Curriculum Units by Fellows of the National Initiative
2023 Volume III: Transitions in the Conception of Number: From Whole Numbers to Rational Numbers to Algebra

A Framework for Problem Solving: Schema Development and Discourse

Curriculum Unit 23.03.04, published September 2023
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Introduction

Children need to be mathematically literate, as their world is increasingly more dependent on technology, or they will be destined for a lower economic class.[1]

> *More than any other subject, mathematics filters students out of programs leading to scientific and professional careers. From high school through graduate school, the half-life of students in the mathematics pipeline is about 1 year, although various requirements hold some students in class temporarily for an extra term or year. Mathematics is the worst curricular villain in driving students to failure in school. When mathematics acts as a filter, it not only filters students out of careers, but frequently out of school itself.*[2]

Currently, my school district is one of the lowest performing in the state of Virginia, and mathematics is no exception. The graduation rate for 2022 was 74.2%.[3] The lack of mathematical literacy is an increasing problem across the country that has been compounded by the learning loss due to COVID. Students are acutely aware of their abilities regardless of their grade. Early experiences, successes and failures, impact not only students' attitudes toward math, but their future relationship with math. Mathematics in elementary school lays the foundation for middle and high school. I hope this unit will engage students as they work collaboratively to examine word problems through mathematical discourse and the use of schemas.

Rationale

I teach fourth grade and serve as a mathematics Academic Architect for the school district. As an Academic Architect I work with the math department to develop the grade 4 pacing and curricular resources for the district.

My school district uses Eureka Math (the Virginia version, which is only slightly different from the Eureka Engage NY version). Most of the changes occur in the fluency component of each lesson. The district develops a pacing calendar and common assessments that teachers across the district follow. The pacing calendar shows which lesson teachers teach each day and omits lessons that do not align with the Virginia Standards of Learning (SOLs). Additional lessons are added to support the alignment to the state-mandated curriculum. The district's expectation is that teachers use the district's curated resources and stay within five days of the recommended pacing. This curriculum unit is designed with these expectations in mind. The ideas and strategies can be infused into a problem-solving lesson or application problem and should be useful for teachers who have similar parameters to follow.

I developed this curriculum unit not only to support students as they tackle single and multi-step word problems, but also in a way that will work seamlessly with the district's curricular resources. This curriculum unit is designed to help students analyze word problems and categorize them based on similarities and differences. The word problems in Eureka Math fit the problem types described in the Common Core State Standards Taxonomies.[4] As students work with word problems, they will begin to recognize the single-step components that make up the multi-step problem, thus enabling them to solve these more difficult problems. Virginia uses a version of the taxonomies in the Standards of Learning.

Identifying patterns, similarities, and schema among word problems foster students' problem-solving skills and algebraic thinking. This curriculum unit is designed as a companion to the curriculum unit I developed in 2017 titled *Understanding Problems: Using Bar Models with Common Core Taxonomies,*[5] which explains and provides examples of all the problem types.

The unit will examine word problems within the grade 4 Eureka Virginia curriculum for schemas. A feature of this unit will be a framework to guide students' identification of schemas through mathematical discourse. See below for discussion of the term "schema".

I plan to lay out the argument that it is beneficial to teach students to recognize and apply schemas as a segue into algebra. By having students discuss, create, and categorize word problems, they will look for patterns and develop their algebraic thinking and algebraic readiness. "Problem solving is a relevant and significant perspective and context through which to introduce students to algebra."[6]

Background Content Knowledge

U.S. vs Chinese Mathematics Education

On international studies, the United States underperforms compared to other countries, especially East-Asian countries. In most countries around the world, education is a part of the national government. However, in the United States, education is a state right. With 50 states who independently decide the standards, the policies and the materials, there is little consistency compared to other countries with national control.

Yan Ping Xin, a professor at Purdue University, examined textbook series in the U.S. and China to determine their relation to student performance on problem solving tasks. He used previous research by others to draw some conclusions about successful problem solvers. In general, successful problem solvers can:

1) identify the mathematical structure, that is broadly generalizable to similar problems;

2) retain the problem structure for a long time; and

3) discriminate between relevant and irrelevant information.

Research also indicates that solving problems that appear different but have a similar structure will promote schema development and foster one's generalizable problem-solving skills.[7]

The study determined that Chinese textbooks provided a variety of problem types using the same story. They have a name for this: teaching with variation. In addition, the Chinese textbooks required students to solve the problems with arithmetic as well as symbolically using a variable. Chinese textbooks also have students modify the problems to fit a different schema. Similar opportunities were not evident in U.S. textbooks.[8] However, Xin's study was conducted prior to the implementation of the Common Core State Standards (CCSS) which are now used by 41 states and the District of Columbia. CCSS have brought more consistency to education in the United States. The Common Core State Standards have clearly articulated types of schemata for problem solving which will address the shortcomings of U.S. mathematics instruction. Following the introduction of the Common Core State Standards, Eureka Math was developed as a curriculum to teach the new standards.

Schema

A goal of this unit is to improve algebraic thinking and readiness through problem solving. As a part of my research, I read a lot of articles about the use of schema to develop problem solving abilities. A problem schema is a general description of a collection of problems that have a common structure and require similar solutions.[9] When students are taught to recognize the schema and see that it can apply to other problems, they improve their problem-solving success. In this unit, students will develop algebraic thinking as they look for patterns among word problems and find connections between them. As a part of the framework that I employ in this unit, students create various word problems for a single-story situation (in the terminology of Xin), similar to mathematics textbooks used in China and described by Xin. Once the students create a variety of word problems, they then categorize and sort them based on the structure and the solution. According to schema-construction theory, a major obstacle of problem solving is the development of schemas that rely on the same solution. Superficial changes can make problems seem novel without impacting the problem type or the solution. Examples of superficial changes from one study are 1) format, 2) key word vocabulary, 3) additional question posed, and 4) placement into a larger problem context.[10] Superficial changes can complicate problems and disguise the problem type, thus making it more difficult for students to discern the schema. Exposure to problems with superficial differences has been shown to improve grasp of schemas and mathematical problem solving in third graders.[11] When students sort novel problems based on their superficial features, they developed broader schema. Chen concurs that solving problems with varying surface features will improve students' chances of noticing the similarities between familiar problems and novel problems.[12] When students become more adept at identifying the schema, they allocate less working memory toward the solution and more toward identifying connections between novel and familiar problems.[13] It is valuable to have students work on identifying schema within word problems because the schema is like a template that can be applied to a wide variety of novel problems.[14] "Schema based instruction fosters skill transfer and generalizable problem-solving skills."[15]

