Curriculum Units by Fellows of the National Initiative
2023 Volume III: Transitions in the Conception of Number: From Whole Numbers to Rational Numbers to Algebra

# Numerical Development: from whole numbers to fractions

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## Introduction and Rationale

Dealing with fractions, decimals and ratios has always objectively been one of the hardest topics in Math that humankind has encountered since prehistoric times. This struggle was so great that the Germans have embedded it in their famous idiom “in die Bruche kommen” which literally means “to get into fractions.” This expression refers to a very difficult general life situation one may get into. So, it is not a surprise that in modern times, students (like their ancestors hundreds of years ago) are scared of this topic and are ready to give up upfront on the problems involving fractions, decimals and ratios. Even the fraction notation is intimidating to students. “First, it was used in Ancient India, then the Arabs started using it. Finally, in XII-XIV centuries the Europeans borrowed this notation from the Arabs. In the beginning the fraction bar was missing in the fraction notation. So, number $1/5$ looked like 1. Number 2 $1/3$ looked like 2

5 1

3.

The fraction bar first appeared in fraction notation only about 300 years ago. The first European mathematician who started using and popularized the fraction notation was Italian merchant and traveler, the son of the clerk, Leonardo Fibonacci (Leonardo of Pisa). In 1202 he introduced the word “fraction”. The words “numerator” and “denominator” were introduced in the 1200s by Greek monk and mathematician Maxim Planud.”[1] So, fractions have never been easy even from the historic perspective. In the modern world the importance of fractions is difficult to overestimate. Not knowing this topic will most definitely result in failure in Finance, Physics, Chemistry, Biology, Medicine, Pharmacy, and, not to mention, your local grocery store. Therefore, it is so common to see conversion tables at the major department stores. They just do not want to lose their customers who do not know how to calculate discounts given in percents (a.k.a. ratios). So, knowing fractions is a pass to a better future for my students.

## Demographics

I teach at a small magnet-like high school where kids have to apply to be admitted. The total minority enrollment is 54%. A big part of the admission policy is to attract students with different backgrounds. Therefore, the student body is very diverse. We have kids from elite private schools as well as kids from really disadvantaged urban middle schools. As a result, students` skills vary a lot. I have to adjust my teaching style accordingly and approach my students in an individual way. While some of them take the most rigorous AP classes, a lot of them struggle with simple mathematical concepts. My goal, as a teacher, is to address every student`s needs and give all of them an opportunity to succeed.

## Content Objectives

Indeed, operating with fractions is an extremely challenging task for Math students. Most of them are ready to give up without even trying. The origin of this struggle goes all the way back to the lack of understanding that fractions are the important part of the unified system of numbers. So, eventually students will realize that fractions are not some mysterious numbers. There is a perfect correspondence between the principles of dealing with whole numbers and fractions, something in the realm of "Numerical development: from whole numbers to fractions." This will help young people dealing with algebraic expressions and mental computation in all classes I am currently teaching (Geometry, Math Analysis, and even AP Calculus). Subsequently, it will help them be successful on the SOLs, AP tests, college placement tests, and college Math since a big portion of modern assignments require them to do mental math without use of a calculator.

Operations with fractions are essential for student success in upper level math, physics and chemistry. Unfortunately, most of them have very modest, if any, skills of that kind. So, my goal in designing this unit is to help teachers and students fill in this gap. I will make every effort to explain to my students the essence of fractions using the number line and the area model, but my goal will also include teaching the most effective strategies and techniques that will enable my students to add, subtract, multiply and divide fractions under time pressure.

## Unit Content

While teaching my Geometry, Pre-Calculus and AP Calculus students, I have realized that my students have a lot of gaps in their knowledge of fractions. My AP Calculus students can solve a definite integral, plug in the limits represented by fractions, but fail to subtract fractions at the final step. Let's say students should solve the following problem:[2]

$$\int_{-2}^{1}(y^2-y+2)dy = \left(-\frac{y3}{3}-\frac{y2}{2}+2y\right)\Bigg|_{-2}^{1} = (-\frac{1}{3}-\frac{1}{2}+2)-(\frac{8}{3}-2-4) = \frac{9}{2}$$

All of my students will find an antiderivative and apply the First Fundamental Theorem of Calculus by plugging in the limits, but later many of them will fail to compute the sum involving fractions. In fact, the situation is so desperate that AP Calculus students are not required to complete this step in the free -response section. As long as they plug in the limits correctly, the problem is considered to be done, and they receive full points for it on the AP test. However, to get their points for a multiple-choice problem student should be able to complete calculations with fractions. Most of them will fail due to the lack of skills with basic Math.

My Geometry students do not do any better. At the end of the year they take a pretty hard standardized test. The test is being constantly modified to become more rigorous. The newer version includes a lot of problems on ratios and proportions be it similarity or relationships between the volumes and/or surface areas of space figures when their dimensions change. Solving such problems requires a lot of skills on operations with fractions. As a result, students feel insecure and unconfident. My unit is designed to fill in the gaps in adding, subtracting, multiplying and dividing fractions to help students succeed at all Math levels.

