Curriculum Units by Fellows of the National Initiative
2019 Volume V: Perimeter, Area, Volume, and All That: A Study of Measurement

Measuring All Around, Inside and Out: A Unit about Perimeter and Area

Curriculum Unit 19.05.09, published September 2019
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Introduction

Student Demographics

My school, Mary Munford Elementary School, is located in the west end of the city of Richmond in Virginia. It is currently the highest performing elementary school in Richmond Public Schools. The students mostly come from middle class homes with a lot of parental support, but there are also immigrants and students from low-income homes. My school also serves low-incidence autistic students. The range of skill level in fourth grade is huge. A few students barely speak English, some perform two years below grade-level, and other students perform in the 99th percentile on norm-referenced tests for reading and math. Math is always an area of focus throughout my school and district. Math scores are very weak throughout the district, which is the lowest performing school district in Virginia. My school performs well in math, but measurement and geometry is definitely a weaker area.

Rationale

Often my fourth graders confuse area and perimeter, both the meaning and how to calculate the two measurements. "The results of the National Assessment of Educational Progress (NAEP) testing indicate clearly that students do not have a very good understanding of formulas. For example in the sixth NAEP, only 19 percent of fourth grade and 65 percent of eighth grade students were able to give the area of a carpet 9 feet long and 6 feet wide."[1]

After conducting research[2] , I have discovered the need to deemphasize the use of formulas $A = l \times w$ for a rectangle and $P = 2l + 2w$ or $P = l + w + l + w$ and further develop their conceptual understanding. To me, as a veteran teacher, this seems so obvious. However, in the everyday world of teaching, I know why I often neglect the conceptual development: time! Area and perimeter is always taught toward the end of the year, when as a teacher I am itching to get through the last of the curriculum in order to preserve some time for review before the state assessment. Since I always rush this unit of study, I only use simple shapes as prescribed in the state objectives. Since I do not explore deeply into the relationships between area and perimeter, the students end up with a limited understanding of both concepts. If I expose my students to more complex figures and allow them time to grapple with both area and perimeter of these more varied shapes, I will help them develop a better understanding of these two measurements.

After spending two weeks diving into area, perimeter, and geometry at a high level, one thing is clear: the students need many opportunities to explore geometric shapes in a hands-on manner. They need to perform transformations and turn basic geometric shapes into other shapes. They should decompose them, rotate them around a central point, or make multiple copies to create a different shape. I plan to use a variety of manipulatives throughout the explorations to provide a variety of exposure. Many of the higher-level theorems depend on decomposing and recomposing shapes, so this foundation is really crucial.

The curriculum unit will address the issues I have laid out in the previous paragraph. It will focus on perimeter and area and will utilize many opportunities for covering the surface and measuring around polygons to develop a strong conceptual understanding. In teaching general math, I always strive to connect concepts to students' prior knowledge. I try to make connections to my previous teachings, but geometry always seems like such an isolated, stand-alone topic. In this curriculum unit I want to intertwine geometry and measurement in a meaningful way. I plan to teach aspects of the curriculum unit throughout the year, thus providing more opportunity to strengthen these two weak areas. According to Van de Walle and Lovin, "The relationship between measurement and geometry is most evident in the development of area and volume formulas for measures of geometric figures."[3]

General Strategies

Tasks

A strategy that the curriculum unit will employ is rich and meaningful math tasks. Rich math tasks are a way to incorporate the Common Core math practices into daily routines. Rich math tasks are accessible to all learners. Rich math tasks use real-world applications and have multiple representations or approaches. They have various entry points so learners at all levels are able to contribute. Math tasks are collaborative, engaging, and make connections across topics. Based on district-wide and regional math meetings, and conferences, and research, it is clear to me that the education trend encourages the use of meaningful math tasks.

My research led me to the book *What Does Math Have to Do with It?* by Jo Boaler. She shares the results of her longitudinal studies in a narrative form and sheds light on the benefits of teaching with rich math tasks. She describes the classrooms that utilize task models where students of mixed abilities work in cooperative groups. The students are challenged to solve complicated, open-ended tasks. Then she observes students working in a more traditional setting. In the traditional setting students tend to work in isolation using problems in a textbook. Boaler conducts her study twice: each time in two different schools on two different continents. The results are the same. The students who worked with a task model outperformed the traditional methods. The task model also improved girls' achievement. This is believed to be because girls really want to understand *why* a math formula or method works. In traditional settings, girls became less successful and the answers to the *why* question were more fleeting. Based on this research, I want to incorporate more tasks into my teaching. Tasks will be sprinkled throughout the curriculum unit, sometimes as warm-ups and sometimes as the main activity.

Cooperative Learning

I will use cooperative learning groups of mixed abilities throughout this curriculum unit. The students will benefit from talking math and from working together to find solutions. Through the use of cooperative groups, students will learn from each other and benefit from each other's insights.

Hands-On Explorations

This entire curriculum unit involves hands-on learning. Covering surfaces with squares, tracing the outside of a shape with a linear unit, drawing area models on graph paper, to cutting apart and reconfiguring quadrilaterals almost the entire curriculum unit will involve hands-on activities.

Background Knowledge

The van Hiele's Levels of Geometric Thought[4]

Level 0 – Visualization

Level 0 is the recognition of basic shapes based on how they look. As students begin to approach Level 1 they begin to understand classes of shapes. Students at this level can name basic shapes, but may be confused if the shape is oriented in a different way.

Level 1 – Analysis

At Level 1, the students start out with an understanding of the classes of shapes and move toward identifying the properties of shapes. Students at this level can recognize different representations of the shape, such as triangles that have 3 sides, but may appear different because of the lengths of the sides or the way the shape is oriented. In the analysis phase, students recognize and describe properties of the shape, such as a rectangle has two pairs of parallel sides, opposite sides have the same length, and there are four right angles. At Level 1 students may have difficulty understanding that all squares are rectangles or all rectangles are also parallelograms.

Level 2 – Informal Deduction

Students are able to think about the properties of the shapes and begin to see and develop relationships between these properties. Students may use if-then statements at Level 2. For example, if a shape is a rhombus with right angles then it is a square. At this level, students may understand that a shape may meet the definition with minimum characteristics. For example, if a parallelogram has one right angle, it is a rectangle. Or if a rhombus has one right angle, it is a square. At Level 2, students progress from understanding properties of shapes to understanding relationships among properties.

Level 3 – Deduction

At Level 3, the students begin to use a system to logically think about parameters that lead to the derivation of other truths. At this level, students are able to not only make conjectures, but also examine if they are true. This is the type of thinking that takes place in high school geometry. At the deduction level, students move from relationships among properties to deductive systems of properties.

Level 4 – Rigor

Students at Level 4 move from deductive systems of properties to analysis of deductive systems. This is typically the level of college math students focusing on geometry.