Transfer

Transfer involves students applying knowledge, skills, and strategies to novel problems. There is significant research about transfer, and it indicates that transfer is complex. Experience with problems of a given schema help enhance one's ability to transfer, while a lack of transfer often hinders one's ability to problem solve. There are three transfer-inducing variables: a) master the rules of arithmetic, b) classify problems based on their solutions, and c) make connections between novel problems and previously solved problems.[16] The challenge as a teacher is to help students develop broad schemas. Broader schemas increase the chance that a connection will be identified, thus enabling transfer. As a teacher it is helpful for me to understand that schema theory considers that irrelevant information, combining problem types, and mixing transfer features makes schema identification and transfer more challenging for students. On the other hand, exposure to these more rigorous schema features should not only broaden problem solving schemas, but also increase transfer distance.[17]

Xan Ping Xin proposes a four-step model to schema-based problem solving. The stages are:

1) identify or recognize the schema

2) represent the problem

3) plan and select the operation or determine the equation

4) carry out the plan.[18]

This is similar to George Polya's 4 step model, except it has a preliminary step that replaces Polya's final "Look back".[19] Students should have opportunities to experience a variety of problems that vary in context and structure to help them recognize structures which will enable them to transfer their skills. An example is to utilize the Common Core State Standard Taxonomies, or the similar one in the Virginia Standards of Learning, to provide many opportunities for students to solve all the different single-step types. Then when the students are presented with a two-step problem composed of the familiar single-step types, the students should be able to apply the one-step schemas and transfer their knowledge to the novel problem. The importance of schema transfer is evident because there are fourteen addition and subtraction schemas and nine multiplication and division schemas, which affords the construction of thousands of multi-step problems that are composed of the single-step schemas.

Eureka Math is a coherent program with many problem-solving opportunities. The word problems in Eureka Math fit with the Common Core Taxonomies, so they can easily be used for schema development. The curriculum promotes the RDW method to solve problems. The "R" stands for read. The students read the problem carefully so that they can "D": draw a representation of the problem with a tape diagram. The way students draw the tape diagram and the context of the problem help the students determine the operation needed to solve the problem. Then, the students go back to the question and "W": write their answer in a complete sentence. RDW can also be related to Polya's steps: the goal or purpose of reading is to understand the problem. The drawing is helpful for devising a plan. And the write is what you do after you carry out the plan. The looking back stage is not really there, but can be promoted by your plan to develop awareness and understanding of schemas.

Almost every lesson in Eureka Math has an Application Problem at the start of the lesson. The Application Problems may be a review of a previously taught skill or a preview of an upcoming skill. A review problem that can be done with automaticity is considered a “low-road” transfer and a novel problem would be a “high-road” transfer problem.[20] In addition to the Application Problem, Eureka Math includes problem solving as a part of many of the lessons. There are also lessons that are exclusively problem solving.

National Council of Teachers of Mathematics Process Standards

The National Council of Teachers of Mathematics (NCTM) developed five Process Standards that “Highlight ways of acquiring and using content knowledge.”[21] NCTM identifies the following Process Standards: 1) Problem Solving; 2) Reasoning and Proof; 3) Communication; 4) Connections; 5) Representations. It also describes how these process standards should be infused throughout all domains of mathematics. The five domains of mathematics referenced by the NCTM are: 1) Number; 2) Algebra; 3) Geometry; 4) Measurement; and 5) Data Analysis and Probability. In my school district we are reminded of the process standards and encouraged to embed them into our math instruction. Below, we discuss the Process Standards individually.

Problem Solving

The NCTM describes how these process standards should be infused throughout all domains of mathematics. Problem solving is described as a process in which students engage and grapple with a task to develop mathematical understandings. Problem solving should require effort and include reflection. Students should have frequent problem-solving opportunities, so that they will develop curiosity, perseverance, and confidence in tackling unfamiliar situations.

Reasoning and Proof

The NCTM defines reasoning as a “habit of mind,” and habits develop over time from repetitive use. If we want students to develop habits of mind, then students should have opportunities to explore and to develop ideas and conjectures. When students make assertions, they should also provide a reason to justify their thinking. In elementary school, it may be as simple as folding a paper to reason that ½ equals 2/4. A goal of the Reasoning and Proof standards is that students will see that math makes sense.

Communication

Communication is a part of creating a mathematical community where students can exchange ideas to test their thinking. Ideas must be communicated so that they can be accepted or refuted. It is through the back-and-forth dialogue that students learn to listen, paraphrase, and question. As students share their thinking, they will develop better communication skills that will help them to refine their ideas and explain them more coherently. As students engage in mathematical discourse, they will become more comfortable with the language of math and expand their repertoire of everyday language to include precise mathematical vocabulary. Communication helps students to think critically about mathematics as they articulate their understanding.

Connections

Mathematical connections are important and should be interpreted in many ways. Mathematics is interwoven into other subjects and the idea that mathematics permeates other areas of learning and life will help students to see the importance and value of mathematics. Students and teachers need to see and make connections not only to prior knowledge such as mathematics learned in earlier grades, but also to look ahead to the mathematical concepts of subsequent grades. It is important to make relevant connections between the domains of math as well as with the NCTM process standards. When students make and see the interconnectedness of mathematics they recognize patterns, develop a deeper understanding, and can apply skills to novel situations. As teachers, we need to foster opportunities for our students to see relationships and apply them to their mathematical thinking. By recognizing the interrelatedness of the subject, students will see mathematics as an integrated subject, not as a series of isolated parts.