## Teaching Strategies

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**Number line.**

Different students use different ways to understand and memorize math concepts. I have observed that most of them are visual learners. Therefore, one of the best tools to study arithmetic is a number line. "The number line is a terrific tool for helping us understand arithmetic in terms of geometry. This can help you get a much more unified picture of what arithmetic is about than working only in symbols, and it can help arithmetic make more sense." [3] This will help students to look at the fractions as part of a unified system rather than a separate concept. So, the first stop in our wonderful journey to the magic world of fraction will be a number line. We should actually build a number line.

To define a number line, we start with an unmarked line "(see Figure 1)."

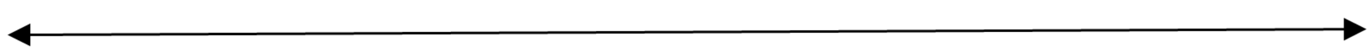


Figure 1

Next, we choose a "starting" point – a point to be the origin or zero point- and a unit length "(see Figure 2)."

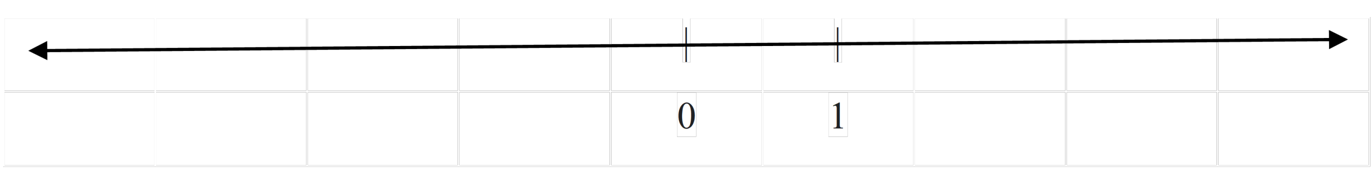



Figure 2

It is a very important thing to emphasize is that the number line is about length: a point is labeled by the number that tells how far it is from the origin, as a multiple of the unit distance "(see Figure 3)." I will make sure that my students are clear about it before doing anything more advanced.

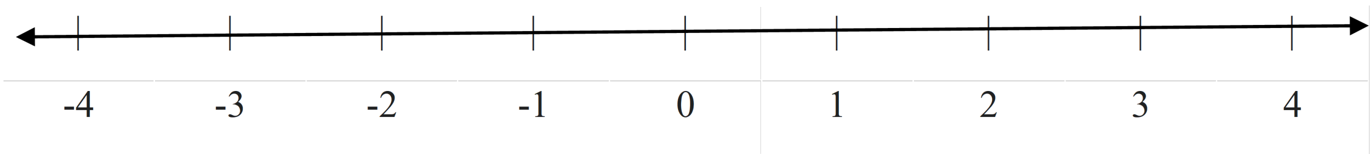



Figure 3

Students should also have a clear understanding that a number line continues indefinitely in the positive (to the right of the origin) and negative (to the left of the origin) directions according to the rules of geometry which define a line without endpoints as an infinite line. The integers are often shown as specially-marked points evenly spaced on the line. Although most of the time we only label whole numbers on the number line, it is actually made up of all real numbers including fractions. The same concept with length can be used to label fractions on the number line.

Example 1: to put an interval of length $\frac{1}{2}$, we divide the unit length between 0 and 1 by two "(see Figure 4)." Then, each piece has length $\frac{1}{2}$. The number "$\frac{1}{2}$" goes at the end of the segment of length $\frac{1}{2}$ starting at 0.

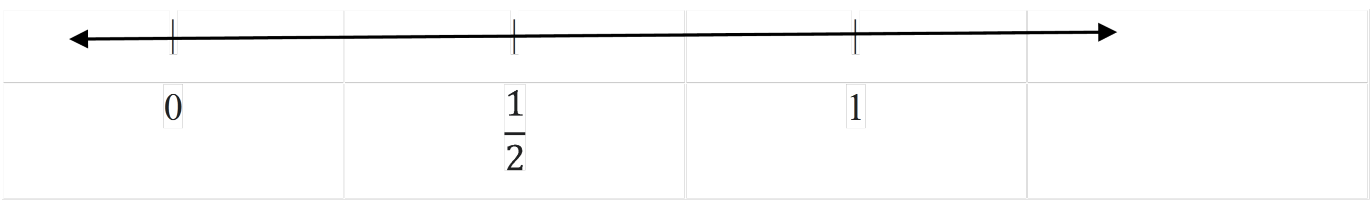



Figure 4

Example 2: to put an interval of length $\frac{1}{3}$, we divide the unit length between 0 and 1 into three equal pieces. Then, each piece has length $\frac{1}{3}$"(see Figure 5)." Similar to the situation with halves, the number "$\frac{1}{3}$" goes at the end of the segment of length $\frac{1}{3}$ with its other endpoint at 0.

