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From Polyominoes to Planters: Using Manipulatives and Project-Based Learning to Explore Measurement

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Introduction and Rationale

At Albert Hill Middle School of Richmond Public Schools, the pattern where black-male children are over represented as students with special needs is all too plain. In my self-contained mathematics classroom of 2018-2019, the portion of black boys fluctuated between two thirds and three quarters of the classroom population while the overall percentage of black-students in the school was just under one half.[1] Students within such special-educational settings are often offered lessons that have been modified substantially and that contain activities incorporating the use of what are customarily called “manipulatives,” or hand-held objects connected in some way to the lesson. In observing the use of these manipulatives in the self-contained setting, I have concluded that teachers do not always introduce manipulatives in effective ways for these students. While Van de Walle states, “hands-on, reflective, and interactive experiences are at the heart of good geometry activities at the elementary and middle school levels,” I have seen manipulative-based pedagogy that leads students away from mastering concepts of geometry.[1] Indeed, in their review of research, Laski et al. find that negative results stem from improper use of manipulatives.[2] In view of this issue, I argue that improper manipulative use makes the fact that geometry is under-emphasized within our curriculum framework more problematic for students with special needs. As Karp and Wasserman state, while geometry is emphasized less in middle-school curricula, it is a substantially vital field of study because it serves as a bridge between visual geometry learned in elementary school and abstract geometry taught at the high-school level.[3] Again, this is especially important for those students who are over-represented in special education settings. The use of effective, evidence-based methods such as manipulatives in the mathematics classroom is critical if we hope to meet the educational needs of students who have substantial learning disabilities. Further, to ensure that such students can access mathematics in visual ways, ways which geometry facilitates, is to take a step toward equity. Considering the above conditions, this curriculum unit is built as an intervention for such students who are learning geometry and measurement.

Overall Learning Goal

The overall learning goal of this unit is to lead students both to mastering the concepts of perimeter, area, and volume of rectangles and rectangular prisms and to computing the area of a rectangle and the surface area and volume of a rectangular prism. These activities will develop students' understanding of the properties of rectangles and rectangular prisms. For simplicity, we will restrict attention to shapes with length, and width (and for prisms, height) that are whole numbers. More specifically, in using such manipulatives as unit squares and unit cubes and in using these tiles as polyominoes, students will be able to develop both concepts of and definitions for perimeter, area, and volume while also solving problems before deriving standard formulas (i.e. $P = 2l + 2w$; $A = lw$; and $V = lwh$). When they have derived standard formulas, students will work on their final project where they explore volume of rectangular prisms by constructing planters.

All activities in the unit will help students understand that rectangles are so simple a shape that formulas can take the place of counting with tiles. The final goal is then for students to make a connection between values derived from measurement with unit squares or unit cubes and values derived from the use of formulas. In other words, students should see that patterns occur when working with rectangles, allowing for the use of formulas. The result of achieving the curriculum unit's main goal will be to emphasize concepts of measurement that can often be quite vague without concrete reinforcement. To do this, the curriculum will seek to re-teach foundational concepts of measurement such as those presented in Beckmann's *Mathematics for Elementary Teachers with Activities* while expanding on knowledge necessary to master Virginia Standards of Learning regarding the area of rectangles.[4] While this curriculum unit will center only on rectangles and rectangular prisms, students can apply what they have learned about these more rudimentary shapes to a wider array of polygons or solid shapes.

General Learning Strategies

The strategies which will help students achieve the overall learning goal include the use of manipulatives, visualization through project-based learning, and direct-instruction paired with both open-ended tasks and open-ended prompting. In this curriculum unit, open-ended activities and project-based learning should draw out prior knowledge of the properties and definitions of rectangles and expand student understandings of these properties before they begin processes of measurement in later activities. However, having open-ended tasks and using their own prior knowledge, some students may choose to measure their shapes early on with rulers. Research has shown that strategies such as increasing spatial ability through manipulatives and through student-generated illustrations can also increase student performance and understanding.[5] Therefore, students will use a variety of manipulatives as unit squares to create and to draw rectangles with various perimeters and areas. They will do so in order to strengthen foundational understandings of perimeter and area measurement, specifically regarding the comparison of rectangles with fixed perimeter or fixed area. In creating visual representations of their rectangles, students will be able both to compare the measurable characteristics of their rectangular shapes and form a spatial understanding of rectangular objects. Students will, for instance, have ample opportunity to draw various rectangles from written prompts and then to practice measurement of perimeter and area. The method of creating rectangular shapes out of whole unit grids will then be connected to previously learned knowledge of arrays and multiplication. This will be imperative in developing a sufficiently strong conceptual understanding of two-dimensional measurement so that students can adeptly manipulate formulas. It will also lead students to the idea that deriving area of rectangles is comparable to deriving area of other two-dimensional shapes.

