 = 200 + 90 + 10 = 200 + 100 = **300**

Due to the basic arithmetic ideas of the distributive, commutative, and associative rules, numbers are able to be written in any way (as long as the place value is maintained), while arriving at the same product. In the numerical example above, you can see that the products of the various place value parts are just the products of the powers of ten times the products of the digits. To better illustrate that statement, here is a visual representation below:

Example #13:

***(25)(12)***

$(2.5 \times 10^{1})(1.2 \times 10^{1}) = (2.5 \times 1.2)(10^{1} \times 10^{1}) = (3) \times (10^{2})$ = **300**

This can be justified because of the Product Rule of Exponents as defined as: “The powers of 10 stand in a multiplicative relationship to each other. If we multiply one of them by 10, we get the next one. ***Product rule for exponents***: $\mathbf{10^{m} \times 10^{n} = 10^{m+n}}$, for any whole numbers *m* and *n*.” [4] The Law of Exponents is not required to be explored in depth in this unit as it will be previously explored and taught prior to the start of this unit, as it governs how powers of the variable are multiplied together. However, there are vital ideas as it relates to the quantities of exponents. These ideas will be investigated prior to this unit to ensure the background knowledge is acquired that will allow students to accurately apply the operations of polynomials.

## Teaching Strategies

I want to offer my students the opportunity to discover multiplying polynomials in a different and self-guided way. By exploring the multiplication of polynomials using the box method and area models, students will be able to execute area, volume and perimeter problems with polynomial side lengths. I will use a number of teaching strategies to introduce and reinforce the process of multiplying polynomials. It is important to note that the above strategies (area and box models, distributive, associative, and commutative rules) will be formally reviewed with students using numerical models first to make connections to previously learned strategies. Using these principles while emphasizing the difference in whole numbers, base ten and polynomials, will let students use visual representation of polynomial multiplication. It is suggested that tasks assigned to students should be meaningful, accessible, and relevant in a number of ways. According to Jo Boaler, these tasks should "…open tasks and make them broader- when we make them what I refer to as 'low floor, high ceiling'- this becomes possible for all students."[6]

**Number Talks**

A number talk is a classroom conversation that typically ranges from 5 – 15 minutes; although in my experience number talks can be extended throughout the lesson to deepen content knowledge and understanding. These conversations are carefully crafted and structured in a way that each one answers specific questions and collects precise information from students. Number talks will be designed to be flexible to tap into learners of all modalities and academic abilities. Number talks should be student centered and presented in a classroom that welcomes productive student discourse. It is essential that students are familiar with the classroom norms and experience them consistently in order for them to be implemented effectively.

Regarding the topic of multiplying polynomials, there are a few observations that I would like my students to make at specific points in the unit. The main takeaways are that students should be able to justify that the associative and commutative rules for multiplication, in combination with the distributive property connecting multiplication with addition, allow for polynomials to be multiplied in different ways, yet all these ways result in the same product.

**Kagan's Cooperative Strategies**

I like to promote community and a cooperative learning environment in my classroom. When using traditional math teaching strategies, I find that engagement is limited for many of my students. Many problems are simply not accessible and many students cannot make connections with what they have previously learned to new ideas. Kagan[7] has offered my students the opportunity to find their place in activities and allow them to feel comfortable doing the math. My activities will be carefully structured to incorporate one or more Kagan Strategy.

## Activities

**Side – By – Side Analysis/ “And then what”**

Objective: Justify how the associative and commutative rules prove their calculations of products of polynomials to be correct. Demonstrate multiplying polynomials using area models and the box method.

Days Covered:  3–4 Days / 6-8 Rotation Schedule

Using this activity will provide parallel learning and exploration for students. This strategy is intended to be used in the beginning of the unit. Before formally introducing the idea of multiplying polynomials but after reviewing area models with numerical values.