Representations

Images, models, drawings, equations, tables, and graphs are some of the ways mathematics is represented. Representations convey meaning and understanding of concepts. They also help students expand not only their mathematical ideas, but also their ability to think mathematically. Students should have opportunities to create their own representations as this builds their understanding, but they also should learn conventional representations, which will improve how they communicate about mathematical ideas. Student representations also provide the teacher with valuable insight into students' thinking and understanding that may not be gleaned from only an answer. Certain representations are better than others to convey mathematical ideas. For example, a number line is a linear model, so it conveys distance, while an array clearly displays the commutativity of multiplication. Since certain representations portray different aspects of mathematics more effectively, it is important for students to have many opportunities to experience multiple representations. Additionally, students should reflect on the strengths and weaknesses of various representations for different purposes. Representations can enhance students' development with all the other process standards. It is important for teachers to know that a given representation may not explain itself, and to make sure students grasp the way to think about it appropriately. An example is to make sure students understand that a number line is about length. When students make sense of the representations, they can use them to strengthen problem solving, reasoning and proof, communication, and connections.

Framework for Mathematical Discourse and Schema Building

I have created a framework to address all Process Standards as students discover patterns and schema. Students are provided a word problem scenario. Then they are asked to work in collaborative groups to formulate questions that could be answered using particular operations. After the questions are developed, the students share them. The collaborative groups then figure out how to organize them based on the number of steps and the solutions. Throughout the process students will communicate with each other to explain their thinking. They will reason as they determine how to represent, solve, and categorize the problems. As patterns emerge between the problems that are solved with the same tape diagram, connections will be made.

Module 1: Place Value, Rounding, and Algorithms for Addition and Subtraction

The first scenario is from Module 1, Lesson 12 Problem Set. I chose this problem specifically because it presents the information in a chart. The chart will provide students a method to represent data and serve as a visual model to organize their thinking. The chart also makes problem solving more accessible for students as it will seem less daunting for English Language Learners and reluctant readers. The simplicity of the chart will provide a level of comfort as students embark on their journey creating problems, identifying similarities, finding solutions, and discussing them.

During National Recycling Month, Mr. Yardley's class spent 4 weeks collecting empty cans to recycle (see Table 1). During Week 2, Mr. Yardley's class collected 1,256 more cans than they did during Week 1.

First the students should solve for the number of cans collected in week 2. It would be helpful at this point to put the number of cans for week 2 on the chart.

Week	Number of Cans Collected
1	10,827
2	
3	10,522
4	20,011

Table 1: Cans collected by Mr. Yardley's class.

The tape diagram (see Figure 1) shows how Eureka Math would approach using the tape diagram. The equation is written using a variable and solved.

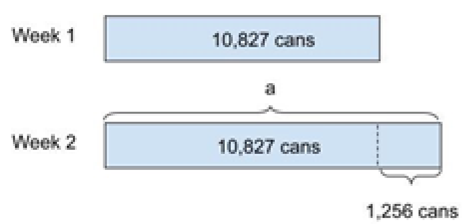

Figure 1: Tape diagram to find the number of cans collected during week 2.

a = 10,827 cans + 1,256 cans

= 12,083 cans

With the chart displayed for students, I will have them work in collaborative groups to create word problems using addition and subtraction. Once they have written some problems, I will ask some groups share them with the class. I will record the shared problems on a Google Slide or a PowerPoint to display them. Then, I will have the collaborative groups decide which problems are similar and how to categorize them. If students need more scaffolding, I will suggest that they first solve them. Then they could sort them by the operation, or the number of steps required.

Below (see Table 2) are the most likely one step addition problems they could develop.

One-Step	Operation	Answer
How many cans were collected in weeks 1 and 2 combined?	+	22,910
How many cans were collected in weeks 1 and 3 combined?	+	21,349
How many cans were collected in weeks 1 and 4 combined?	+	30,838

How many cans were collected in weeks 2 and 3 combined?	+	22,605
How many cans were collected in weeks 2 and 4 combined?	+	32,094
How many cans were collected in weeks 3 and 4 combined?	+	30,533

Table 2: One-step questions for can collection scenario

The tape diagram (Figure 2) below illustrates one way to represent one-step addition.

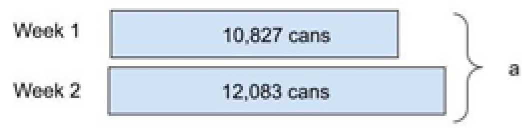

Figure 2: Tape diagram for one-step addition problems

Here is another way a one-step addition problem could be modeled with a tape diagram.

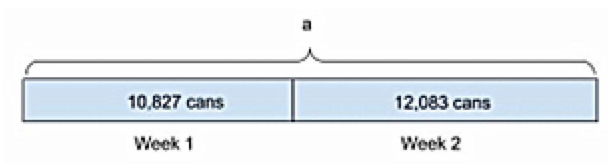

Figure 3: Another tape diagram for one-step addition problems

No matter which tape diagram model is selected, the equation would be the same.

$$a = 10{,}827 \text{ cans} + 12{,}083 \text{ cans}$$

Students could also create the following one-step subtraction problems (see Table 3).

One-Step	Operation	Answer
How many more cans were collected in week 1 than in week 3?	-	305
How many more cans were collected in week 2 than in week 1?	-	1,256
How many more cans were collected in week 2 than in week 3?	-	1,561
How many more cans were collected in week 4 than in week 1?	-	9,184
How many more cans were collected in week 4 than in week 2?	-	7,928
How many more cans were collected in week 4 than in week 3?	-	9,489
How many fewer cans were collected in week 3 than in week 1?	-	305
How many fewer cans were collected in week 3 than in week 2?	-	1,561
How many fewer cans were collected in week 1 than in week 2?	-	1,256
How many fewer cans were collected in week 1 than in week 4?	-	9,184
How many fewer cans were collected in week 2 than in week 4?	-	7,928
How many fewer cans were collected in week 3 than in week 4?	-	9,489

Table 3: One-step subtraction problems for can collection scenario

All the one-step subtraction problems in Table 3 could be solved with a similar tape diagram to the one below, but the labels and the numbers would be different as would the equation.