The development of taxonomical frameworks of two- and three-dimensional shapes will also be helpful in having students compute measures in this unit. In the sixth-grade Virginia Standards of Learning, there is relatively little emphasis on the taxonomy of common shapes into typologies.[6] Nonetheless, students should come to understand that rectangles are a subset of polygons which have reflectional symmetry of two degrees and with diagonals that have equal lengths.[7] While not often emphasized, such ordering is quite instrumental in developing connectional knowledge of shapes. In *Teaching Children Mathematics,* Siegel and Ortiz describe the need for correct sorting of shapes before moving into measurement application.[8] They explain that before discussing perimeter with elementary students, they reviewed classification because "some students had struggled naming quadrilaterals, I first asked the class to name the polygon."[9] In the case of Siegel and Ortiz, the relatively informal process of identification was done with a "thumbs-up" math conference and produced the satisfactory result of students coming to the idea of multiplying both measures by two and then adding the products together to find a rectangle's perimeter.[10] While Siegel and Ortiz had moderate success in such a review, this success may not be met in classes with students with disabilities who may have impairments in cognitive processing or memory retrieval. As this is the case, this curriculum unit will include direct-instruction and scaffolded elements that reteach classification as opposed to informal review. Additionally, it will use graphic organizers such as those featured in *Teaching Student-Centered Mathematics* to explore patterns in area and perimeter.[11] More specifically, during practice in categorization, the square will be presented as having a reflectional symmetry across four different lines, while the rectangle will be presented as having two lines of symmetry.

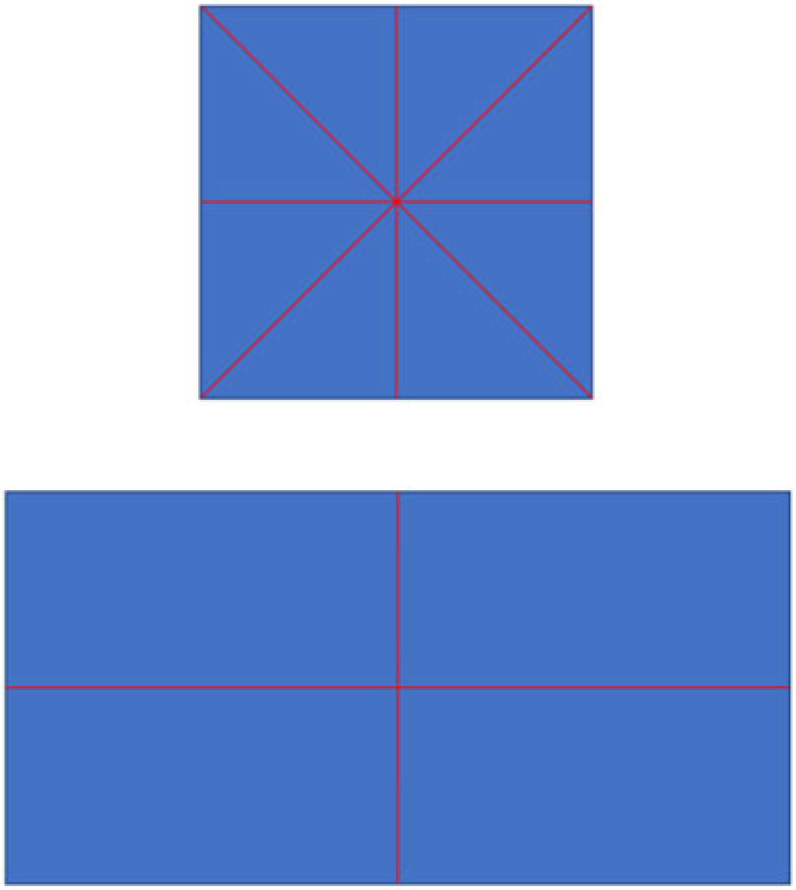

Figure 1. This illustration depicts a square’s four lines of symmetry and a rectangle’s two lines of symmetry.

Another significant process in creating foundational understandings of area measurement is having students create grids on squares of the same length and width and then having them count the units. This procedure fits within the larger strategy of having students explore concepts with multiple means of representation and can also be connected to previously learned knowledge of addition, arrays, and multiplication. Alternatively, it can be done through the open-ended task of having students create polyominoes out of two-dimensional unit squares. Students will then focus on creating polyominoes that are nets of three-dimensional figures. According to Van de Walle and Lovin, it is important to use whole number units and to provide students with rulers and unit squares while doing this.[12] As polyominoes are iterations of a single unit square (i.e. each tile used to make a polyomino has the same dimension), they are versatile tools in teaching operations such as addition and multiplication and, also, in teaching geometrical concepts such as distance, perimeter, and volume. Additionally, Van de Walle and Lovin explain that the process of students’ using rulers and a square unit to determine area without explicit instruction is critical in developing a strong conceptual understanding of two-dimensional measurement.[13] The following quote explains that instructors should guide students’ own determinations toward the main idea of area as a measurement of arrays before introducing formulas:

> “As students discuss their various approaches, focus on those ideas that are closest to this idea: The length of a side will determine the number of squares that can be fit on the side. The length of the other side will determine how many rows of these squares will fit in the rectangle all together. Multiply the length of a row by the number of rows.”[14]

The Learning Progression for Geometrical Measurement

For students with disabilities severe enough to significantly impede learning, concepts and processes of measurement can often be quite vague if they are introduced without concrete reinforcement. Geometrical measurement involves many separate procedures and takes regular practice to master. Without such mastery, there can be little expectation for a student in a self-contained setting to make marked improvement in mathematics instruction or to overcome significant learning challenges in general. Geometrical measurement itself is defined by the National Council of Teachers of Mathematics as "the assignment of numerical value to an attribute of an object, such as the length of the pencil" through unit partition and unit iteration. As stated above, middle-school geometry serves as a bridge between visual concepts taught in elementary school and abstract, high-school concepts. Felix Klein provides a useful and elegant definition of geometry: geometry is the study of the properties of figures that are unchanged by rigid motions.[15] If a student has yet to develop counting skills and has also not developed an understanding of the properties of shapes, measurement and manipulation of shapes would be hard to perform with proficiency. Kim et al. offer a framework, titled "Learning Progression for Geometrical Measurement in One, Two, and Three Dimensions," which serves to address the need for effective geometrical instruction and which can be applied to instruction for students with special needs.[16] The framework provides the underlying structure for this curriculum unit and includes the following processes:

1. Unit partition and unit iteration
2. Spatial ability
3. Efficient-sized measurement units and spatial structuring
4. Abstraction

In using unit squares and unit cubes, the activities in this curriculum unit help students understand measurement as partition into units. Further, through activities related to polyominoes, rectangles, and rectangular prisms, they will increase their spatial ability. In manipulating physical objects and in transferring them into visual representations, students will begin to understand that measuring is the decomposing of an object into "efficient-sized measurement units" or, as Kim et al. state, the "taking a part of an object as a unit and then placing the unit alongside the object in terms of unit subdivision and unit iteration, respectively."[17] Having students move between practice in two dimensions and practice in three dimensions will also be a strategy used in this curriculum unit. As Kim et al. show, this movement, from dimension two to dimension three, is valuable in forming spatial reasoning skills and forming what Kim et al. call "spatial ability," or the ability to understand the relationships extending from a two-dimensional representation of an object to the object itself, which is understood according to "terms of their three-dimensional properties."[18] In other words, students with spatial ability know that parts of shapes not represented in a depiction are in reality still present. Open-ended questions that can increase spatial ability and that will be used in this curriculum unit include, "What do you notice about the face of your planter if I turn it around?" and "How would you draw this shape now on your paper?" These questions should make explicit the fact that there are some faces that, while not being visible, are still present, so that students begin to develop an ability to visualize spatial relationships while reading two-dimensional representations of three-dimensional objects.[19] Finally, abstraction will occur when students transition to using the formulas for area and volume after practicing measurement with manipulative units.

The Use of Polyominoes as an Effective Manipulative in the Special-Education Classroom

My research indicates that, when coupled with direct-instruction and open-ended tasks, the proper use of manipulatives can serve as a powerful tool in special-education settings. Carbonneau and colleagues' 2013 meta-analysis of studies on the efficacy of manipulatives explains that when manipulative use is accompanied by high levels of instructional guidance there is an increase in academic performance and student understanding.[20] Additionally, Boaler and colleagues support a pedagogical move away from teaching math as mostly an algorithmic practice where students simply "memorize and calculate." [21] They argue for a movement toward mathematics pedagogy where visual processes are encouraged and where approaches such as the use of manipulatives are coupled to a deep and deliberate thinking process, initiated but not ended by instructors.[22] Yet, while many educators in mathematics rely extensively on their use, data show that the use of manipulatives is not always productive. Again, in their review of research, Laski et al. found that many mathematics strategies involving hands-on activities impede learning.[23] To address this issue, Laski et al. put forth the "Four Principles for Maximizing the Effectiveness of Manipulatives," a list in which they enumerate the requirements for effective use of manipulatives in mathematics. The four principles follow:

1. The consistent and coherent use of objects longitudinally where there are successive observations of a phenomenon and its relations to other phenomena.
2. The use of "highly transparent concrete representations" followed then by the use of "more abstract representations."
3. The avoidance of objects that have what I would call connotative dissonance or, in other words, alternative meanings which may interfere with both the significance of the concept on which the lesson circles.
4. The direct and explicit presentation by the instructor of the connection between the manipulative and the specific mathematical concept it embodies.