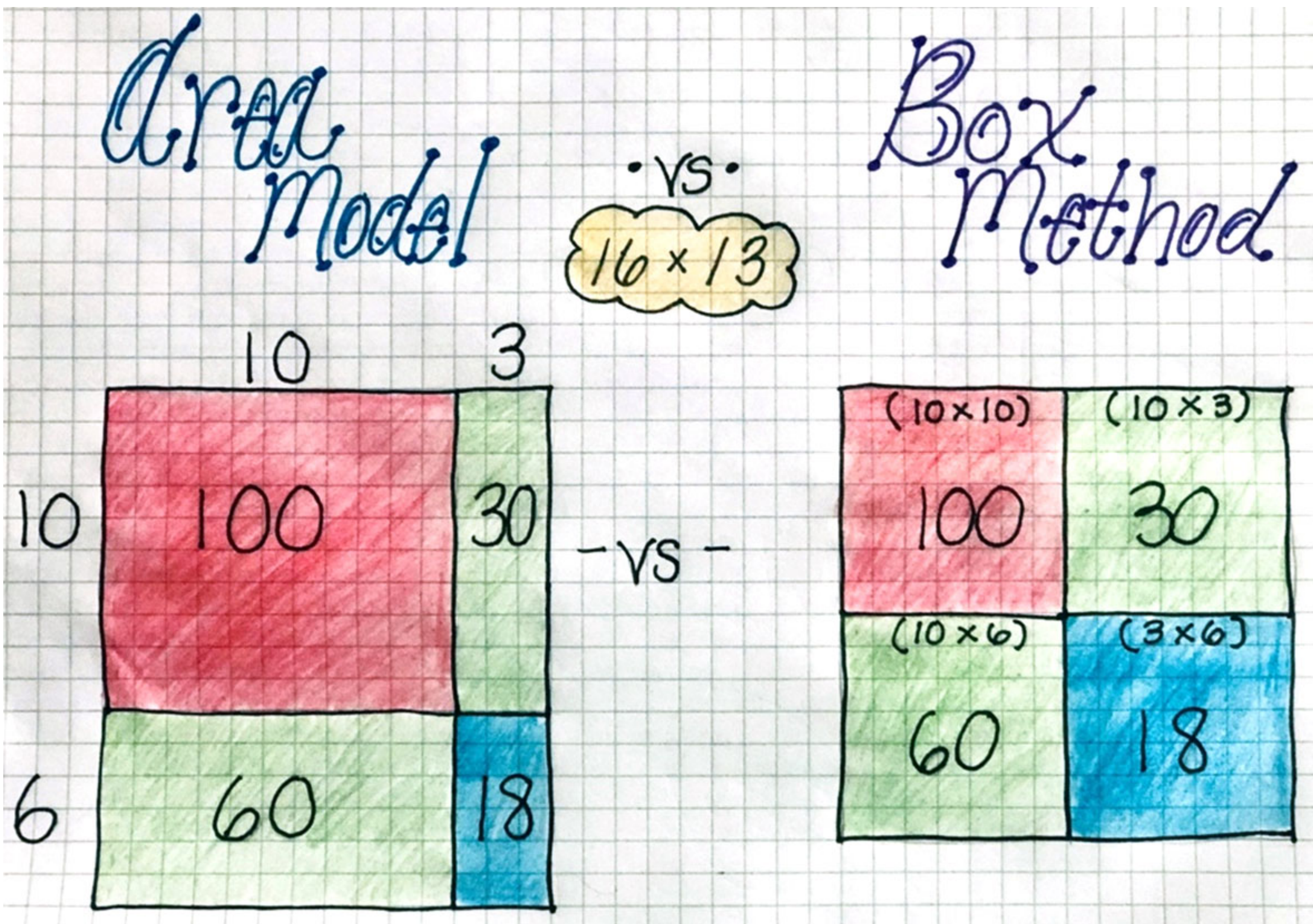



Figure 5: Side by Side Anchor Chart for Student Reference

These activities will allow scholars to see the relationship with the place value/base ten parts and the *x* or other variables used to represent each polynomial. There will also require some time for structure math talk/discourse however, conversation will be concentrated on solidifying the general principles of arithmetic: associative, commutative and Any Which Way rules of multiplication, the Law of Exponents, and the distributive and extended distributive rules. The side by side examples should be scaffolded in a number of ways that allow students to make comparisons and begin justifications. These examples should be organized in a way that explicitly demonstrates that the same multiplicative rules apply for polynomials. Each rule is valid for all numbers and therefore for variables, since variables simply represent numbers. It is imperative that students are familiar with the Law of Exponents, particularly the Product Rule, to guarantee student understanding and accuracy with each task.