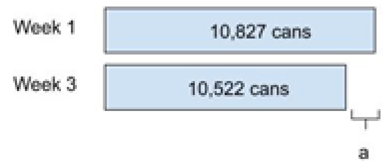

Figure 3: Tape diagram for one-step subtraction

a = 10,827 - 10,522

The following table (see Table 4) includes possible two-step addition problems students may create.

Two-Step	Operations	Answer
How many cans were collected in weeks 1, 2, and 3 all together?	+, +	33,432
How many cans were collected in weeks 1, 2, and 4 all together?	+, +	42,921
How many cans were collected in weeks 2, 3, and 4 all together?	+, +	42,616

Table 4: Two-step addition problems for can collection scenario

All the problems of this type could be solved with the same tape diagrams (see Figure 4 and Figure 5) by substituting the appropriate quantities for the corresponding weeks.

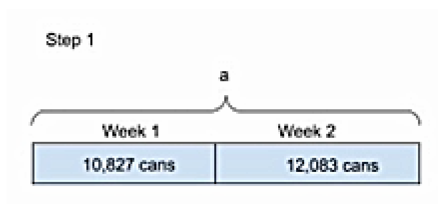

Figure 4: Tape diagram for the sum of 2 weeks

a = 10,827 cans + 12,083 cans

22,910 cans

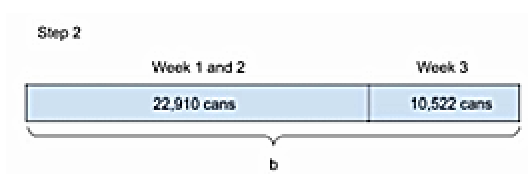

Figure 5: Tape diagram for sum of 3 weeks

b= 22,910 + 10,522 cans

b = 33,432 cans

Below are possible two-step problems (see Table 5) that students may create that use addition and subtraction to solve them. They follow the schema where the number of cans for 2 different weeks are combined through addition, then the sum for two weeks is compared to a third week which requires subtraction.

Two-Step	Operations	Answer
How many more cans were collected in weeks 1 and 2 combined than in week 4?	+, -	2,899
How many more cans were collected in weeks 1 and 3 combined than in week 4?	+, -	1,338
How many more cans were collected in weeks 2 and 3 combined than in week 4?	+, -	2,594
How many more cans were collected in weeks 1 and 2 combined than in week 3?	+, -	12,388
How many more cans were collected in weeks 1 and 3 combined than in week 2?	+, -	9,266
How many more cans were collected in weeks 2 and 3 combined than in week 1?	+, -	11,778

Table 5: Two-step addition and subtraction

The tape diagram below shows joining the number of cans for Week 1 and Week 2, then comparing the sum to the number of cans collected in Week 4. All the problems (see Table 5) could be solved with the same tape diagram by substituting different weeks and the corresponding numbers.

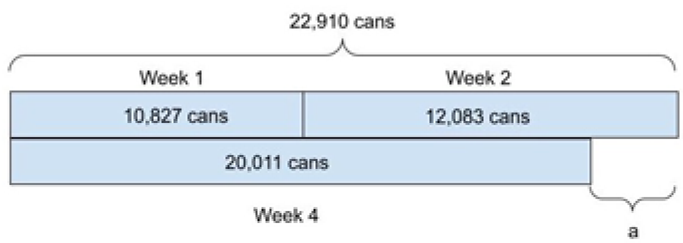

Figure 6: Tape diagram for two-step addition and subtraction

10,827 cans + 12,083 cans = 22,910 cans

a = 22,910 cans - 20,011 cans

a = 2,899 cans

Module 3: Multi-Digit Multiplication and Division

The following scenario comes from the Grade 4 Module 3 Lesson 12 Problem Set problem #2: The Turner family uses 548 liters of water per day. The Hill family uses 3 times as much water per day.

I have removed the question posed, in the text, as I want the students to generate questions that could be asked for the scenario.

Here is a sample of possible questions that students might develop:

1. How much water does the Hill family use in a day?
2. How much more water does the Hill family use than the Turner family in a day?
3. How much less water does the Turner family use than the Hill family in a day?
4. How much water does the Hill family use in a week?
5. How much more water does the Hill family use than the Turner family in a week?
6. How much less water does the Turner family use than the Hill family in a week?
7. How much water do the Turner and Hill families use in a day combined?
8. How much water do the Turner and Hill families use in a week combined?

I will ask students to share out their question and record them on a Google Slide or PowerPoint. Then have the students categorize them by the number of steps required to solve each problem. This will create a wonderful opportunity for some mathematical discourse. The table below (see Table 6) shows how the problems could be categorized. Next, I would have the class discuss the problems in small groups. If there is limited time, a small group could have 2 to 3 problems to solve.

1 Step	2 Step	More than 2 Steps
#1	#2	#5
#7	#3	#6
	#4	
	#5	
	#6	
	#8	

Table 6: Problems sorted by the number of steps.

Students can share their tape diagrams (see Figure 7) and their solution paths with the class.

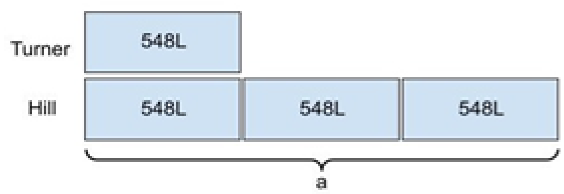

Figure 7: Tape diagram for one-step multiplicative comparison

For Problem #1: How much water does the Hill family use in a day? The students should multiply 548 X 3 (see Figure 7). Students may use addition, but this problem is an example of a multiplicative comparison problem.

The next two problems ask essentially the same question, but they ask it in different ways.

Problem #2:

How much more water than the Turner family does the Hill family use in a day?

Problem #3:

How much less water does the Turner family use than the Hill family in a day?

Both these problems are represented with the same tape diagram (see Figure 8).