The phrase "highly transparent concrete representations" refers to an object or image in mathematical instruction that has significant verisimilitude and which details in an exact manner the concept being taught, so that both a visual and conceptual connection can form between the two. Concrete representations that are fully transparent present neither conceptual static nor distraction for students because their design is devoid of redundant or irrelevant characteristics. In the case of this curriculum unit, the rectangle and the rectangular prism are both objects that must be encountered in everyday life, and that without much effort can display transparent representational coherence or, more succinctly, a representational naturalism. Additionally, simple unit squares and unit cubes will be highly transparent manipulatives. Laski et al. define the common term "manipulative" as "concrete materials used to demonstrate a mathematics concept or to support the execution of a mathematical procedure."[24] As Laski et al. write, the manipulative as concrete object represents the concept as abstract knowledge.[25] Thus, hands-on activities can be the use of what is often called a "foldable" worksheet, where the foldable has no representational value within the framework of the mathematical concept of instruction. On the other hand, manipulatives are objects such as one-hundred squares which represent numerical concepts such as additive properties and decimation and that, therefore, facilitate numeracy. In effect, manipulatives make explicit what in a concept may be implicit or abstract, while hands-on activities often simply impede learning.

Considering the above scholarship, I have distinguished between "hands-on" activities and "manipulative" activities, avoiding the former in designing this curriculum unit. Again, there is a substantial difference between the term "hands-on" and the term "manipulative," which often goes ignored in special educational settings. To be clear, the term "hands-on" designates an activity involving materials that can be obliquely related to or entirely detached from the concept of the lesson. The term "manipulative", in contrast, indicates the use of concrete materials within a lesson where those materials are deeply and inextricably connected to the content and concept of the lesson. In the case of this unit, manipulatives will be one-inch unit squares and one-inch unit cubes which have no other symbolic value for students. With the use of unit squares, which I perceive as highly transparent objects, students will configure polyominoes and decompose planar shapes to explore concepts of symmetry, rotation, and additive properties of area.[27] In order to create a polyomino students will take the middle of a square's side and join it to the middle of at least one other square's side. Citing Solomon W. Golomb, Roger Howe presents the following definition for polyominoes:

> "a figure made by joining squares, with: i) two squares touching either only at a corner, or along a whole edge of each; and ii) connected, in the sense that you can get from any square to any other square by passing through the middle of edges (and avoiding corners)."[27]

The use of polyominoes provides ample opportunity for students to visualize additive properties and to practice measurement using units. Yuan, Lee, and Wang show as much in their 2010 study revealing marked improvement in learning when middle-school children used polyominoes, whether these were digital or physical.[28] And because there is such a large number of activities that can be performed by students using these manipulatives, there is ample opportunity to affect an open-ended instructional style.