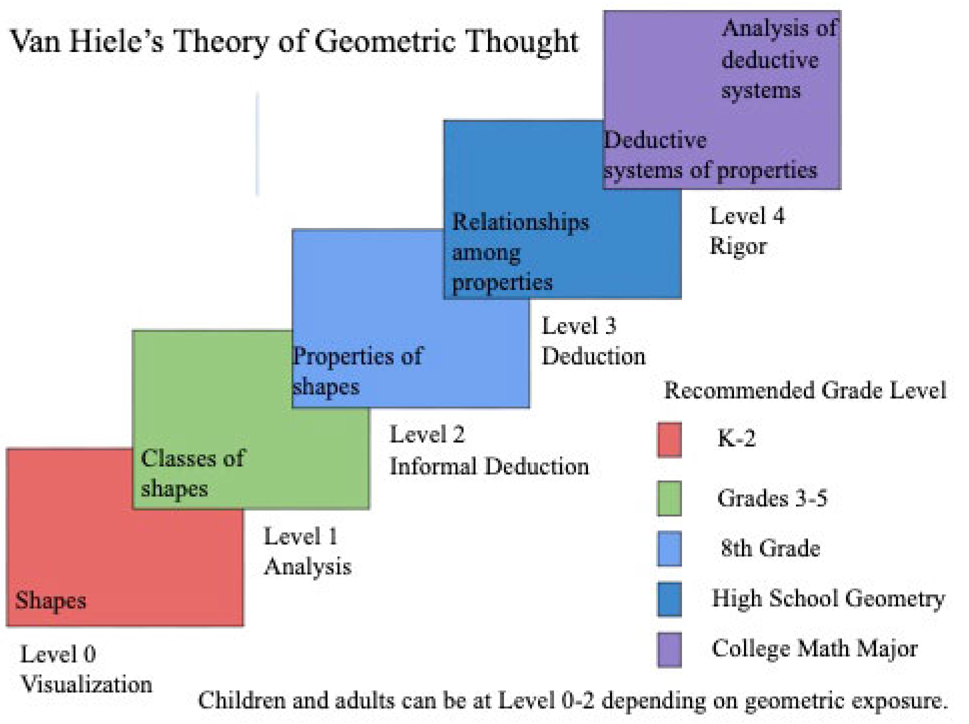


Figure 1: van Hiele's Theory of Geometric Thought

The van Hiele's Theory was developed in 1959. See Figure 1. This theory took a while to catch on in this country,[5] but today it greatly impacts the American geometry curriculum.[6]

The van Hiele's levels are sequential. The levels are not developmental but more experiential. "Geometric experience is the greatest single factor influencing advancement through the levels."[7] Some researchers believe a person can be at different levels depending on the specific topic and their exposure to that topic. Ideally, it is believed that the best way to help students advance through the levels is to instruct them on the students' current level with explorations, interactions, and discussions at the next level. This is basically the same approach used with reading instruction. Students read on their independent level and are exposed to a higher level through read aloud or small group instruction with support. Just as with reading, if the level is too high, the students will have difficulty engaging with the material and will not have much to contribute to discussions.

Implications

According to Van de Walle & Lovin, “Nearly all students in K-3 will be at Level 0, by at least grade 3 teachers certainly want to begin to challenge students who seem able to engage in Level 1 thinking.”[8] The expectation is that students will be at Level 2 by the eighth grade, Level 3 in high school geometry, and Level 4 as a college math major.

I will use the van Hiele levels throughout the curriculum unit to identify the level of each activity involving area, perimeter, and quadrilaterals. Suggestions for differentiation at a lower level as well as ways to move the activity to a higher level to meet the students’ needs will be made.

Quadrilaterals

General quadrilaterals

Quadrilaterals are polygons with four sides. Many classes of special quadrilaterals have been singled out for study. Some are more general, and others are special and have specific properties.

Special Quadrilaterals

A **trapezoid** is a quadrilateral with *at least* one pair of parallel sides. Some people prefer to require a trapezoid to have only one pair of parallel sides, but “at least” is the definition we will use in this unit. See Figure 2.

A **parallelogram** is a quadrilateral with two pairs of parallel sides. This implies that opposite sides have the same length, and vice versa. A **kite** is a quadrilateral with two pairs of adjacent (next to), equal sides. A quadrilateral with four congruent sides is a **rhombus**. A rhombus is both a kite and a parallelogram. A **rectangle** is a quadrilateral with four right angles. A rectangle is a parallelogram. A rectangle that is also a rhombus is a **square**. The square is the most special of all quadrilaterals as it has many properties and lots of symmetry (see Table 1).

Properties of Quadrilaterals

Properties of Quadrilaterals

	One Pair Parallel Sides	Two Pairs Parallel Sides	Opposite Sides the Same Length	Congruent Sides	Right Angles
trapezoid	√				
kite				√ (2 or 4)	
parallelogram	√	√	√	√ (2 or 4)	
rhombus	√	√	√	√ (4)	
rectangle	√	√	√	√ (2 or 4)	√
square	√	√	√	√ (4)	√

Table 1: The properties of quadrilaterals.

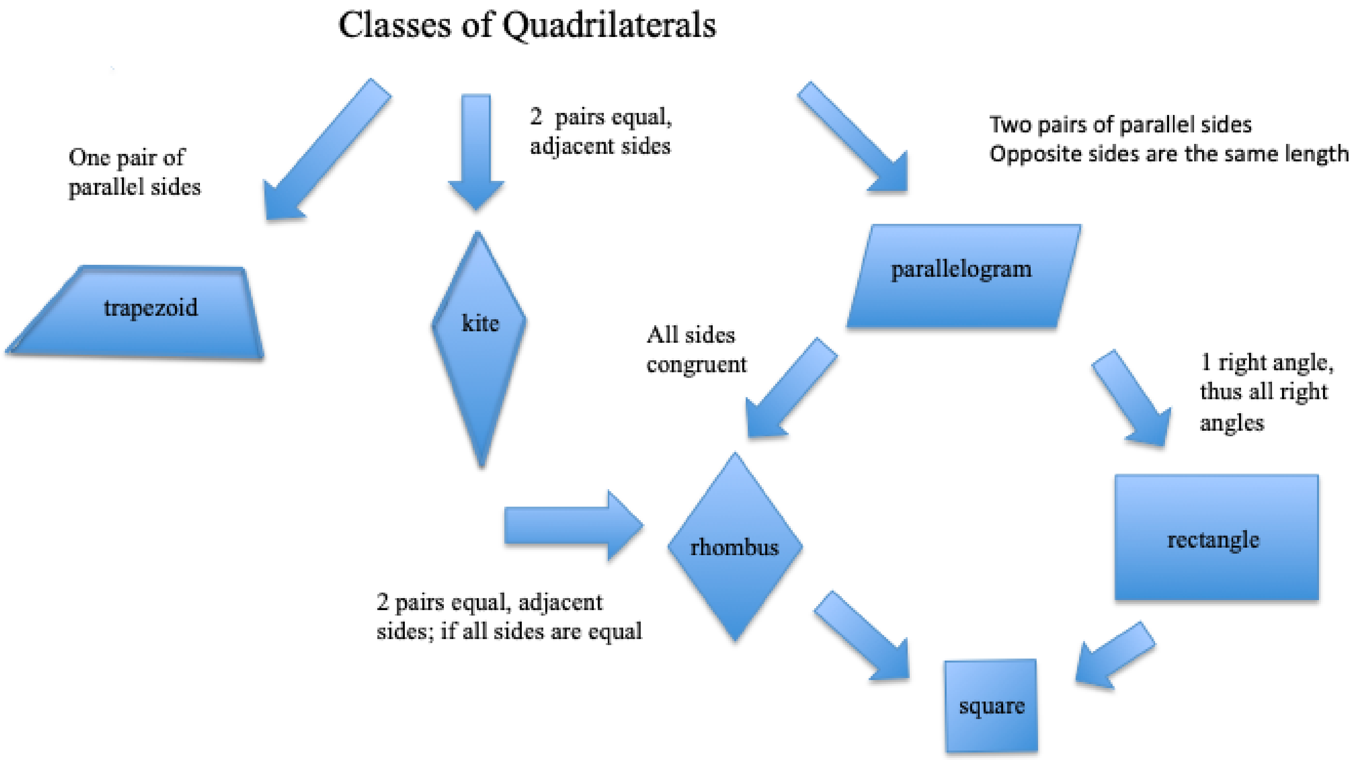


Figure 2: A diagram of special types of quadrilaterals

Symmetry

Symmetry can also be used to distinguish classes of quadrilaterals. Some have lines of symmetry, rotational symmetry, and reflection symmetry across a diagonal.