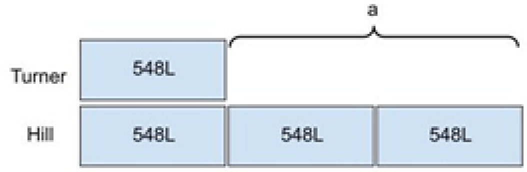

Figure 8: Tape diagram comparing daily water usage by the Hill and Turner families

Problem #4 asks:

How much water does the Hill family use in a week?

The students would multiply 548L by three and determine that the Hill family uses 1,644 L of water in a day (see Figure 9). Then they would multiply 1,644 by seven days to get 11,508 L of water per week (see Figure 10).

Step 1:

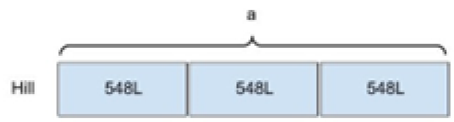

Figure 9: The tape diagram shows daily water usage for the Hill family

Step 2:

Figure 10: The tape diagram shows Hill water usage for 1 week.

Like Problems 2 and 3, Problem 5 and Problem 6 are the same question, but students may approach them differently. It would be worth discussing how the wording makes the problems appear different, but they are the same.

Problem #5:

How much more water does the Hill family use than the Turner family in a week?

Problem #6:

How much less water does the Turner family use than the Hill family in a week?

Both problems compare the difference in the amount of water used. These two problems will likely generate a robust discussion. The problems are "Comparison More" and "Comparison Less" with the same scenario. And both are "Difference Unknown."

The shortest solution path is to multiply 548L by two to get 1,096 L, then multiply 1,096L by seven to find 7,672 liters of water. However, some students will do 548L X 3 = 1,644L as the first step, then 1,644L X 7 = 11,508L as the second step. The third step may be . The fourth step may be to subtract 11,508 - 3,836 = 7,672L.

Another possible way to solve the problem would be 548L X 7 = 3,836. Then multiply 3,836 times 3 to get 11,508. The final step would be to subtract 11,508 minus 3,836, and the answer would once again equal 7,672 liters. Students can discuss the number of steps required to solve the problem which could lead them to understand that while all the solution paths are correct, the one with the fewest steps is better as it reduces the chance for computational errors. Using fewer steps also saves time.

Problem #7: How much water do the Turner and Hill families use in a day combined?

could be solved with multiplication or addition. With multiplication the students would use 548L X 4 = 2,192L. This would involve reasoning that 3 times some amount plus the amount makes 4 times the amount. This is nice reasoning and uses the Distributive Rule. If any student does it this way, it would be worth showcasing it in a class discussion. The Hill and Turner families together use 2,192 Liters of water per day.

Problem #8 could build on problem #7. Problem #8 asks: How much water do the Turner and Hill families use in a week combined? If the Hill and Turner families use 2,192 liters of water in a day, then 2,192 liters times 7 would equal 15,344L. The Turner and Hill families use 15,344 liters of water in a week.

Students could then take the 2-Step problems and categorize them by the operations (see Table 7). Since problems #5 and #6 can be solved with two steps, I included them in the table below. This would provide a wonderful opportunity for student discourse. It would lead to an authentic conversation to help students who used three or four steps to see a higher-level solution.

Problem	Operations	Operations
#2	X; —	
#3	X; —	
#4	X; X	
#5	+; X or X, X	X; X
#6	+, X or X, X	X; X
#7	X, +	+, X
#8	X; X	

Table 7: Problems #1-8 sorted by the operations needed to solve them.

Module 5: Fraction Equivalence, Ordering, and Operations

The next scenario is from Module 5, which is about fractions, and offers the opportunity for multi-step problems with fractions. For Virginia teachers, this activity will go beyond the grade 4 mathematics Standards of Learning (4.5), which only requires students to solve 1-step problems with fractions. The teacher can impose a constraint so the students only write one-step problems to stay within the standard, but this would be detrimental to the thinking and discourse, as there are only three one-step problems that could be written. The exercise could be a small-group activity for students who are ready for a challenge. Or, if students are solving a lot of word problems, engaging in discourse, and thinking critically, they are probably capable of solving these problems. This is problem is from Grade 4 Module 5, Lesson 18 Problem Set problem #3.

In a fish tank ¾ of the fish are guppies, 1/6 of the fish are goldfish, and the rest of the fish are angelfish.

Here are some examples of problems students might create using this scenario. They may word them more simply in a more child-friendly language. Record the problems as students share them. Then have students solve the problems created. For the sake of time, it is probably best to have each group work on one or two problems. There can be some overlap. Then guide the students to discuss and categorize the problems based on the number of steps needed to solve each one (see Table 8).

1. What fraction of the fish are angelfish?
2. What fraction of the fish are guppies or angelfish together?
3. What fraction of the fish are goldfish or angelfish together?
4. As a fraction of the total, how many more guppies are there than goldfish?
5. As a fraction of the total, how many more guppies are there than angelfish?
6. As a fraction of the total, how many more guppies are there than goldfish or angelfish combined?
7. As a fraction of the total, how many more guppies or angelfish combined are there than goldfish?

1 Step	2 Step	More than 2 Steps
	#1	#3
#4	#2	#5
		#6
		#7

Table 8: Problems #1-7 sorted by the number of steps needed to solve

Students should share their solutions and tape diagrams (see Figure 11) as they reason and justify their answers.

Problem #1: What fraction of the fish are angelfish? This is a two-step problem. The first step is what fraction of the fish are guppies or goldfish? This is 11/12. Then the angelfish are 1 - 11/12 = 1/12.

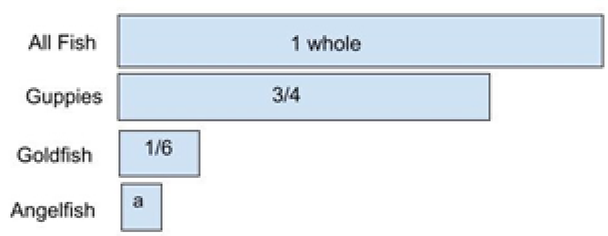

Figure 11: The tape diagram represents the fraction of each type of fish in the aquarium.