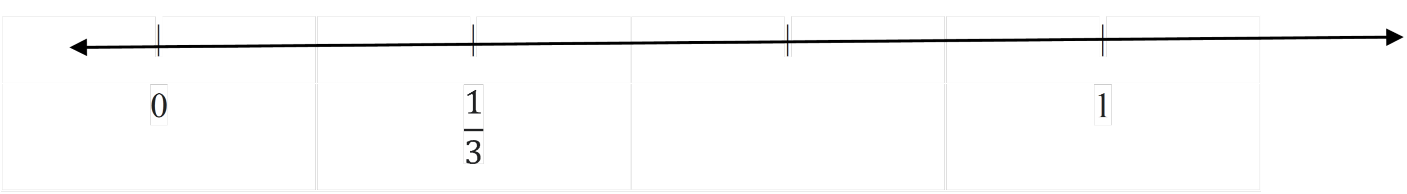



Figure 5

Putting fractions on a number line is a challenging task for many students. Mostly they struggle with figuring out how large or small a fraction is. To solve this problem, students need to know how to compare fractions. I plan to focus on the two major concepts. The first thing to realize is that if the numerator of a (positive) fraction increases while the denominator stays fixed, then the fraction increases. For example,

$$\frac{1}{3} < \frac{2}{3} < \frac{4}{3} < \frac{5}{3}$$

To help my students visualize this concept, I will place all these fractions on the same number line "(see Figure 6)."

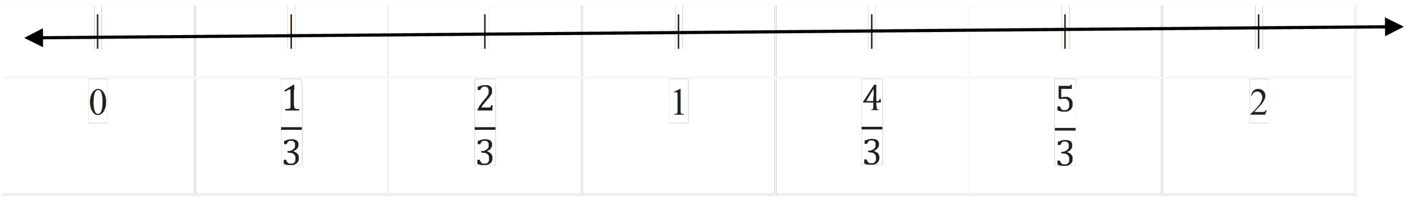



Figure 6

To the contrary, if the numerator stays fixed and the denominator increases, then the fraction decreases.

For example,

$$\frac{1}{4} < \frac{1}{3} < \frac{1}{2}$$

To connect this concept to the real life I suggest that my students imagine one big pizza that they need to share with five friends, or with their four friends, or with their three friends, or two friends. Then I ask them in what case their slice will be the biggest. They never fail. They know that the less people want to eat their pizza, the bigger portion of the pizza they will enjoy. This real-life example creates a good mnemonic rule of this not so obvious math fact. In addition to that, I will get my students place all these fractions on the number line to reinforce the concept that the number line is about length: the smaller the length is, the closer the number is to the origin "(see Figure 7)."

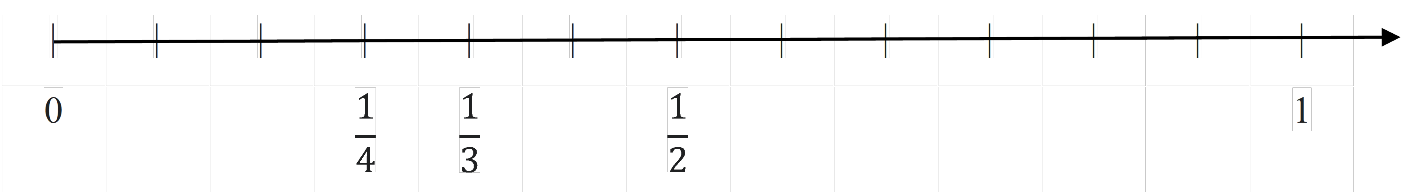



Figure 7

**Bar and Area Models.**

One of the cornerstones of students` literacy in fractions is the idea that different fractions that look different can represent the same number, that is, are equal. This is the bridge to the arithmetic operations with fractions with different denominators. I will start teaching this concept with a bar model that lets students clearly see how it all works.  The top rectangle is divided into two equal pieces, so the yellow region represents $\frac{1}{2}$.

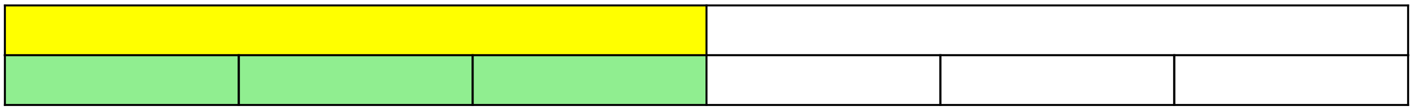


Figure 8

The bottom rectangle is divided into six equal pieces, so the green region represents $3/6$.

Since the areas of both rectangles are obviously equal, the conclusion that students will be led to make is that $1/2 = 3/6$ "(see Figure 8)." This can also be illustrated systematically with the number line.

Another way to show the same is a circle. Let`s draw a circle and divide it into four equal parts.