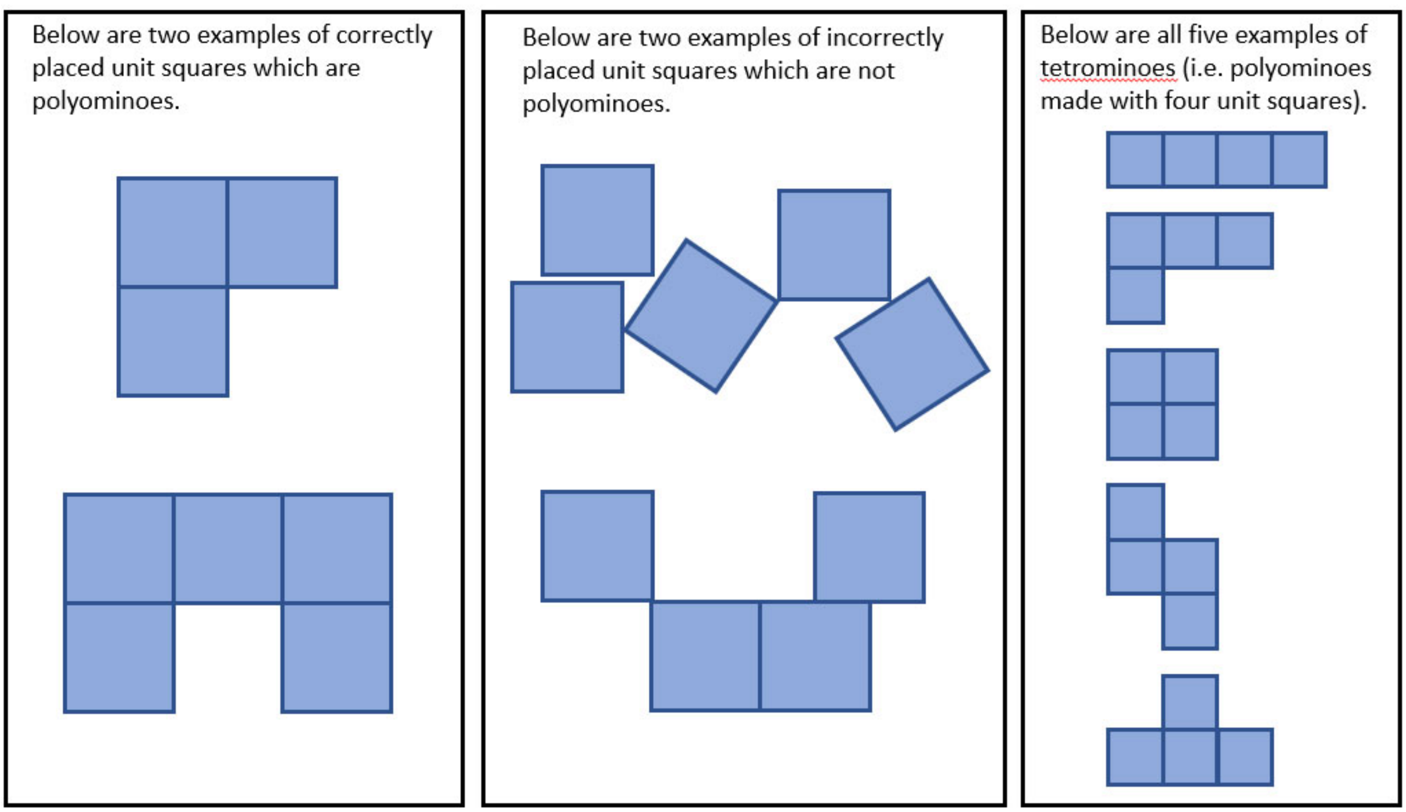

Figure 2. The above illustration presents two examples of polyominoes, two non-examples of polyominoes, and all five versions tetrominoes (i.e. polyominoes made of four unit squares). It should be noted here that there are twelve possible versions of pentominoes and 35 different versions of hexominoes where rotations and reflections of these shapes are not considered distinct.

Project-Based Learning and Open-Ended Tasks in Teaching Area, Perimeter, and Volume

As delineated by Stephanie Bell, the definition of Project-Based Learning (PBL) is “a student-driven, teacher-facilitated approach to learning” where students “pursue knowledge by asking questions that have piqued their natural curiosity.”[29] Bell explains further that PBL raises the academic performance of students and increases their self-efficacy. Both increases can be understood as resulting from learning centered on individual responsibility, group accountability, independence, self-monitoring and sharing out student work with significant community members. While peer-to-peer collaboration can be an important component of PBL, it is not an essential one. That is, through the actual construction of the planters the individual student can learn and engage this curriculum in meaningful ways by working independent of others. In the case of this curriculum unit, the process of guiding student’s planter-building process and the process of directing students to calculate the volume, area, and perimeter of the objects individually can be understood as both a collaborative and project-based process.

Along with developing an understanding of the relationships of shapes and their components through both student-generated representations and ordering tasks, this curriculum unit will rely heavily on open-ended tasks similar to those espoused by Dorothy Varygiannes in her recent article “The Impact of Open-Ended Tasks.”[30] In this article, Varygiannes shows that there was an explicit push to include open-ended tasks in classroom instruction upon the adoption of Common Core standards. She writes that providing opportunities for “outside-the-box” thinking was a way of “inviting our students into the mathematics conversation…and providing a ‘safe haven’ for them to contribute and to increase confidence.”[31] Student application of conceptual understandings spurred by higher level thinking within an open-ended framework was considered a prime indicator of student success in Common Core. Thus, open-ended tasks should not be undervalued in planning. In this unit, open-ended tasks will include asking students, “How many rectangles can you make with an area of 16 (and whole number side lengths)?” This open-ended activity will invoke prior knowledge of the properties and definitions of rectangles and expand student understandings of these properties before beginning processes of measurement in later activities. Furthermore, questions such as “Why did the perimeter of your shape change?” and “Why did the area of your shape stay the same?” will be posed to students during their activities. A noteworthy open-ended task from Siegel and Ortiz’s lesson mentioned above was having students draw a series of rectangles with different dimensions on graphing paper in order to assist their calculations and measurement. Students can decide for themselves the shape and size of the rectangles, and once done, find different ways to measure area and perimeter. This open-ended process of creating visual representations is so valuable that the authors express the following: “Pictorial representation is an incredibly important tool…that my students are encouraged to implement on a near-daily basis.”[32]

In conclusion, the review of taxonomy, student’s daily production of images, PBL, evidence-based manipulative use are all significant features in the “From Polyominoes to Planters” curriculum unit. Combining these strategies will demand of students a high level of cognitive engagement and will allow them to understand shapes, their interwoven relationships, their dimensions, and how to measure them. Further, the categorization and reproduction of shapes has importance for students with disabilities, because without this, the process of measurement and identification can be a nebulous set of procedures to them. Only after students master the representation and classification (e.g. students’ creating property lists of rectangles) will they learn the formulas needed to calculate values of perimeter, area, and volume. In using these strategies, the curriculum unit will move from concrete to abstract instruction in accordance with “The Learning Progression for Geometrical Measurement” discussed above.[33]

Classroom Activities

Activity One - Property Lists and Minimal Property Lists of Rectangles

Activity-One Objective

This open-ended activity will reengage prior knowledge of the properties and definitions of rectangles and expand student understandings of these properties before beginning processes of measurement in later activities. Students will be provided with rulers and unit squares for measurement. The intent of this activity is for students to realize that the opposite sides of any rectangle are equal in length.