A shape has rotational symmetry if, when it is rotated, or turned, it appears to be in the same position. The only possibilities relevant for quadrilaterals are 90 or 180 degrees. A quadrilateral with 180-degree rotational symmetry is a parallelogram (and vice versa) (see Figure 3). The only kind of quadrilateral with 90-degree rotational symmetry is the square.

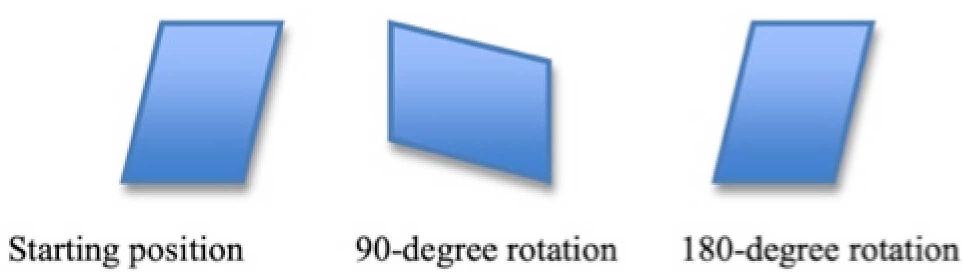


Figure 3: This shape has rotational symmetry because it will get back to the original position when rotated less than 360 degrees.

A shape has symmetry across a line (aka reflection symmetry) if, when it is folded across the line, the two halves match exactly (see Figure 4).

Figure 4: This figure (an isosceles trapezoid) has reflection symmetry. If the line of symmetry goes through a side, it will pass through two opposite parallel sides, and you have an isosceles trapezoid.

Reflection symmetry across a diagonal is when a figure is folded on a diagonal, and the two halves line up exactly (see Figure 5).

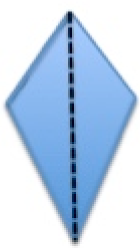


Figure 5: If the line of symmetry is a diagonal, you get a kite.

A diagonal of a quadrilateral separates it into two triangles. If the triangles are congruent, then the quadrilateral is a kite (if the halves are reflections of each other across the diagonal), or it is a parallelogram (if the halves are congruent by 180 degree rotation around the midpoint of the diagonal). If all four triangles are congruent when both diagonals are drawn, then the shape is a rhombus.

A rhombus is the only quadrilateral with reflection symmetry across both diagonals (see Figure 6).

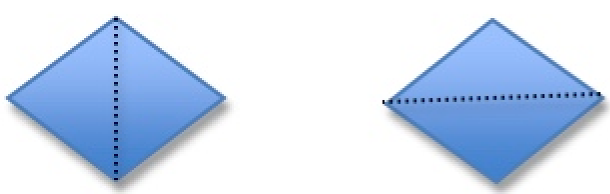


Figure 6: A rhombus is a kite with all sides equal. A rhombus is symmetric across both its diagonals.

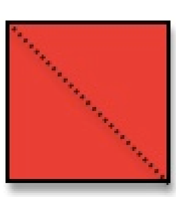


Figure 7: If the rhombus is folded along the diagonal line, the two triangles will lie on top of each other and will be aligned evenly. A rhombus with right angles is a square.

By drawing the diagonals of quadrilaterals students can discover characteristics and relationships between the different classes (see Figure 7). Students can describe the characteristics that result from the two diagonals.

The diagonal forms two congruent triangles (see Figures 8 and 9).

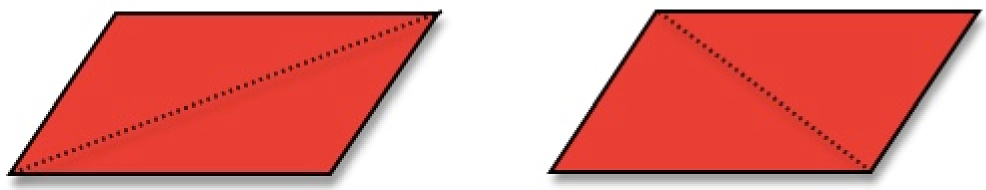


Figure 8: In a parallelogram each diagonal forms two congruent triangles.

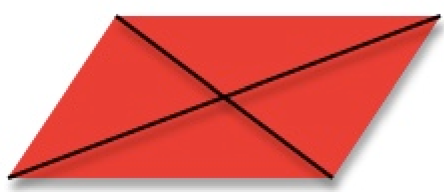


Figure 9: In a parallelogram, diagonals also bisect: each cuts the other into two line segments each of the same length.

Another characteristic of the diagonals (of parallelograms) is that the point of intersection of the diagonals is a central point, that bisects or splits each line segment into two segments of equal length (see Figure 9). Also the parallelogram is symmetric under 180- degree rotation around this point. Thus, although the typical parallelogram does not have any reflection symmetry, it does have 180-degree rotational symmetry, and this accounts for all the commonly stated properties of the parallelogram (opposite sides are equal and parallel, opposite angles are equal, diagonals bisect each other, etc.) (see Figure 10).

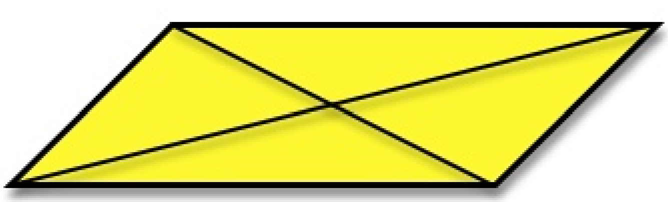


Figure 10: The parallelogram shows diagonals that bisect.

In a kite, diagonals intersect perpendicularly (see Figure 11). The diagonals can be perpendicular without bisecting each other. In fact, they can be perpendicular and equal without bisecting each other. In a kite, one of the diagonals will bisect the other one, but not necessarily vice versa. Diagonals that are perpendicular and bisect each other reveal that the shape is a rhombus.

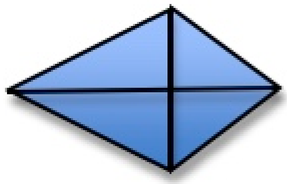


Figure 11: In a kite, the diagonals intersect at right angles.

An analysis of diagonals is another way to classify and explore relationships among quadrilaterals.

I have created a “tool” that my students will use to investigate the diagonals of quadrilaterals. I name the tool a “Crisscross Vise.” I tweaked a design with paper to turn it into more of a tool. I crafted it using two straws joined by a brass fastener. I punched equidistant holes in the straws (see Figure 12).