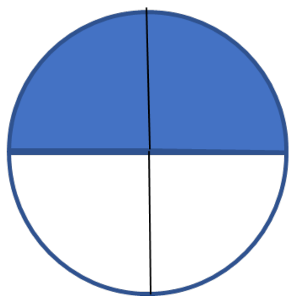


Figure 9

Two of these parts (the blue region on top) make half of a circle, so we can conclude that $1/2 = 2/4$ "(see Figure 9)."[4] Students need to have a clear understanding of the fact that dividing (or multiplying) the numerator and denominator of a fraction by the same non-zero number will always yield an equivalent fraction. We will be using this important property to add and subtract fractions with different denominators, as well as reducing a fraction to the lowest terms. It is helpful for students to realize that on the number line equal fractions are represented by the same point "(see Figure 10)."

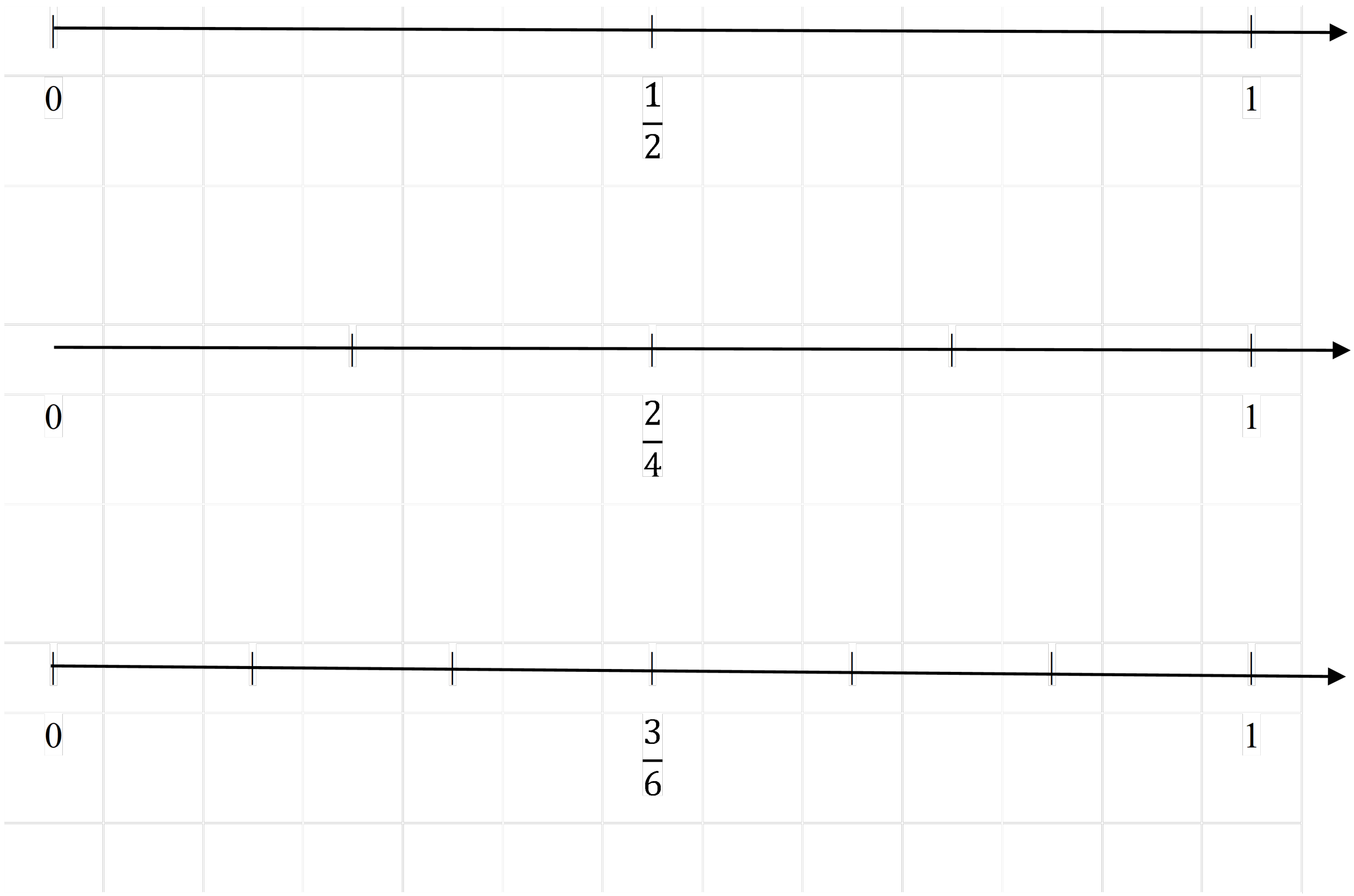



Figure 10

Now it is time to make a general statement that if we multiply both the numerator and the denominator of some fraction by the same number (not equal to zero), the result will be the fraction equal to the original one.

$$\frac{a}{b} \times \frac{c}{c} = \frac{a \times c}{b \times c}.$$

The same is true if we divide both the numerator and denominator by the same number (not equal to zero). This is also known as writing a fraction in lower terms.

Arithmetic operations with fractions.