Students would add the fraction of guppies to the fraction of goldfish. Students would have to find like denominators to calculate the sum of guppies and goldfish.

$$a = \frac{3}{4} + \frac{1}{6}$$
$$= \frac{9}{12} + \frac{2}{12}$$
$$= \frac{11}{12}$$

Problem #2: What fraction of the fish are guppies or angelfish together? Problem #2 is a three-step problem. First, students need to find the fraction of fish that are angelfish. This is a two-step problem, with solution as described above. Then they subtract the resulting fraction from 1 whole to determine the fraction of fish that are angelfish. The third step is to add the fraction of angelfish to the fraction of guppies.

$$\frac{3}{4} + \frac{1}{6} = \frac{11}{12}$$

$$1 - \frac{11}{12} = \frac{1}{12}$$

$$\frac{1}{12} + \frac{3}{4} = \frac{1}{12} + \frac{9}{12} = \frac{10}{12}$$

$$\frac{10}{12} = \frac{5}{6}$$

If students reason guppies and angelfish are the whole minus the goldfish, it is one step. I will ask whether any students solved it differently, in hopes of eliciting this solution. However, if no student came up with this solution, I will not introduce it myself.

Problem #3: What fraction of the fish are goldfish or angelfish together? Problem #3 is a three-step problem. The students follow the steps from Problem #2 to determine the number of angelfish. Then they should add the fraction of angelfish to the fraction of goldfish (see Figure 12).

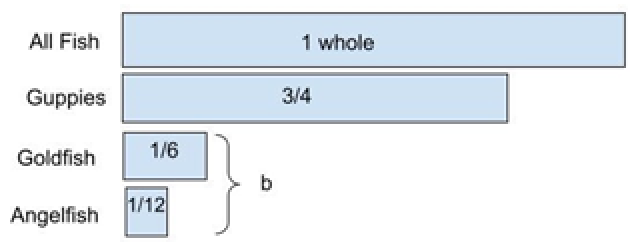

Figure 12: The tape diagram depicts the sum of goldfish and angelfish as the unknown, b

$$= \frac{1}{6} + \frac{1}{12}$$

$$= \frac{2}{12} + \frac{1}{12}$$

$$= \frac{3}{12} = \frac{1}{4}$$

Again, it can be found in one step: the whole minus the guppies. I will again query students to see if any came up with this method. If someone did, but no one had done it with problem 2, I will revisit problem 2 and point out the possibility there.

Problem #4 How many more guppies are there than goldfish? Write the answer as a fraction. This is a one-step problem (see Figure 13). The tape diagram would be a comparison, and looks similar to bars on a bar graph, but in a horizontal position.

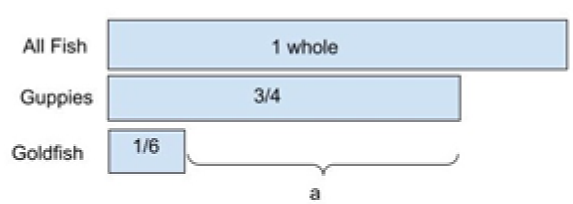

Figure 13: The tape diagram compares the fraction of goldfish to the fraction of guppies.

$$a = \frac{3}{4} - \frac{1}{6}$$

$$\frac{9}{12} - \frac{2}{12} = \frac{7}{12}$$

Problem #5:

As a fraction of the total, how many more guppies are there than angelfish?

The tape diagram (see Figure 14) shows how to represent the problem. This problem uses three steps. The first two steps, addition followed by subtraction, would have been done in Problem #1. Then third step is to subtract the fraction of angelfish from the fraction of guppies. Having worked with fourth graders for over two decades, I anticipate that some students will go through all the steps, instead of using the fraction of angelfish from Problem #1.

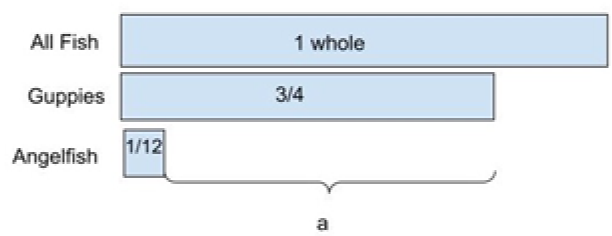

Figure 14: Tape diagram shows the third step which compares the fraction of angelfish to the fraction of guppies.

The solution is obtained by subtracting the fraction of angelfish from the fraction of guppies.

$$a = \frac{3}{4} - \frac{1}{12}$$

$$\frac{9}{12} - \frac{1}{12} = \frac{8}{12} = \frac{2}{3}$$

Problem #6:

As a fraction of the total, how many more guppies are there than goldfish or angelfish combined? Four steps are required to solve this problem. The first two steps follow problem #1 to determine the number of angelfish. Then the fraction of goldfish is added to the fraction of angelfish (see Figure 15). Or you could subtract guppies from the whole, leaving ¼ for goldfish and angelfish combined, then subtract the ¼ from ¾ to get 2/4 or ½. Once again, some students will go through all the steps even though they already found the number of goldfish or angelfish in Problem #3. By working in collaborative groups, some students will recognize the repetition, and show other students how to use the work they have already done in previous problems to save time.

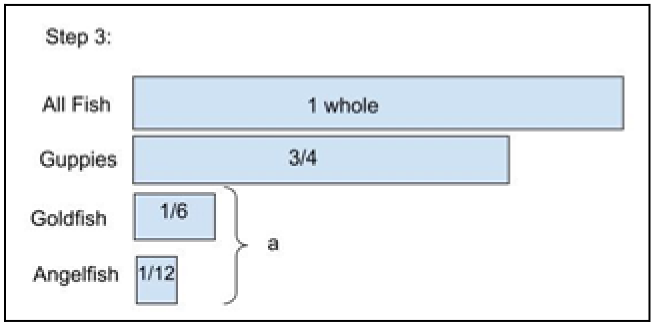

Figure 15: The tape diagram shows the third step which combines the fraction of angelfish and goldfish.

$$a = \frac{1}{6} + \frac{1}{12} = \frac{2}{12} + \frac{1}{12} = \frac{3}{12} = \frac{1}{4}$$

The final step is to subtract the combined amount of goldfish and angelfish from the fraction of guppies (see Figure 16).