**Multiplicative Rules with Polynomials**

**The Distributive Rule of Multiplication:** For non-negative integers, $a$, $b$, and $c$

$a \times (b + c) = (a \times b) + (a \times c)$

let $a, b, c$ be defined as $a = 2$, $b = x$, $c = 2$

$2\ (x + 2)$

$(2 \bullet x) + (2 \bullet 2) = (2x) + (2 \bullet 2) = \mathbf{2x + 4}$

**The Associative Rule of Multiplication:** real numbers, $a$, $b$, and $c$

$a \times (b \times c) = (a \times b) \times c$

let $a, b, c$ be defined as $a = 2$, $b = 3$, $c = x$

$2 \bullet (3 \bullet x) = (2 \bullet 3) \bullet x = \mathbf{6x}$

It should be noted that when thinking of the associative rule of multiplication, we could look at the rule as a symbolic way of saying that we can compute the volume of a cube as base times height, or as length times area of side. The associative and commutative rules often get used together to justify the *any which way rule*.

**The Commutative Rule of Multiplication:** For non-negative integers, *a and b*

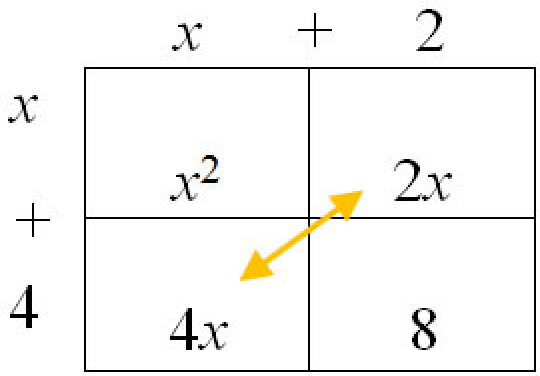
$a \times b = b \times a$

let $a, b$ be defined as $a = x$, $b = 3$

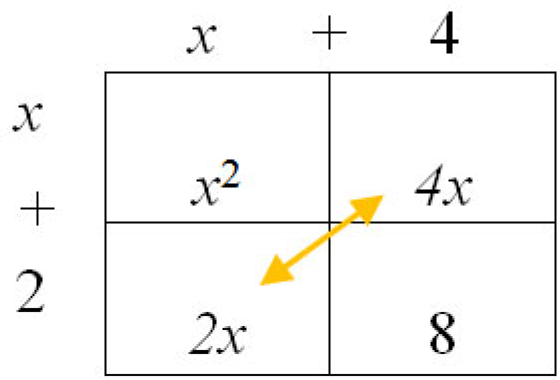
$x \bullet 3 = 3 \bullet x = \mathbf{3x}$

let $a, b$ be defined as $a = (x + 2)$, $b = (x + 4)$

$(x + 2)\ (x + 4)$




$$= x^2 + (4x + 2x) + 8$$
$$= x^2 + 6x + 8$$




$$= x^2 + (2x + 4x) + 8$$
$$= x^2 + 6x + 8$$

Example #13:

**If…** 2 (3+7) = (2)(3) + (2)(7)

= 6 + 14

= 20

**Then…** $x$ ($x$ + 2)

Example #14:

**If…** 4 • 3 • 8 = (4 • 3) • 8 = 4 • (3 • 8 )

96 = 12 • 8 = 4 • 24

= 96 = 96

**Then…** $x \cdot x^2 \cdot 2x$

1. After completing the above examples, what did you notice was similar? What about the process/equation was different? What evidence can you provide/offer to clarify your statement?
2. Can you and/or your partner find an alternative way to arrive to the same product? Draw/create your new model in the space provided.

Example #15:

($x^5$ )

define $x^5 = x \cdot ((x \cdot x) \cdot (x \cdot x))$

1) Show that $x^2 \cdot x^3 = x^5$ , using the associative rule.

Example #16:

**($x$ + 4) ($x$ + 2)**

A teacher asked 2 students to find the product of the above polynomials. Raven and Lillie demonstrated their work and responses below. Note your observations about their work, be prepared to discuss.