Developing good skills with equivalent fractions is essential for teaching how to compare, add and subtract fractions with different denominators. However, in the beginning we should practice comparing, adding and subtracting fractions with the same denominators. The rule of comparing fractions with the same denominators has been discussed above. To add fractions with the same denominators we just add the numerators, keeping the same denominator:

$$\frac{a}{b} + \frac{c}{b} = \frac{a + c}{b}$$

The same is true for subtraction:

$$\frac{a}{b} - \frac{c}{b} = \frac{a-c}{b}.$$

Again, these operations can be represented uniformly on the number line.

Once students feel comfortable with adding and subtracting fractions with the same denominators, it is time to teach them how to do these basic operations with fractions with different denominators. This is a difficult task that requires time and effort, but it is essential for students` success not only in Math, but in Science and other Math-related subjects. So, I make the effort to be patient, thorough and persistent in teaching these concepts.

Multiplying fractions is a pretty easy task for my students as long as they know the rule: to multiply fractions, we should multiply the numerators and multiply the denominators of these fractions:

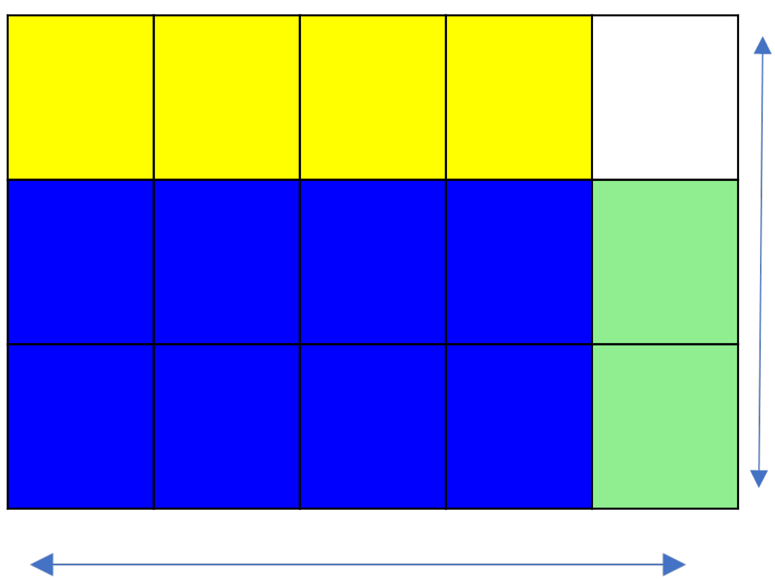
$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}.$$

To visualize this rule for students using an area model can be effective.

“Problem 2. Suppose, the length of some rectangle is $^4/_5$dm, and its width is $^2/_3$dm. What is the area of the rectangle? “

To answer this question, let`s draw a diagram. To create such rectangle, we can divide one side of a square (whose side length is 1 dm) into 5 equal parts and shade 4 of them, and the other side of a square – into 3 equal parts and shade 2 of them. In this case the total number of equal parts will be 15, and the original rectangle will be made up of 8 parts. So, the area of this rectangle will be equal $^8/_{15}$dm$^3$ . As we know, the area of a rectangle is equal to the product of its length and the width. Therefore, we decide that $\frac{8}{15} = \frac{4}{5} \times \frac{2}{3}.$ . Let`s summarize $\frac{4}{5} \times \frac{2}{3} = \frac{4 \times 2}{5 \times 3} = \frac{8}{15}$ .” “(see Figure 11).”[5]

Square side = 1 dm



$^4/_5$dm

Figure 11

On the contrary, dividing fractions has always been challenging, since it is a multi-step procedure that requires a lot of practicing.

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}.$$

"Dividing by a fraction is the same as multiplying by its reciprocal." [6]

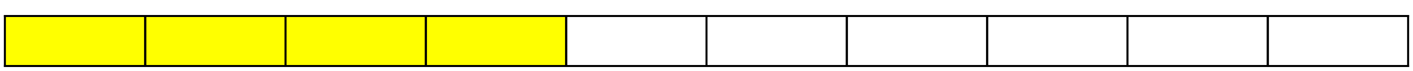
Note: A very special attention should be drawn to the fact that division by whole number b (not equal to zero) is the same as multiplication by $^{1}/_{b}$ and vice versa. This equivalence follows from the definition of 1/b. This is also a point worth making. Many students struggle with understanding of this concept. For example, each time I ask my students to name a slope of line y = $^{x}/_{4}$ + 3, their answers vary. Most common are: slope is equal to 4, slope is equal to 0, slope is equal to 1. Obviously, in this example they should understand that $^{x}/_{4}$ = $^{1}/_{4}$ X x, so the slope of the line is equal to $^{1}/_{4}$. It is easy to predict the struggle with linear functions, parallel and perpendicular lines, derivatives and many other topics in Math that comes simply from misunderstanding of the relationship between division by b and multiplication by 1/b.  As we can see, "Fractions" is a substantial and challenging topic. The more persistent we, teachers, will be with this topic, the better results students will achieve.

## Progressive problem set

For this unit, I have written a collection of problems that progress from very simple to more complex problems that contain a variety of operations with fractions.