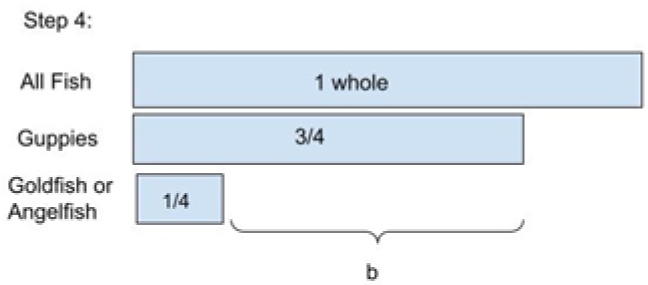

Figure 16: The tape diagram shows the fourth step which compares the fraction of goldfish or angelfish to the fraction of guppies.

$$\frac{3}{4} - \frac{1}{4} = \frac{2}{4} = \frac{1}{2}$$

Some clever students might reason that, since guppies are ¾ of the fish, the others combined are 1/4, so the difference is ½. These problems should generate rich mathematical discourse as some students recognize that previous problems can be used to reduce the number of steps. Through student-to-student conversations, students will not only grapple with different ways to solve these problems, but also come to some conclusions about which ways are more efficient. The student discourse and ownership will be more powerful than if I stood in front of the class and led a discussion about the various ways to solve these problems.

Problem #7 is similar to Problem #6 and can involve up to four-steps. Problem #7: How many more guppies or angelfish combined are there than goldfish? The initial two steps are finding the fraction of angelfish (Problem #1). The third step (see Figure 17) would be to find the sum of guppies or angelfish (Problem #3). Students would add ¾ and 1/12, which would require that they find a common denominator. By multiplying the numerator and the denominator by 3, ¾ can be transformed into 9/12. The sum of 9/12 and 1/12 is equal to 10/12.

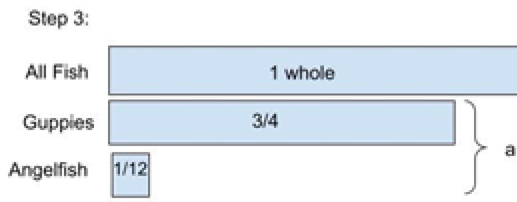

Figure 17: The tape diagram shows step 3 combining the fraction of guppies and angelfish.

$$\begin{aligned} \text{a} &= \frac{3}{4} + \frac{1}{12} \\ &= \frac{9}{12} + \frac{1}{12} \\ &= \frac{10}{12} \end{aligned}$$

The final step (see Figure 18) would be to subtract the fraction of goldfish from the fraction of guppies or angelfish. Again, this could be solved by subtracting 1/6 from 1, then subtracting 1/6 again. Students may recognize and add 1/6 to 1/6 then subtract 2/6 from 1. It will be interesting to see students develop more efficient ways to solve this problem after working through the previous problems. I suspect some will; in which case, they will demonstrate schema recognition and transfer to a novel problem!

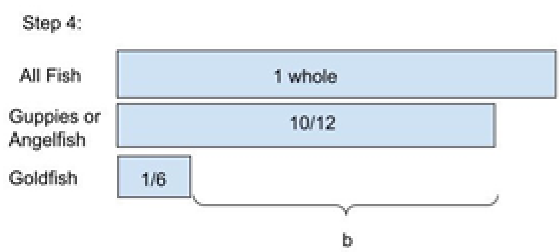

Figure 18: The tape diagram shows step 4, the difference between the fraction of goldfish and the fraction of guppies or angelfish.

$$\begin{aligned} \text{b} &= \frac{10}{12} - \frac{1}{6} \\ &= \frac{10}{12} - \frac{2}{12} \\ &= \frac{8}{12} = \frac{2}{3} \end{aligned}$$

The equation 10/12 minus 1/6 has unlike denominators, so the students would have to find the common denominator between 6 and 12. Or they might reduce 10/12 to 5/6. It might be valuable for the teacher to ask, at a strategic time, to express all the fractions with a common denominator (of 12). The Least Common Denominator (LCM) is 12. The numerator and denominator in the fraction 1/6 can be multiplied by 2. So, 1/6 equals 2/12. Then 10/12 minus 2/12 equals 8/12. The fraction 8/12 can be simplified by finding the Greatest Common Factor between 8 and 12 which is 4. Then the numerator and the denominator are each divided by 4. The resulting fraction is 2/3 which is in simplest form. This can give students a lot of good practice with fractions.

Once the problems are solved, categorized, and shared, the students can sort the multi-step problems based on their operations (see Table 9).

Problem	Operations	Operations (fewer steps)
#2	+; —; +	—
#3	+; —; +	—
#5	+; —; —	—
#6	+; —; +; —	—
#7	+; —; +; —	—, — or +, —

Table 9: The operations needed to solve Problems #2-7.

Strategies

The three main strategies that this curriculum unit utilizes are mathematical discourse, the RDW strategy, and small group collaboration. A brief description of each is provided below.

Mathematical Discourse

Throughout the lesson, students will engage with each other in mathematical discourse. As students discuss, explain, reason, and justify their ideas they learn to use precise vocabulary and articulate their thinking clearly. Through the process, they help each other to understand math and deepen their own knowledge. If students are reluctant, it is easier to discuss with a partner before sharing with a group of four or five students.

Read Draw Write (RDW)

The RDW strategy is used in the Eureka Math curriculum. As described earlier in this unit, the “R” stands for read. The students read the problem carefully so that they can “D”: draw a representation of the problem with a tape diagram. The way students draw the tape diagram and the context of the problem help students determine the operation needed to solve the problem. The next step is to write an equation and solve the problem. The “W” stands for write and students are taught to go back to the question and write their answer in a complete sentence.

Small Group Collaboration

Small groups are used during all phases of this curriculum unit. Students work together to discuss, solve, and categorize word problems. Students learn to help each other and to use each other’s strengths as they work as a team and develop a learning community.