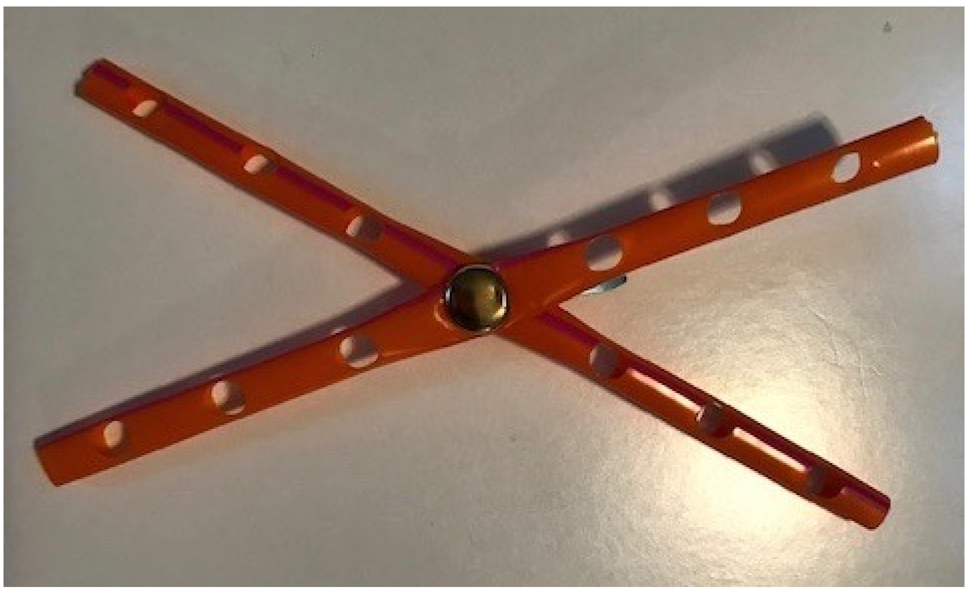


Figure 12: A crisscross vise made with straws and a brad fastener.

Then I searched on Google® and discovered a real tool for construction called a Crisscross Vise. The “real” device was invented in the mid 19th century and is used to maintain parallelism.[9] The exploration of quadrilaterals using the Crisscross Vise is in the activity section of this unit.

Properties of the Diagonals

	Diagonals are perpendicular	Diagonals bisect each other	Form 4 congruent triangles
Trapezoid			
Kite	√		
Parallelogram		√	
Rhombus	√	√	√
Rectangle	√	√	
Square	√	√	√

Table 2: Properties of the diagonals by quadrilateral

The diagonals of a rectangle do not form 4 congruent triangles. The pairs of opposite triangles are congruent, but the two pairs are not congruent, unless the rectangle is a square.

Type of Symmetry

	Rotational	Reflection	Diagonal
Trapezoid		√ *	
Kite		√	√ (1)
Parallelogram	√		
Rhombus	√	√	√
Rectangle	√	√	
Square	√	√	√

Table 3: Depicts the type(s) of symmetry of each quadrilateral.
** Note: Only an isosceles trapezoid has reflection symmetry.*

Earlier I stated that the square is the most special quadrilateral because of all of its properties and its symmetry. If you look back at the three tables, the square has checks in each column. The square has 10 out of 10 properties (see Tables 2 and 3). That shows clearly that the square is so special!

Related Symmetry

Trapezoids, except for the isosceles trapezoid, do not have any type of symmetry but they are related to symmetry.[10] The isosceles trapezoid is the most prevalent trapezoid in elementary school as it is the one in the set of pattern blocks that has two congruent sides. That isosceles trapezoid is half of a regular hexagon. It has three equal sides.

Trapezoids are related to symmetry in the following ways: 1) A trapezoid is half of a parallelogram. A parallelogram does not have a line of symmetry, unless it is a rhombus or a rectangle. A parallelogram has a point of central symmetry. Any line through the central point will cut the parallelogram into two trapezoids, which are congruent under the 180-degree rotation around the central point. (Exception: if the line is a diagonal of the parallelogram, the two halves are triangles.) 2) Triangles are related to parallelograms in a similar way. A triangle is half of a parallelogram in three different ways.[11]

Area and Perimeter

The measure of the space inside a region is called area, and it is measured in square units. The perimeter is a measure of length around the boundary of the region, and it is a linear measurement. My students have a difficult time with area and perimeter, as they often confuse the two measurements and which units to use for each.

I suspect there are several reasons why this is a source of confusion. First, area and perimeter are often taught at the end of the school year as I am trying to get through the curriculum and preserve some time for review prior to state testing. In my haste, I often quickly jump to the formulas, thus skipping the hands-on process of measuring area and perimeter. Since fourth graders are new to algebraic formulas, the formulas are easily confused or forgotten. Typically students measure the area and perimeter of rectangles and squares, which are simple shapes in terms of calculating area and perimeter. The procedures of adding up all four sides or multiplying the length by the width are not tied to deeper meaning, thus the confusion.

In the latter stage of Level 1 in van Hiele's Theory of Geometric Thought, students should analyze the properties of shapes. Analyzing the properties of shapes is also included in the early stage of Level 2 Informal Deduction. By working with area and perimeter the students will not only develop their measurement skills, but also develop a deeper understanding of the properties of shapes.

Paving the Area

The idea of paving a road or a parking lot conjures up images of a large steamroller covering an entire space with hot, black macadam. This is the image I want my students to have in their head when they think about area. There is a short video listed in the student resources section to help the students visualize the process.

Once my students have the image, the strategy is to pave squares and rectangles around the school and schoolyard with square units. It is important to remind the students that the squares must touch, but not overlap. Once a rectangle or square is covered with square units, the representation looks like, and is in fact, an array. The commutative law holds true, that the rectangle can be rotated and the area will stay the same. A 4 x 6 rectangle has an area of 24 square units. A 6 x 4 rectangle also has an area of 24 square units.

The students will use a variety of square units such as squares of paper, 1 inch tiles, Cheez-It® , floor tiles, construction paper squares, or Starburst® to cover a variety of surfaces throughout the classroom, school, and schoolyard. The hands-on nature and the use of "square units" will help the students remember the proper units for area.

Measuring Perimeter

Perimeter is the linear distance around the outside of a shape. The class may start out by taking a walk around the perimeter (a city block) of the school grounds while identifying other real world objects that represent perimeter. I will provide multiple opportunities for the students to determine perimeter with

Cuisenaire® rods, toothpicks, pencils, rulers, string, yardsticks, tape measures, coffee stirrers, and Wikki Stix® around the school and schoolyard. The idea is to help the students visualize perimeter and have a conceptual understanding of the units for measuring it.

Is There a Relationship between Perimeter and Area?

In order for the students to develop a deeper understanding of area and perimeter, it is important for them to explore area and perimeter deeply to distinguish between the two measures. There is a very weak relationship. There are figures with arbitrarily small area and arbitrarily large perimeter. There is a minimum perimeter for a given area; the exact value depends on the type of figures you allow. The van Hiele's Theory of Geometric Thought includes the "relationships among properties" in the later stage of Level 2. When I look at these Levels of Geometric Thought and think about my students, they are expected to understand the properties of shapes. Some of the standards certainly include relationships among the properties. I see my students at the upper stage of Level 1 and the early end of Level 2. In order to move my students forward in their level of geometric thought, I need to meet them where they are and push forward. Because of this, it makes sense once my students understand the properties of area and perimeter to explore the differences between area and perimeter.

The next few strategies that I have selected for this curriculum unit are specifically designed to help students distinguish between area and perimeter. The general strategies are to use multiple representations, specifically visual models and charts to record, display, and analyze the data. In one of the tasks the students will explore a problem with a "fixed area."

In the "fixed area" task the students are to make as many rectangles as they can make using 36 square tiles. Once the students create the rectangles, they sketch them onto grid paper. Then they calculate the area and perimeter of each. If they understand the idea of area; they won't have to calculate it: they will know it is 36. It is helpful to provide a chart for collecting their data. After careful analysis of the data, the students should notice some patterns.

The questions asked were the key component of the lesson, which took the task to a new level. The students are to compare the smaller perimeters and find the pattern. I went through the task myself, and this was when the light bulb came on for me. First I drew the rectangles in my notebook in a random manner. Then, I ordered the perimeters from greatest to least then looked at each figure. The pattern I noticed was that the "tall and skinny" rectangles had the greatest perimeter, while the "short and stout" rectangles had smaller perimeters (see Table 4).