**Problems on fraction notation.**

**Problem I.**



1) Write down a fraction that represents the area of the shaded region on the diagram above, compared with the whole diagram. Reduce the fraction to the lowest terms.

2) Write down a fraction that represents the area of the shaded region on the diagram below, compared with the whole diagram. Reduce the fraction to the lowest terms.

**Problem II.** [7]

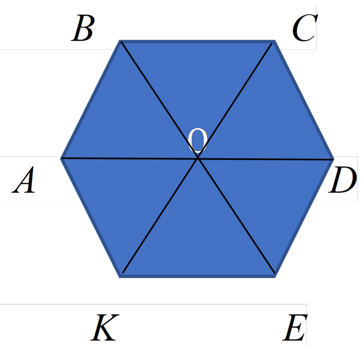
Use the diagram below to answer the following questions:

1) What fractional part of quadrilateral ABCO is triangle ABO?

2) What fractional part of quadrilateral ABCD is triangle ABO?

3) What fractional part of quadrilateral ABCD is quadrilateral ABCO?

4) What fractional part of hexagon ABCDEK is quadrilateral ABCO? Express this fraction in lowest terms.




In Problem II 4), I ask students to reduce the answer to the lowest terms, because I think teachers should use every opportunity to have students practice on reducing the fractions.

**Problem III.**

1) What part of the (standard calendar) year is 1 day?

2) What part of the week is 1 day?

**Problem IV.**

1) What part of an hour is 45 min, 12 min, 15 min, 40 min, 35 min? Write down each answer as a fraction. Reduce the fraction to the lowest terms.

2) What part of a straight angle is $30^0$ , $45^0$ , $60^0$ , $90^0$ , $120^0$ , $135^0$ , $150^0$ ? Write down each answer as a fraction. Reduce the fraction to the lowest terms.

In Problem IV 1) students should understand that first they need to convert 1 hour into 60 min. In Problem IV 2) they need to know that the measure of a straight angle is $180^0$ .

**Problem V.**

There is a 5-liter jar. Mrs. Brown poured *a* liters of water into the jar. What part of the jar is filled with water if a = 1, 2, 3, 4? What part of the jar remains empty?

In Problem V first, students should be able to plug in the given numbers for a and represent their answers in fraction notation. To answer the second question, then they should subtract these fractions from 1.

Problems involving a number line.

**Problem VI.**

1*)* On graph paper draw a number line. Let the unit length be equal to 5 squares. Put the following numbers on this number line:

$^{1}/_{5}, ^{2}/_{5}, ^{3}/_{5}, ^{4}/_{5}, ^{5}/_{5}$

2) On graph paper draw a number line. Put the following numbers on this number line:

$^{1}/_{8}, ^{3}/_{8}, ^{5}/_{8}, ^{7}/_{8}$

In Problem VI 1) students have a prompt regarding the scale of the number line. In Problem VI 2) this prompt is missing, and students are invited to figure out the best way to put the fractions on the number line based on their experience with Problem I 1). I also like to use this problem to reinforce the concept that if two proper fractions have the same denominator, the smaller fraction is closer to 0, while the bigger fraction is closer to 1.

**Problem VII.**

1) Without drawing a number line, decide which of the two points lies to the left. Explain.

A ($^{3}/_{7}$) or B ($^{5}/_{7}$).

2) Without drawing a number line, decide which of the two points lies to the right. Explain.

C ($^{11}/_{13}$) or B ($^{9}/_{13}$).

In Problem VII students are invited to practice on mental Math. They should "draw" a number line in their imagination and answer the question. Their explanation should also be based on the principle of comparing fractions with the same denominator by comparing their numerators.

**Problem VIII.**

1) Use a piece of graph paper. Draw a number line. Let the unit length be equal to 6 squares. Place the following numbers on the number line:

$3\,^{1}/_{3}, 2\,^{5}/_{6}, 2\,^{2}/_{3}, 1\,^{1}/_{2}$

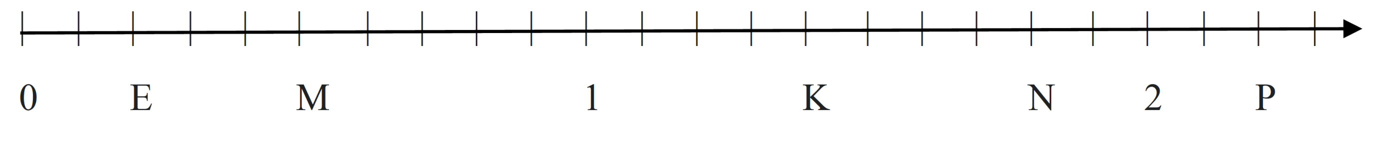
2) 1) Use a piece of graph paper. Draw a number line. Place the following numbers on the number line:

$1\,^{1}/_{8}, 2\,^{3}/_{4}, 2\,^{5}/_{8}, 3\,^{1}/_{4}$

Similar to Problem VI, students are given a prompt in Problem VIII 1) regarding the unit length. In Problem VIII 2) they should choose the unit length by themselves.

**Problem IX.**

Find the coordinates of points M, N, K, P, and E on the number line below:




This problem has a reverse assignment. A number line is given. Students should be able to write down the coordinates of each point in the form of a fraction or a mixed number.