Gradual Release

The curriculum unit will roll out using gradual release. The unit will span the first semester and is spread throughout three Eureka Math modules. It will begin by simply asking students to write one more question to go with the daily Application Problem. Gradually students will take on more responsibility, so they are prepared to succeed when it is time for the culminating activity. See below where the gradual release is described in more detail.

Activities

As the school year starts, students will get used to the format of Eureka Math. Each lesson begins with Fluency and an Application Problem prior to delving into the Concept Development. The progression will begin the second week of school with the Application Problem for Module 1 Lesson 3. I will ask them to create one additional question to go with the Application Problem. Students may work with a partner initially to develop students' comfort with the process and with mathematical discourse. After a few days, students will be asked to look for and share similarities or differences between the two questions. I plan to carry out this routine throughout the year. By the time we reach Lesson 12 in the first Module, students will be accustomed to creating additional questions and thinking about the similarities and differences between problems.

Activity 1

This activity will coincide with Module 1 Lesson 12. I will teach as described in the Framework for Mathematical Discourse and Schema Building section above. The following day, students will follow the Framework with the problem and data chart (see Table 10). Students will receive the following novel story situation:

Last year on Ted's farm, his four cows produced the following number of liters of milk:

Cow	Liters of Milk Produced
Daisy	5,098
Betsy	
Mary	9,980
Buttercup	7,087

Table 10: Milk produced on Ted's farm.

Betsy produced 986 more liters of milk than Buttercup.

Students will work in small, collaborative groups. They will create at least four questions that use one-step addition or subtraction. Once the problems are created, they will create tape diagrams to solve each problem. They will sort them based on the solution path. The process will repeat with two-step problems. Students should create at least three, two-step problems using addition or subtraction. Then they will draw a tape-diagram and solve each problem. Next, they will sort them. The exit ticket will be for the students to write on an index card what they noticed and what they wonder.

Activity 2

The second activity will follow Module 3 Lesson 12 Problem 2 as explained in the Framework for Mathematical Discourse and Schema Building section above. Following the instructional day, the students will be given a novel problem which includes a multiplicative comparison. Students will complete a task using the following situation from Module 3 Lesson 12 Problem 3:

Jayden has 347 marbles. Elvis has 4 times as many as Jayden. Presley has 799 fewer than Elvis.

This time, students will develop as many one-step problems as they can. Then they will solve the problems with tape diagrams and classify the problems according to their operations. As with Activity 1, they will then write two-step problems with addition, subtraction, or multiplication. Students will solve these, sort them, and share out their solutions explaining what they notice.

Activity 3 (culminating activity)

For this activity, students will complete Module 5 Lesson 18 Problem Set #3 through a guided lesson as explained above in the Framework section. The following day, students will have to create a story situation for the following tape diagram:

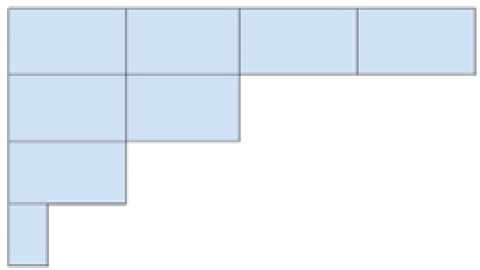

Figure 19: The tape diagram for the culminating activity.

This tape diagram will lend itself to fractions or to multiplicative comparison. Students should be able to tap into the schemas they developed from their previous work and apply them to a novel problem.

Then they will create as many single-step problems as they can in the allotted time. The problems will be solved and sorted. Next, students will create two-step problems that they will solve and sort. Students will complete a poster template provided by me (see Figure 19). For the final section of the poster they will write at least one paragraph explaining what they learned. Additional time will be provided daily to allow students to complete their posters. I hope these projects will become part of the district-wide STEAM Fair.

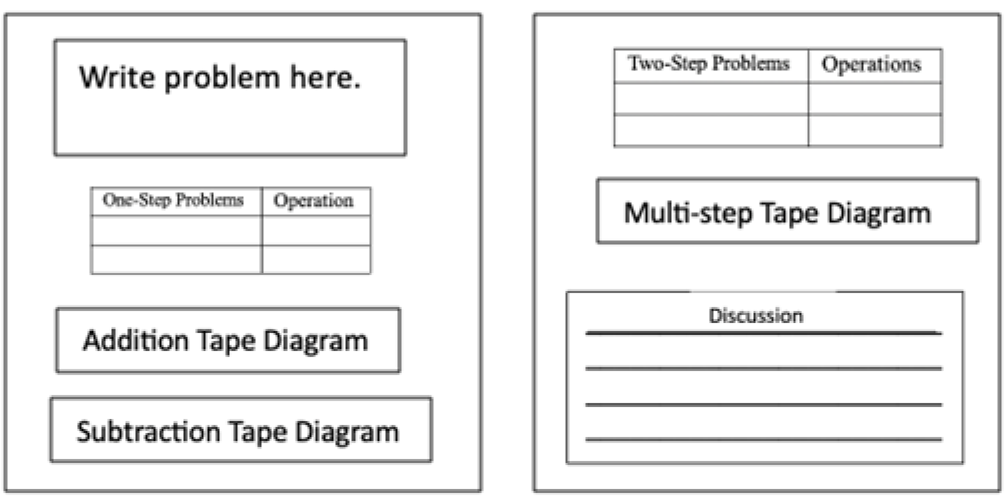

Figure 19: Template for the culminating project

Appendix on Implementing District Standards

Virginia uses the Virginia Standards of Learning (SOL) and an adaptation of the CCSS Taxonomies for problem solving is included in the Virginia SOLS. The two main standards that are addressed in this curriculum unit are mathematics standards 4.4d and 4.5c.

4.4 The student will

d) create and solve single-step and multistep practical problems involving addition, subtraction, and multiplication, and single-step practical problems involving division with whole numbers.

4.5 The student will

c) solve single-step practical problems involving addition and subtraction with fractions and mixed numbers.

These two standards are addressed through the curriculum unit. SOL 4.4d is used through the instruction for the word problems from Module 1 and Module 3, as these modules address problem solving with whole numbers. Standard 4.5 is addressed for the instruction of Module 5 which involves problem solving with fractions.

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Notes

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