Activity-One Procedure

Teacher will provide a word bank of properties (including terms such as vertices, line segments, angles, parallel, congruent, opposing, intersections, perpendicular), and also poster boards, rulers, and various sheets of colored paper cut into rectangles of various sizes, that students will be asked to measure.

Activity-One Mini-Lesson

Students will work in pairs or small groups to select all properties from the word bank in order to form their property lists for their unique rectangles, re-writing their lists on the right side of their poster boards. Students who complete this task will be encouraged to think of properties and vocabulary terms not included in the word bank.

Students will redraw their rectangles on the left side of their poster board, creating a visual with both text and image which clearly enumerates the properties of their shapes in multiple forms of representation. While students present their posters, the teacher will provide open-ended prompts to facilitate discussion of each poster board.

Students will then work in pairs and small groups to create what Van de Walle calls “Minimal Property Lists,” or “subsets of the property lists that, if true, would guarantee a shape of that type, and with no unnecessary properties. If any property is removed from a minimal list, the list should no longer guarantee the shape.”[1] An example of the property list is below:

Rectangle Property List

- Four sides
- Four right angles
- Opposite sides parallel
- Diagonals bisect each other
- Adjacent sides perpendicular
- Opposite sides equal in length
- Diagonals equal in length
- Lines of symmetry through opposite sides
- At least one right angle
- Four vertices connected by straight lines

Activity Two – Creating Polyominoes and Rectangles with Unit Squares

Activity-Two Objective

Students will use unit squares to create polyominoes and rectangles with various perimeters and areas in order to strengthen foundational understandings of perimeter and area. In forming polyominoes and rectangles, students will be able both to compare the measurable characteristics of their rectangular shapes and to form a spatial understanding of rectangular measurements.

Activity-Two Procedure

Students will be provided unit squares and asked to create as many different polyominoes and rectangles as they can with a given number of tiles, and will record the measurements of their shapes on graphic organizers. Open-ended prompting for this activity will include variants of the following statements:

- Make as many different-sized rectangles and polyominoes with four, unit squares that you can.
- Draw a sketch of each one of your shapes and record the area and perimeter of the rectangles beside your drawing.
- Now make as many as you can with six, unit squares. Measure and draw those.

This prompting would go on to include as many unit squares as students need to understand the process of measurement. All squares should be used in a given prompt. For instance, students should use six squares if the open-ended prompt directs them to do so.

Activity-Two Mini-Lesson

Students will first be directed to build as many polyominoes and rectangles of different lengths and widths with their squares as they can. Students will work independently before sharing the perimeter and area of a few of their shapes with the class. After sharing out, they will be provided illustrations of five rectangles that they have to recreate with their units. Students will measure the lengths of these shapes and their areas to verify that they are congruent and to compare their shapes with the original examples.

Activity Three – Drawing Rectangles and Squares on Graphing Paper and Multiplication Charts

Activity-Three Objective

In this activity, students draw various rectangles from written prompts to practice measurement of perimeter and area. At this point in the curriculum unit, the process of creating rectangular shapes out of whole unit grids will be connected to previously learned knowledge of arrays and multiplication. This will be critical in developing a strong conceptual understanding of two-dimensional measurement needed to adeptly manipulate formulas. A multiplication table such as the one below will be used here.

	1	2	3	4	5	6	7	8	9
1									
2									
3									
4									
5									
6									
7									
8									
9									

Figure 3. This multiplication table will be used so that students connect the process of creating rectangular shapes to measuring area and, importantly, to multiplication.

Activity-Three Procedure

Students will receive task cards of written prompts which direct them to draw rectangles with specified bases, heights, perimeters, and areas on graphing paper. After they present their work to the class, they will then be provided a multiplication chart consisting of a square grid. Students will color in rectangles on this grid and will have to determine the area and perimeter of each shape.

Activity-Three Mini-Lesson

Students will work individually and then in pairs to determine if they correctly depicted the rectangle according to its description on the assigned prompt. At the end of this lesson, the formula $A=lw$ should be taught explicitly to students.