Problems involving comparing fractions.

**Problem X.**

Arrange the given fractions first in ascending order, and then in descending order:

$$7/12,\ 1/12,\ 5/12,\ 9/12,\ 11/12,\ 4/12$$

**Problem XI.**

1) List at least five fractions that are greater than $1/9$

It is obvious that the answers to Problem XI 1) will vary.

2) List at least three fractions that are less than $5/6$.

3) List at least four fractions that are less than $1/2$.

In Problem XI 3) students often represent $1/2$ as a fraction with a larger denominator. For example, $5/10$. Then, their answer contains the list of fractions with number 10 in the denominator and numbers 1, 2, 3, and 4 in the numerator. Though this solution is correct, I will use the discussion of this problem to reinforce their understanding of the fact that all fractions with number 1 in the numerator and any natural number greater than 2 in the denominator will also be a solution to the problem.

**Problem XII.**

1) Find all natural number values of x such that fraction $x/7$ will be less than 1; equal to 1.

2) Find all natural number values of x such that fraction $8/x$ will be greater than 1; equal to 1.

3) What natural numbers can be plugged in for x to make a true compound inequality: $7/13 > x/13 > 4/13$?

Changing denominator of a fraction.

**Problem XIII.**

Find equal pairs of numbers in this set of fractions:

$^1/_3$, 1, $^3/_6$, $^4/_{12}$, $^{10}/_{25}$, $^1/_2$, $^3/_9$, $^7/_7$, $^{10}/_{30}$, $^{11}/_{11}$.

To answer the question students will have to reduce some of these fractions to the lowest terms to find equal fractions. They also should know that a fraction is equal to 1 if its numerator and denominator are represented by the same number.

**Problem XIV.**

The following points are marked on a number line:

A ($^2/_8$), B ($^1/_7$), C ($^1/_4$), D ($^2/_{14}$), E ($^5/_{20}$), K ($^{10}/_{70}$).

How many distinct points are there?

Problem XIV builds on Problem XIII. Students are not required to draw a number line in this problem. They should be able to find the same coordinates represented by equal fractions and draw a conclusion that equal fractions correspond to the same point on the number line.

**Problem XV.**

Prove the following inequalities:

1) $\frac{123}{800} > \frac{1}{8}$

2) $\frac{361}{6000} < \frac{1}{15}$.

**Problem XVI.**

For what value of x is the equality true?

1) $\frac{15}{35} = \frac{x}{7}$

2) $\frac{x}{6} = \frac{40}{48}$

3) $\frac{26}{65} = \frac{2}{x}$

4) $\frac{6}{x} = \frac{30}{35}$

To solve Problems XV and XVI, students should reverse their plan of action and represent fractions with the smaller denominator as fractions with the bigger denominator. They could also put the given fraction in lowest terms.

**Problem XVII.**

Arrange the following numbers in ascending order.

1) $\frac{4}{5}, \frac{7}{10}, \frac{8}{15}, \frac{11}{30}$

2) $\frac{11}{12}, \frac{5}{24}, \frac{5}{6}, \frac{3}{8}$

In Problem XVII 1) and 2) it is pretty easy to identify a common denominator, and then represent all the members of the sets as fractions with the selected denominator and rearrange them in the required order. This question can be posed using descending order as well.

Now that students have practiced changing denominators of fractions, it is time to practice adding and subtracting fractions with different denominators.

Problems on adding and subtracting fractions.

In the beginning students should practice on fractions with the same denominators. In my experience, most students add and subtract fractions with the same denominators correctly. They understand these operations pretty well and feel comfortable. I would also use this opportunity to reinforce the usage of the Associative Rule of addition. This rule will let students save time and effort when calculating expressions with fractions. Students are given very little time to complete their tests. It adds to their stress and anxiety. A lot of problems they should complete without use of a calculator; and unfortunately, they are not used to it. First, let`s review the rule.

Associative Rule of addition:

If a, b, and c are any real numbers, then (a + b) + c = a + (b+ c).

**Problem XVIII.**

Find the value of the expression

1) $\frac{4}{11} + a$, if $a = \frac{1}{11}, \frac{3}{11}, \frac{5}{11}$

2) $b - \frac{1}{10}$, if $b = \frac{7}{10}, \frac{5}{10}, \frac{3}{10}$

3) $\frac{3}{14} + (\frac{6}{14} + c)$, if $c = \frac{1}{14}, \frac{2}{14}, \frac{5}{14}$

4) $\frac{12}{17} - \frac{3}{17} - d$, if $d = \frac{4}{17}, \frac{5}{17}$

In Problem XVIII 3) first of all, students should apply the associative rule of addition to add $\frac{3}{14}$ and $\frac{6}{14}$ .And in Problem XVIII 4) first, students should combine like terms, and only after that plug in for the variable. I will always encourage my students to reduce the answer to the lowest terms (sometimes it is already in lowest terms, so no work is necessary, except to check to be sure.)