Dimensions	Area	Perimeter
1 x 36	36 square units	74 units
2 x 18	36 square units	40 units
3 x 12	36 square units	30 units
4 x 9	36 square units	26 units
6 x 6	36 square units	24 units

Table 4: The data for a fixed area problem

Then I worked through a Fixed Perimeter problem where the perimeter was 36 units. I started with a square with sides that were 9 units. So my first rectangle that I drew was 9 x 9 units. Immediately, I decided that I needed to have a system to find all of the rectangles. I increased the length by one and decreased the width by one. Soon I had a nice "chart" (see Table 5) that looked like this:

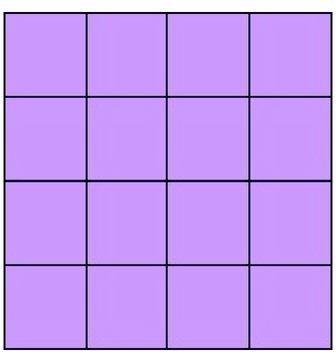
Dimensions	Perimeter	Area
9 x 9	36 units	81 square units
10 x 8	36 units	80 square units
11 x 7	36 units	77 square units
12 x 6	36 units	72 square units

Table 5: The data for a fixed perimeter problem

This chart continued until I reached 17 x 1. A pattern emerges in the chart: as the perimeter gets longer and skinnier, the area decreases. I want the students to discover that with a fixed perimeter, the square is the rectangle with the largest area. This idea is important because it can be generalized. With a fixed perimeter, the polygon with the largest area is always the regular polygon, or the polygon with equal sides and angles.

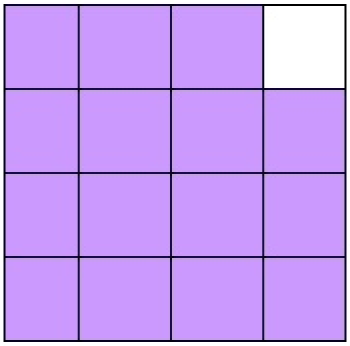
However, once I went through this exercise, I saw the pattern emerge and realized why this task is meaningful. It integrates many great math skills. The students have to calculate area and perimeter, record data, analyze data, make generalizations, and explore the relationship. The students will work in small groups and engage in discourse to discover a pattern, and come to a conclusion. This type of exercise requires critical thinking, collaboration, and leads the students to understanding the relationship between a fixed perimeter and the largest area. This process will lead my students to a deeper understanding of area and perimeter. That is certainly more meaningful then finding the area and perimeter of a rectangle over and over again!

The next activity begins by finding the area and perimeter of a square. Then one unit square is removed, then another, and another. Each time a square is removed, the area is calculated. The area is decreasing by 1 square unit every time a square is removed. As you follow the progression, pay attention to the perimeter. It may surprise you!



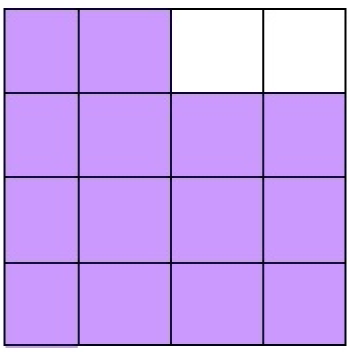
Progression 1.1

The area is 16 square units and the perimeter is 16 units.



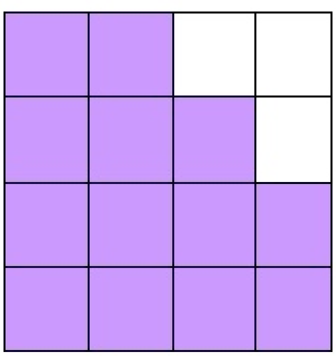
Progression 1.2

Then one square unit is removed. The area is reduced by 1 and equals 15 square units and the perimeter remains the same (16 units).



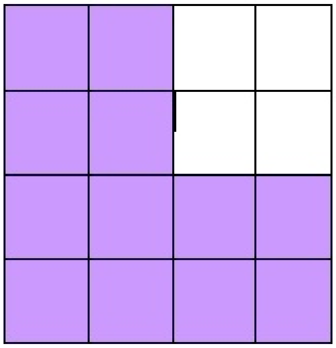
Progression 1.3

Another square unit is removed so the area goes to 14 square units and the perimeter is still 16 units.



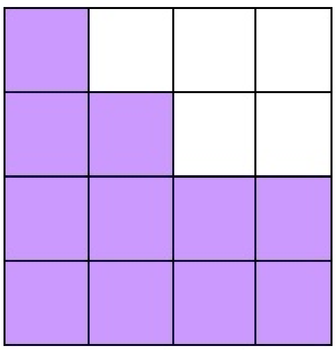
Progression 1.4

Remove another square. The area equals 13 square units and the perimeter is still 16 units.



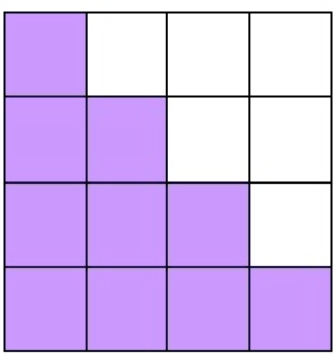
Progression 1.5

When four squares are removed the area is 12 square units and the perimeter is 16 units again!



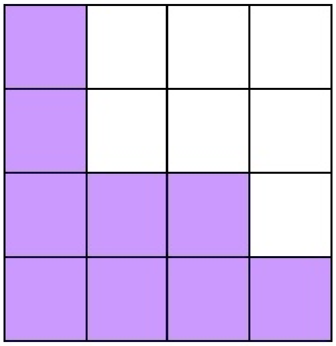
Progression 1.6

Remove another square and the area is equal to 11 square units while the perimeter remains 16 units.



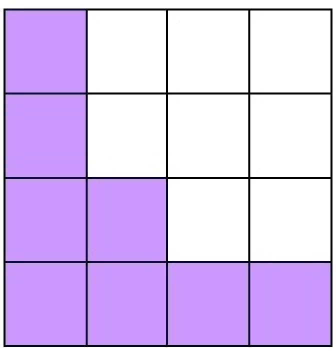
Progression 1.7

When another square unit is taken away, the area becomes 10 square units and the perimeter is still 16 units.



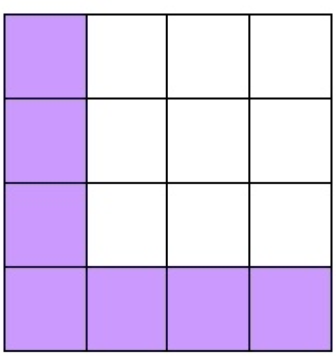
Progression 1.8

You guessed it. Take away another square. The area will equal 9 square units and the perimeter will equal 16 units.



Progression 1.9

Take away another square and the area reduces by one again and equals 8 square units. The perimeter continues to equal 16 units.



Progression 1.10

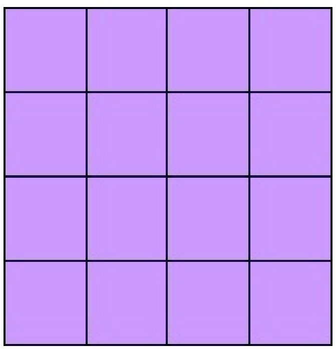
When one more square unit is removed the area is 7 square units and the perimeter is…you guessed it…16 units.