Activity Four – Perimeter and Area of a Rectangle and Triangle Sort

Activity-Four Objective

In this activity, students will use the formula introduced in the third activity. It is important for students to note that the process of deriving area established for rectangles is a common process which can be used for other two-dimensional shapes. Therefore, the area of triangles will be briefly discussed here.

Activity-Four Procedure

Provided with rectangles that have a large range of areas and perimeters, students will sort a variety of these shapes into groups of rectangles that have equal perimeters and equal areas. They will also find all triangles that have half the area of each rectangle in the sort. Before introducing the step of the activity where students will match triangles with half the area of rectangles, the teacher will demonstrate by folding with paper folding and paper cutting that the area of a triangle is just half that of a rectangle. Paper folding and paper cutting will start with right triangles made from rectangles and squares and will then include a few examples of general triangles made from parallelograms.

Activity-Four Mini-Lesson

Through open-ended prompting students will develop a process of calculating the bases and the heights of their rectangles during this sort. A prompt for this activity would include “The area of this rectangle is 4 square units and its perimeter is 10 units long. Which rectangle is the correct one?” or “Which triangle do you think has an area of six? Why”

Activity Five – Calculating the Planting Area and Volume of Seedling Planters

Activity-Five Objective

The objective of this activity is to have students use processes to find the volume of rectangular prisms that are similar to those used when finding the area of two-dimensional rectangles earlier in the unit. Specifically, students should be able to apply the formulas $V = l\ w\ h$ or $V = b\ h$ to find volumes of all rectangular prisms while remembering that finding the volume of a three-dimensional prism related to finding the area of its base.

Activity-Five Procedure

This activity will include the creation of nets of rectangular prisms using techniques and formulas learned over the previous four lessons, to determine the planting area and volume of student-made planters. The definition of a net will be provided explicitly by the instructor.