**Raven:**

| | $x$ | $+\ 4$ |
| --- | --- | --- |
| $x$ | $x^2$ | $4x$ |
| $+\ 2$ | $2x$ | $4$ |

= $x^2 + 2x + 4x + 4$ = $\boldsymbol{x^2 + 6x + 4}$

**Lillie:**

$(x + 4)\ (x + 2)$

$(x \cdot x) + (x \cdot 2) + (4 \cdot x) + (4 \cdot 2)$

$x^2 + (2x + 4x) + 4$

$\boldsymbol{x^2 + 6x + 4}$

With your partner please use the guided questions to make conjectures about the above examples.

1. What similarities do you notice in their work? What differences do you notice?

2. How were both students able to arrive at the same product if they multiplied the factors differently? How can your explanation be justified?

3. Are there other ways to multiply these factors and prove the product to be accurate?

4. Come up with at least one other way to demonstrate this process.

It should be noted that these types of problems will increase in difficulty, allowing my students the same experience with examples where the coefficients are greater than 1. An extension idea for students who grasp this concept quickly would follow:

5. Suppose the students were asked to multiply: ($2x$ + 2) ($2x$+4); recreate the methods used by both students to find the product.

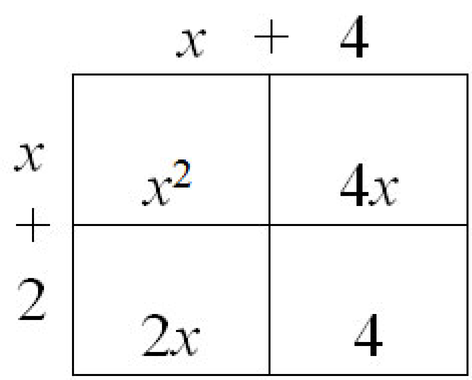
6. Aries noticed that $2x+2 = 2(x+1)$ and $2x+4 = 2(x+2)$. She said: I can get the same answer by multiplying $(x+1)(x+2)$, and then multiplying all the terms by 4. Do you agree with Aries? Carry out her procedure.

There will also be a number of opportunities for students to solve problems and make connections with the parallels of multi-digit and polynomial multiplication. Below are a few examples that I will use to demonstrate multi-digit multiplication in parallel with polynomial multiplication. Examples like these will be used to make a point to ensure students agree when $x = 10$. Using examples that we've previously used will assist in making this point, as shown below.

Example #17:

**If…**

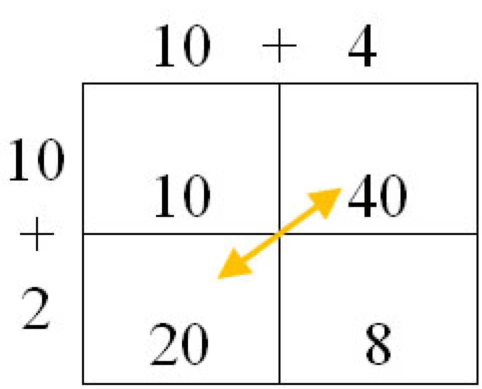
$(x + 4)(x + 2) =$ $\boldsymbol{x^2 + 6x + 4}$




**Then…** when $x = 10$

$\boldsymbol{(x + 4)(x + 2) = 14 \times 12}$

$= (10 + 4)(10 + 2)$




$10^2 + 6(10) + 8 = 100 + 60 + 8$ $\boldsymbol{= 168}$

Example #18:

**If…**

$x (x + 2) = (x)(x) + (x)(2) =$ $\boldsymbol{x^2 + 2x}$

**Then…** when $x = 10$

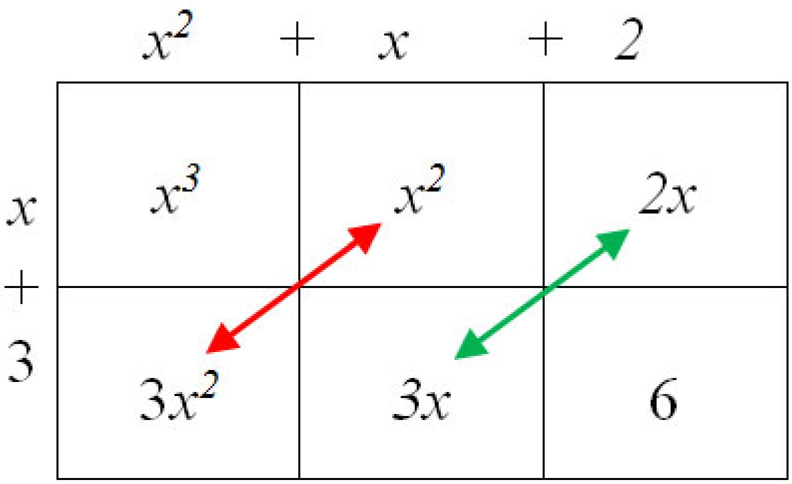
$\boldsymbol{x (x + 2) = 10 (10 + 2)}$

$= 10^2 + 2(10) = 100 + 20 =$ **120**

Example #19:

**If…**

$(x + 3) (x^2 + x + 2)$




$\boldsymbol{x^3 + 4x^2 + 5x + 6}$

**Then…** when $x = 10$

$\boldsymbol{(x + 3) (x^2 + x + 2) = 13 \times 112}$

$= 10^3 + 4(10^2) + 5(10) + 6$

$= 1000 + 4(100) + 50 + 6$

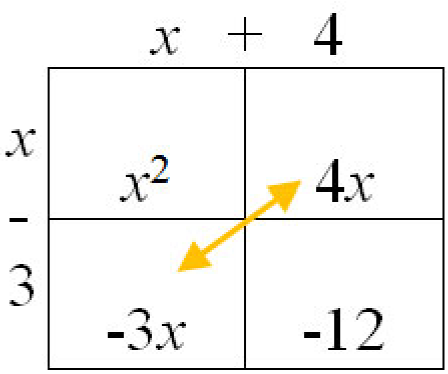
$= 1000 + 400 + 50 + 6$

**= 1456**

Example #20:

**If…**

$(x - 3) (x + 4)$




= $\boldsymbol{x^2 + x - 12}$

**Then…** when $x = 10$

**$(x-3)(x+4) = (10-3)(10+4)$**

$= 10^2 + 10 - 12$

$= 100 + 10 - 12$

**= 98**

**Flip-Side**

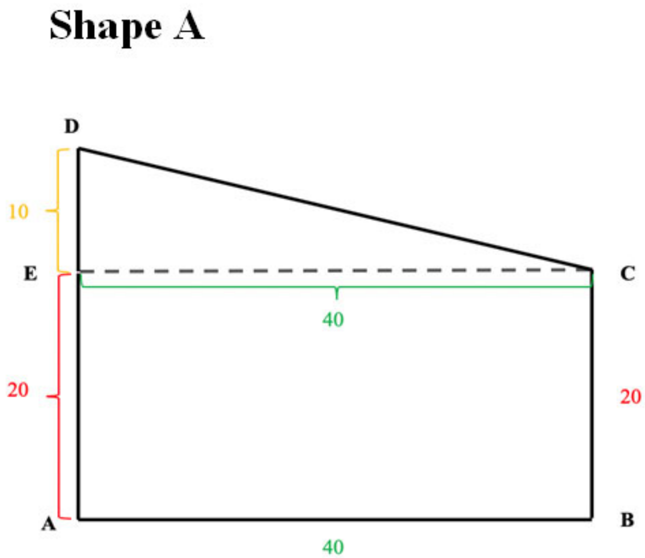
Objective: Find the area and perimeter of a rectangle with polynomial side lengths