Area (in square units)	16	15	14	13	12	11	10	9	8	7
Perimeter (in linear units)	16	16	16	16	16	16	16	16	16	16

Table 6: The area and perimeter are compared for the progression

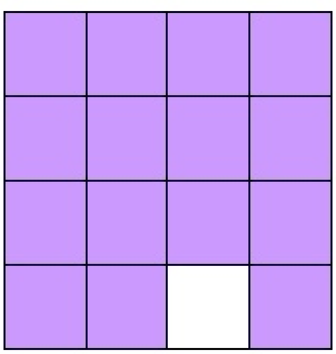
The progression shows that the area can vary significantly for a fixed perimeter. In the chart the area decreases from 16 square units to 7 square units, or decreases by more than half. Thus, the perimeter remains unchanged. See Table 6.

In the next progression, the area decreases one square unit at a time, but the perimeter increases.



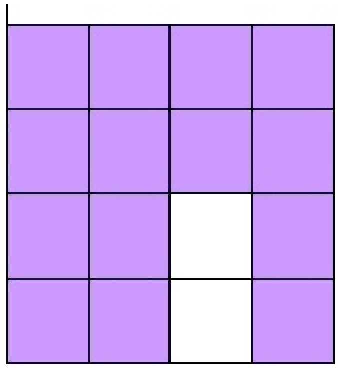
Progression 2.1

In the figure above, the area is 16 square units and the perimeter is also 16 units.



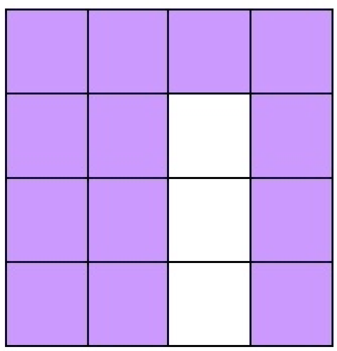
Progression 2.2

One square is removed. The area is 15 square units and the perimeter is 18 units.



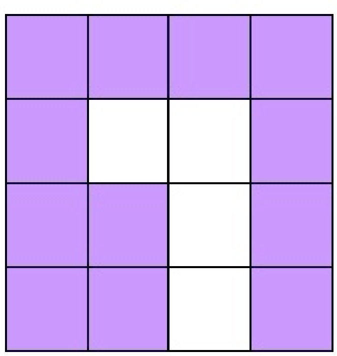
Progression 2.3

Again a tile is removed, the resulting area is 14 square units and the perimeter is 20 units.



Progression 2.4

Remove another tile. The resulting area is 13 square units and the perimeter is 22 units.



Progression 2.5

Another tile can be removed and the area is12 square units and the perimeter is now 24 units. See Table 7.

Area (in square units)	16	15	14	13	12
Perimeter (in linear units)	16	18	20	22	24

Table 7: The table summarizes the progression.

The two progressions above show that area and perimeter are two distinct measurements, and one has no bearing on the other. The area and perimeter problems above exemplify the type of thinking that I will have my students engage in to develop deeper meaning and distinguish between area and perimeter.

More Complicated Area and Perimeter Problems

Next, I want my students to explore a variety of perimeter and area problems that have nuances. This is a strategy used in Singapore math to help the students develop critical thinking skills. In the United States, the tendency is to have students complete multiple problems with different numbers, which results in rote skills.

The problems I encountered in a Singapore math book require that students push the sides' lengths outward and create an enclosing rectangle. Once the enclosing rectangle is drawn or envisioned, the calculation of the perimeter is the same as the calculation for the perimeter of a rectangle.

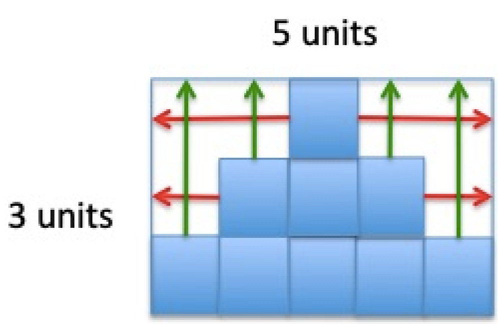


Figure 13: A model of stair steps shows the enclosing rectangle strategy for perimeter.

The diagram (Figure 13) shows the strategy of making the enclosing rectangle. The length and the width of the enclosing rectangle are congruent to the length and width of the "stair step" figure.

I want students to realize that it isn't always necessary to count every linear unit. By "pushing" the lengths out, students can create the enclosing rectangle or square with an equivalent length. This method will reduce the potential for errors caused by miscounting.

The next type of problem that I will present to my students is also from Singapore math. See Figure 14. I think of these problems as "paved with rectangles." The area of the square or rectangle is subdivided into smaller rectangles that require problem solving and critical thinking to determine the area or perimeter. The students will work in small groups not only for collaboration, but also to make them use math language and so I can gain insight into their thought processes. The Singapore Math Challenge Grade 3+ book listed in the bibliography offers various problems that may look similar, but have subtle differences.

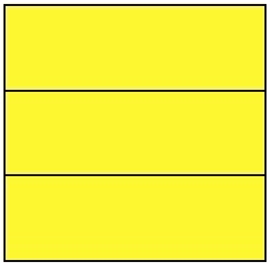


Figure 14: David made a square by putting three identical rectangles together. The perimeter of each rectangle is 32 cm. What is the perimeter of the square?

The final type of problem that my students will explore involves finding the area and perimeter of the special classes of quadrilaterals, and particularly, parallelograms and trapezoids. The strategy is to decompose a parallelogram into a rectangle and two triangles. See Figure 15. The typical way is to drop two perpendicular segments thus revealing a rectangle and two triangles. Then remove one triangle, translate it next to the other triangle to form a rectangle. Once the parallelogram has been recomposed into a rectangle, the students can proceed to either pave the area with tiles or measure the sides and apply the formula for the area of a rectangle: A = l × w or A = b × h (aka base × height). Although the rectangle has the same area as the parallelogram, it has a smaller perimeter. So, parallelograms are good shapes to show the independence of area and perimeter, because the side lengths clearly determine the perimeter, but give few clues as to the area. If you fix the side lengths of a parallelogram, the area can be as small as you want. The largest possible area for the parallelogram is when it is a rectangle; then it is the product of the side lengths. Rhombi are special parallelograms, so this same technique will work for them as well.

Some mathematics educators suggest that the formula A= l × w is limited because it only applies to rectangles. The formula A= b × h is more unifying than A = l × w because it can be generalized for all parallelograms, and also is a springboard to understanding the area formula for a triangle A= ½ b × h.