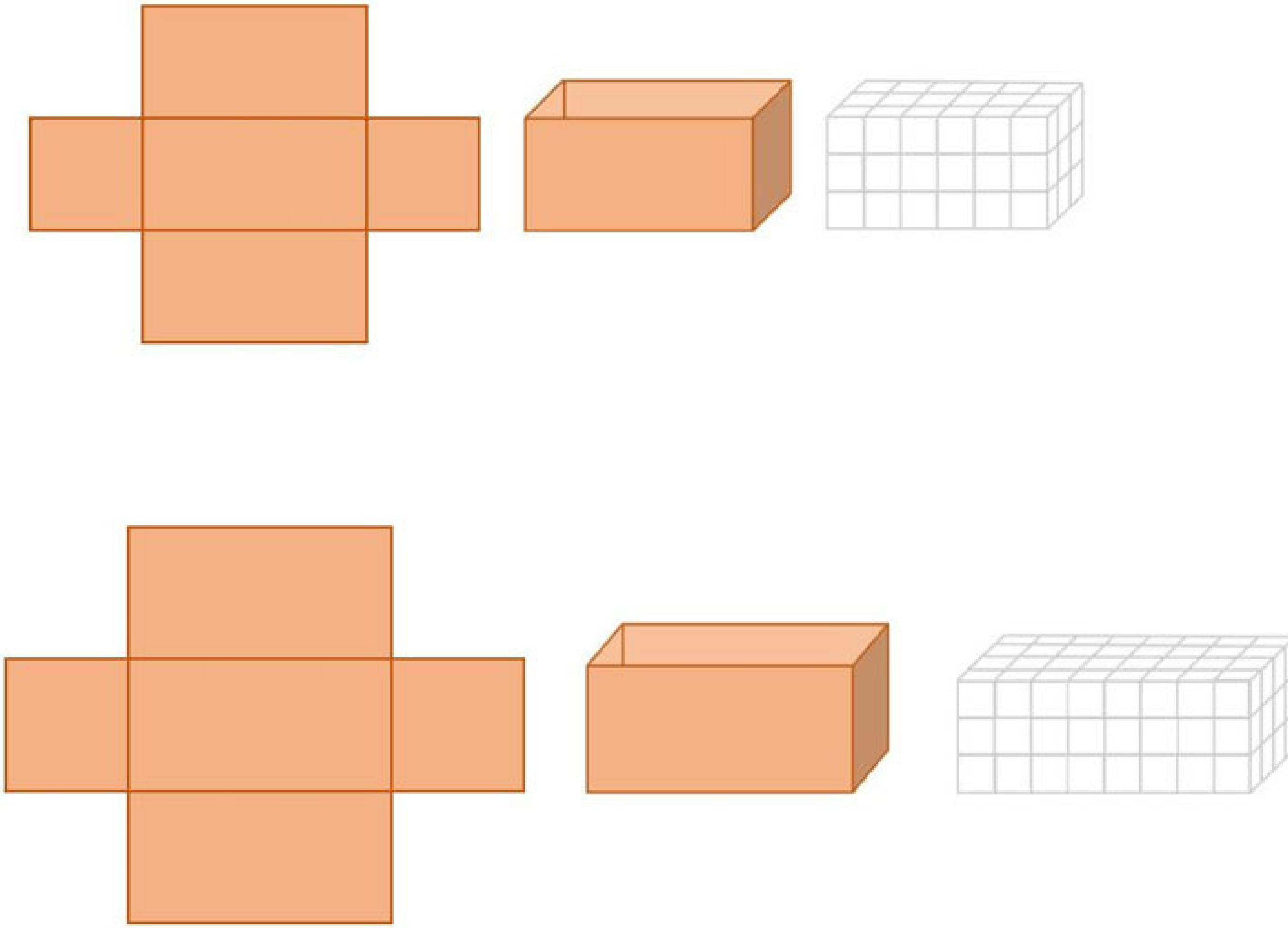

Figure 4. Above are examples of two different-sized nets of rectangular prisms, the planters constructed out of these nets, and the rows, columns, and layers of unit cubes which students will use to calculate the volume of their shapes.

Activity-Five Mini-Lesson

To begin, students will be provided four different nets made of chipboard along with worksheets that depict the nets' faces as two-dimensional planes (i.e. rectangles). Before constructing the nets, students will be tasked with finding the perimeter and area of each rectangle that makes up the net. They will then use their worksheets to input measurements, specifying if they note any patterns or similarities regarding the shapes' perimeter and area measurements. Students will discuss what similarities they found in their measurements. After discussion, students will begin constructing their nets to make their planters.

Once the planters are finished, the teacher will provide unit cubes to the class, demonstrating with a pre-made box how the unit cubes can be used to find its volume. Students will calculate the volume of one of their planters with unit cubes, inputting the measurements into their worksheets. At this point students should use their data and their volume to derive either a formula or a systematic method without the use of unit cubes to produce the volume of each of the four planters. Students will be asked the following questions in order to assist in the formulation of the volume formula:

- "How many cubes fit on the bottom of your box?"

- “How many layers of cubes did you have?”
- “What do you think the number of layers can tell you about the height of the planter?”
- “What can you tell me about the number of cubes running along the inside of the planter?
- Is there any relationship here with what you see on the inside and what you noted about the outside measurement?”

Students should recognize that the measurement of the interior array that they created shares the dimensions of the nets they used to create their planters. Once the line of prompting leads students to make connections about the relationships of their measurements, they will need to use the formulas of $V = l\ w\ h$ and $V = B\ h$ to derive the volume of the other three. Specifically, students should be able to apply the formulas of volume, $V = l\ w\ h$ (where l stands for length, w stands for width, and h stands for height) or $V = B\ h$ (where B stands for the area of the base and h stands for height. to find volumes of all rectangular prisms while remembering that finding the volume of a three-dimensional prism related to finding the area of its base. Students will input their calculations into their worksheets and, once this is finished, can then request the correct volume of soil from the instructor to fill up all their planters.

Resources

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26. For polyomino activities related specifically to perimeter and area, see Henri Picciotto's section "Perimeter and Area" in *Polyomino Lessons* (1986) at www.MathEducationPage.org.
27. In a draft of this unit with his comments within, Roger Howe provided the definition quoted here. This definition is derived from Golomb's 1965 work *Polyominoes.* Roger Howe, email correspondence with the author, August 3, 2019. See generally Solomon W. Golomb, *Polyominoes* (New York: Scribners, 1965).
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Appendix – Implementing District Standards

Richmond Public Schools follows the Virginia Department of Education's (VDOE) Standards of Learning (SOL). According to the *Mathematics Standards of Learning for Virginia Public Schools*, sixth-grade students will be expected to solve area and perimeter problems of rectangles while seventh-grade students will be expected to solve problems related to volume of rectangular prisms. This curriculum unit focuses on the following sixth- and seventh-grade standards within the "Measurement and Geometry" sections of the VDOE's *Mathematics Standards of Learning*:

SOL 6.7c The student will solve problems, including practical problems, involving area and perimeter of triangles and rectangles.

Activities one through four are open-ended activities that will help students define the properties of rectangles and then measure the shapes. They will use manipulatives as well as multiple means of representation to find the perimeter and area. Eventually, students will come to understand the meaning and purpose of the formulas used to find a rectangle's area. During these activities they will see that unit iteration is the basis of measurement and will abstract the process of using units so that they can grasp the connection between arithmetic processes of formulas and physical processes of measuring using manipulatives.

7.4a The student will describe and determine the volume and surface area of rectangular prisms.

The fifth activity extends into the seventh-grade curriculum and is a project-based, practical project where students will use unit cubes to solve the volume of various rectangular prisms that they will then fill with the correct volume of soil. The focus here is to have students grasp the relationship between volume and area. The last activity will help students develop their spatial reasoning and spatial ability while also seeing the connections between forms of measurements of rectangles and rectangular prisms.