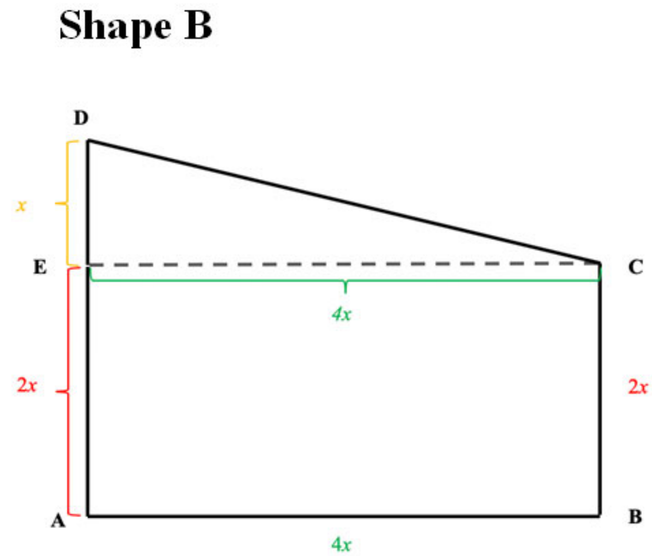
Days Covered: 1-2 Days / 2-4 Rotation Schedule

During this activity students will be given the opportunity to use their knowledge of finding the area and perimeter of rectangles and irregular shapes with both numerical and variable side lengths. Students will be either paired up or in small collaborative groups to allow discourse and accountable math talks to happen. Each pair/group will be given approximately 5-10 sheets of paper that have the same rectangle or irregular polygon on it. Side A will represent numerical side lengths while on the flip-side, Side B will display the same shape only this time using variable side lengths. This activity is meant to allow my students to realize that the process in determining the area and perimeter is the same no matter the value of each side length. In the example below, you can see the contrast between both sides.

Example #1

1. Find the area of Shape A & Shape B.
2. Describe how Shape A aided in finding both measurements on Shape B.
3. Compare your work with another student; did you use the same process to solve? Arrive at the same answer? Justify how you both got the same answer by defining one of the arithmetic rules for multiplication.




These examples can be customized for any student on any cognitive level. I will use colored paper and letters to label groups of problems based on the needs and academic reach of each of my students. Once identified students will be grouped homogeneously to ensure accessibility and engagement. Through proper facilitation the determination to move into heterogenous groups will be made.

## Appendix

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### Common Core State Standards (CCSS)

This unit was created with the intent of addressing specific Common Core State Standards (CCSS). These CCSS will provide emphasis on specific learning objectives and outcomes of students.

#### Essential Standards

- *MATH.CONTENT.HSA.SSE.A.1* Interpret expressions that represent a quantity in terms of its context.
- *MATH.CONTENT.HSA.SSE.A.1.A* Interpret parts of an expression, such as terms, factors, and coefficients.
- *MATH.CONTENT.HSA.SSE.A.1.B* Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret* $P(1+r)^n$ *as the product of P and a factor not depending on P*.

This unit is primarily focused on the arithmetic rules of addition and multiplication. These rules are used in justifying adding and multiplying polynomials of various degrees.

#### Related Standards

- *MATH.CONTENT.HSA.SSE.A.2*

Use the structure of an expression to identify ways to rewrite it. *For example, see* $x^4 - y^4$ *as* $(x^2)^2 - (y^2)^2$, *thus recognizing it as a difference of squares that can be factored as* $(x^2 - y^2)(x^2 + y^2)$.

- *MATH.CONTENT.HSA.SSE.B.3.C*

Use the properties of exponents to transform expressions for exponential functions. *For example, the expression* $1.15^t$ *can be rewritten as* $(1.15^{1/12})^{12t} \approx 1.012^{12t}$ *to reveal the approximate equivalent monthly interest rate if the annual rate is 15%*.

- *MATH.CONTENT.HSA.SSE.B.4*

Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems. *For example, calculate mortgage payments.*

**Standards of Mathematical Practice (SMPs)**

1. Make sense of problems and persevere in solving them
2. Reason abstractly and quantitatively
3. Construct viable arguments and critique the reasoning of others
4. Model with mathematics
5. Use appropriate tools strategically
6. Attend to precision
7. Look for and make use of structure
8. Look for and express regularity in repeated reasoning

My students will have used the SMPs throughout their experience through this unit. As they are introduced to and explore through the various teaching strategies, they will be demonstrating characteristics from SMP #4, 5, 7, and 8. As students work on/through the planned activities their understanding will be evident through exhibiting characteristics of SMP #1, 2, 3, and 6.

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## Endnotes


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