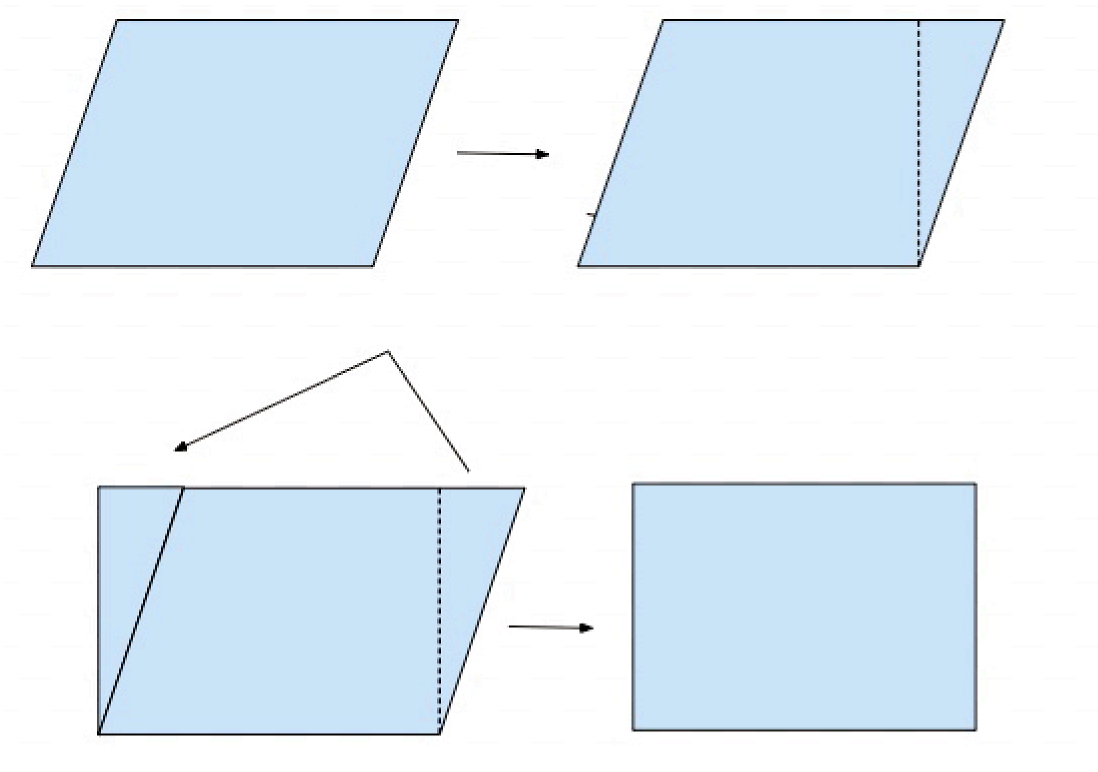


Figure 15: The diagram depicts how to lop off part of a parallelogram, reposition it, and form a rectangle.

Similarly, an isosceles trapezoid can be decomposed into a rectangle (or square) and two triangles. In a similar way, manipulate the subsequent shapes into a rectangle. The students know how to find the area and perimeter of the rectangle.

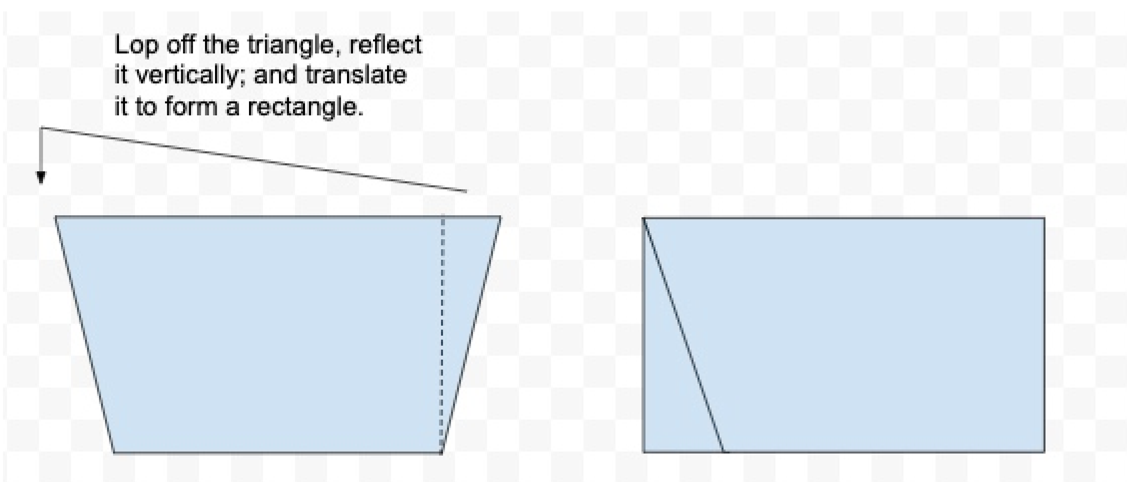


Figure 16: The diagram above shows how to reconfigure an isosceles trapezoid into a rectangle.

The isosceles trapezoid is unique, and this strategy will not work for all trapezoids. A more general technique that works for all trapezoids is to double the trapezoid. Combine the two figures to form a parallelogram, as described above (see Figure 17 below). Then decompose the parallelogram as explained above, to reveal the hidden rectangle and two triangles. Rearrange the parts and turn it into a rectangle. Voila!

In this example, the center of the parallelogram gets constructed. It is the midpoint of one of the (possibly) non-parallel edges of the trapezoid. Rotating the trapezoid around this point makes a second copy. When the two copies are joined, they form a parallelogram (see Figure 17). Once the original trapezoid is doubled, forming a parallelogram, students can work through the steps modeled above to determine the area of the parallelogram. The steps were: lop off two triangles, recompose the two triangles and the rectangle into a larger rectangle, then find the product of the side lengths. It is important to remember that the goal is to find the area of the original trapezoid. So once the product of the rectangle is determined, the product must be halved, since the initial trapezoid was doubled. Thus, the parallelogram reflects the area of two trapezoids. So in order to find the area the students will pave the area and divide it by two. Thus, the formula is ½ (b × h). I will allow my students to use the formula if they can figure it out, otherwise they will tile the rectangle and divide by two. As I believe, “Shortcuts are the privilege of experts!”

Keep in mind that the base of the parallelogram that is the double of a trapezoid has length equal to the **sum** of the parallel sides of the trapezoid. I will not get this deep with my fourth grade students, but a few may figure it out on their own.

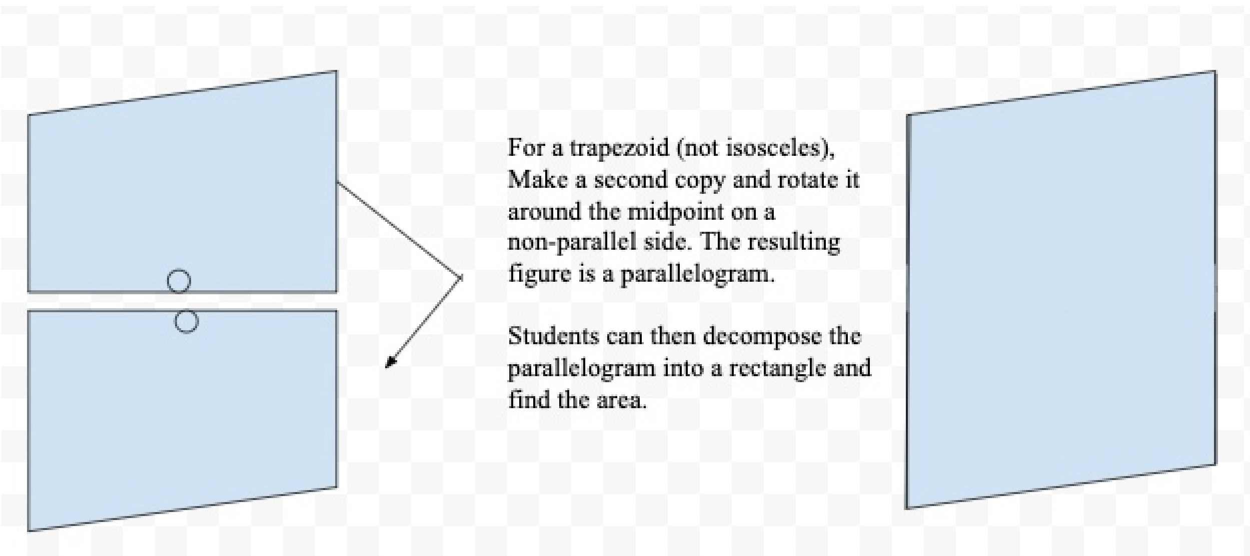


Figure 17: The diagram above shows how to duplicate a trapezoid and reposition it around the bisecting midpoint of a non-parallel side to form a parallelogram.

This formula should look familiar. It is the area for a triangle. The triangle is the case when one of the parallel sides of the trapezoid has length 0! I am not going to go into the area of triangles, because in Virginia that is a fifth grade objective. However, the students should be ready after working through this curriculum unit.

These types of exploration should build a strong geometric foundation with deep understandings around the concepts of area, perimeter and quadrilaterals. Roger Howe says, “If we can understand the area of a parallelogram, we can understand the areas of all of the special classes of quadrilaterals.”[12]

Activities

This activity is listed for day 11. It would be a great task to have the students work with 12 tiles and an enclosing rectangle of area 16. The students will also have graph paper to record their figures. The students will work in small groups to determine a progression where the area stays fixed at 12 square units and the perimeter increases. There are different ways to arrange the tiles for a desired perimeter. So this type of task would generate excellent mathematical discourse and higher level thinking. (Determining area and perimeter – van Hiele Level 1; developing a progression and the mathematic discourse pushes this lesson into Level 2.)

The activities for Day 14-16 are discussed together as they build on one another. The activity for day 14, students will be given a bag of quadrilaterals that are all the same. They also will receive a "tool kit" which will contain an inch ruler, a "Crisscross Vise," wax paper for determining reflection symmetry, and a pushpin and foam core board for determining rotational symmetry. The teacher will model how to use each "tool" and what all of the categories on the data collection sheet mean. (Determining the properties is van Hiele's Level 1 and moves into Level 2.)

The students will examine the shapes and complete a data collection sheet. When all groups are done, the students will share their data as I add it to a master sheet.

On day 15, the students will examine the data, at first in pairs, then small groups, and finally it will be shared with the class. I will collect observations from each group. (Discussing the properties of the shapes found with the tools is van Hiele's Level 1; Examining the data and looking for relationships among the properties is van Hiele's Level 2.)

On day 16, the students will receive the same tool kit with a bag of mixed quadrilaterals. Each group will work to determine whether or not they agree with the observations. Discrepancies will be shared and discussed. (Discussing the properties of the shapes found with the tools is van Hiele's Level 1; Examining the data and looking for relationships among the properties is van Hiele's Level 2.)

Day 1: Students will complete a pre-assessment (vH Level 1 & 2).

Day 2: Students will find the area and perimeter of classroom objects with manipulatives. (vH Level 1).

Day 3: Students will find the area and perimeter of classroom objects with manipulatives. (vH Level 1).

Day 4: Students will go to the hallway with string and construction paper squares and measure the area and perimeter of objects. (vH Level 1).

Day 5: Students will go to the playground with skein of yarn, a yardstick, and construction paper squares (1 foot x 1 foot) to measure area and perimeter. (vH Level 1).

Day 6: Students will work on the fixed area problem in the strategy section. (vH Level 1 & 2).

Day 7: Students will work on the fixed perimeter problem in the strategy section. (vH Level 1 & 2).

Day 8: Students will work on the problem that adds 1 square unit over and over in the strategy section of the unit. (vH Level 1 & 2).

Day 9: Students will work on the take away 1 square unit problem and the perimeter stays the same in the strategy section of the unit. (vH Level 1 & 2).

Day 10: Students will work on the second take away 1 square unit problem and the perimeter increases in the strategy section of the unit. (vH Level 1 & 2).

Day 11: Students will work on activity 1. Given 12 tiles and an enclosing rectangle of 16 units, students will keep the area fixed and make the perimeter increase by rearranging the tiles. (vH Level 1 & 2).

Day 12: Students will work on irregular-shaped perimeter problems. ((vH Level 1 & 2).

Day 13: Students will work on irregular-shaped perimeter problems. (vH Level 1 & 2).

Day 14: Introduce quadrilaterals and have students explore the characteristics of each. (vH Level 1).

Day 15: Students will explore the similarities and differences between quadrilaterals. (vH Level 1 & 2)

Day 16: Students will explore a group of the same shaped quadrilaterals with a "tool kit" and the data will be collected. (vH Level 1 & 2).

Day 17: Students will analyze the data collected and share observations. (vH Level 1 & 2).

Day 18: Each group will receive a bag of mixed quadrilaterals and will examine them with the tool kit. The class will discuss if the observations were correct or if adjustments need to be made. (vH Level 1 & 2).

Day 19: Students will work in small groups to explore how to determine the area and perimeter of parallelograms as shown in the strategies section without being told a formula. Each group will have graph paper and scissors. (vH Level 1 & 2).

Day 20: Students will work to determine the area and perimeter of isosceles trapezoids as shown in the strategy section. Each group may use scissors and graph paper. (vH Level 1 & 2).

Day 21: Students will complete a post assessment. (vH Level 1 & 2).

Student Resources

https://www.youtube.com/watch?v=rHymjhn5m94

This video is just over three minutes long and will provide a lasting image of a road roller paving to help kids connect that area is "paving" the squares.

Appendix

This curriculum unit aligns to the Virginia Standards of Learning and the Common Core standards.

Virginia Standards of Learning

Virginia is one of a handful of states that did not adopt the Common Core standards. However, Virginia's Standards of Learning are "generally aligned" to the Common Core standards.

The Virginia's SOLS Strand Introduction in the Curriculum Framework clearly states that students should actively engage with measurement in their environment. This curriculum unit provides many opportunities for students to measure the area and perimeter of objects in and around the school environment. The curriculum unit closely matches the description in the strand introduction.

SOL 4.7 states, "The student will solve practical problems that involve determining perimeter and area in U.S. Customary and metric units."

The curriculum unit also aligns with SOL 4.12 that states, "The student will classify quadrilaterals as parallelograms, rectangles, squares, rhombi, and/or trapezoids."

The curriculum framework lays out the "essential knowledge" the students should learn. In this section, it specifies that students should do the following: 1) develop definitions for parallelograms, rectangles, squares, rhombi, and trapezoids, 2) identify properties of quadrilaterals including parallel, perpendicular, and congruent sides, 3) classify quadrilaterals as parallelograms, rectangles, squares, rhombi, and/or trapezoids, 4) compare and contrast the properties of quadrilaterals, and 5) identify parallel sides, congruent sides, and right angles using geometric markings to denote properties of quadrilaterals.

Common Core

The Common Core standards focus on area and perimeter in grade 3. These Common Core standards: CCSS 3.MD.C.5; CCSS 3.MD.C.5.A; CCSS 3.MD.C5.B; CCSS. MD.C5C speak to the square unit, covering an area with unit squares, and recognizing area as an attribute, and understanding area as measurement.

Common Core standards CCSS 3.MD.C.6; CCSS 3.MD.C.7; CCSS 3.MD.C7A; CCSS 3.MD.C7B address counting unit squares; relating area to addition and multiplication; and connecting tiling area to multiplying side lengths.

CCSS 3.MD.C.7.C; CCSS 3.MD.7.D; and CCSS 3.MD.D.8 move more into problem solving using multiplication; using area models and the distributive property; recognizing that area is additive (for non-overlapping unions of regions); and finding perimeters of polygons given side lengths, finding an unknown side length, and solving problems with a fixed perimeter, and problems with a fixed area.

In grade 3, the geometry standard, CCSS 3.G.A.1 relates to classifying quadrilaterals. CCSS 3.G.A.3 relates to reflection symmetry.

In fourth grade, CCSS 4.MD.A.3 students apply area and perimeter formulas to solve real world problems. Given the area, the students will find the width or the length by viewing the formula as an equation missing a factor.

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Endnotes

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