**Problem XIX.**

Solve the equations:

1) $^{17}/_{20}$ - x = $^{14}/_{20}$ - $^{3}/_{20}$

2) $^{8}/_{15}$ - $^{7}/_{15}$ + y = $^{14}/_{15}$

3) $^{12}/_{19}$ - ($^{1}/_{19}$ + x) = $^{5}/_{19}$

4) $^{16}/_{27}$ + $^{2}/_{27}$ - y = $^{1}/_{27}$

To challenge students even more, I ask them to solve these equations in their mind.  The first student who receives the correct answer should explain the steps he/she takes to do the mental Math. This small competition helps me train students to calculate as fast as they can to get them ready for a real-life task when students take tests under severe time pressure.

Also, it is a good time to reinforce word problems. I am an absolute advocate of creating problem sets that ask students to practice on multiple concepts, especially on word problems since they have direct connection to the real world and develop students` critical thinking in the best way. So, the suggested problems may look like these ones:

**Problem XX.**

To make 8 pairs of pants a seamstress used 4 yds of fabric. To make 8 skirts she used 3 yds of fabric.

- How much fabric did she use to make a skirt?
- How much fabric did she use to make a pair of pants?
- Did she use more fabric to make a skirt or a pair of pants?
- How much fabric did she use to make a skirt and a pair of pants?

In a problem like this many student are tempted to divide 8 by 4 since it looks way easier and more attractive. Another surprising pitfall is linguistic. Not all of them understand that “a skirt” means “one skirt”. My Geometry students have made such a mistake many times. So, over the years I have developed a habit to make sure that the reading comprehension lets them move forward with this problem in the right direction. Finally, some students do not know that the sign of division can be replaced by a fraction bar. I make a point to review this with students before they attempt Problem XX. For example,

$$3 \div 8 = \frac{3}{8}.$$

**Problem XXI.**

A farmer has 11 greenhouses of the same size. Tomatoes have been planted in 4 of them and cucumbers in 2 of them. What part of the greenhouses are occupied by tomatoes?

1. What part of the greenhouses are occupied by cucumbers?
2. What part of the greenhouses remain unplanted?

To solve this problem, students need to realize that the total number of the greenhouses should be in the denominator of the fraction.

**Problem XXII.**

A farmer has planted $^{11}/_{17}$ of his fields with potatoes. Cucumbers occupy by $^{1}/_{17}$ of the fields more than carrots, but by $^{8}/_{17}$ less of the fields than potatoes.

1. What fraction of his fields are planted with cucumbers?
2. What fraction of his fields are planted with carrots?
3. What fraction of his fields are planted with potatoes, carrots and cucumbers together?
4. What fraction of his fields remain unplanted?

Next come comparing, adding and subtracting fractions with different denominators. We should start with some simple problems and gradually increase the challenge.

**Problem XXIII.**

Find the value of each arithmetic expression. Put your answer in the lowest terms:

1) $$\frac{1}{2} + \frac{5}{8}$$

2) $$\frac{5}{7} - \frac{3}{14}$$

3) $$\frac{19}{20} - (\frac{1}{4} + \frac{2}{5})$$

4) $$\frac{1}{30} + (\frac{3}{5} - \frac{1}{6})$$

5) $$\frac{13}{18} - \frac{1}{24} - \left(\frac{29}{72} - \frac{5}{36}\right)$$

6) $$\left(\frac{7}{8} - \frac{4}{5}\right) + \left(\frac{1}{20} + \frac{1}{4}\right) + \frac{1}{2}$$

7) $$\frac{1}{4} + 0.7 + \frac{1}{2} - \frac{1}{5}$$

8) $$\frac{4}{5} - \frac{1}{3} + 0.6$$

In Problem XXIII 7) and 8) students should replace decimals by fractions to solve the problem.

**Problem XXIV.**

Solve the equation:

1) $$x + \frac{4}{15} = \frac{2}{3} + \frac{2}{5}$$

2) $(\frac{4}{5} - x) + \frac{13}{20} = \frac{25}{30}$

3) $y - \frac{5}{20} = \frac{5}{8} - \frac{3}{10}$

4) $\frac{2}{3} - \left(\frac{7}{9} - a\right) = \frac{1}{3}$

**Problem XXV.**

Find the value of each algebraic expression:

1) $a/_{10} + a/_{15}$, if a = 1, 2, 5, 7

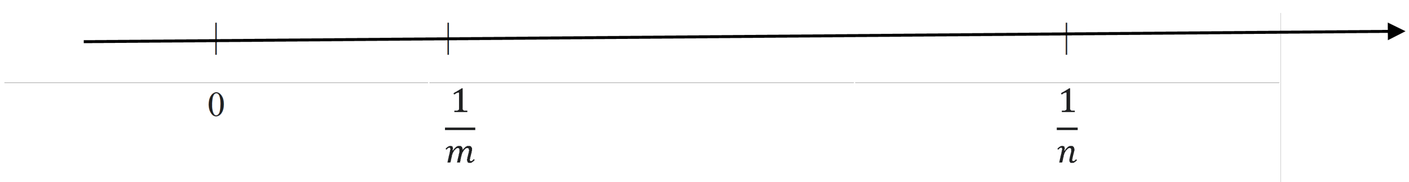
2) $x/_{12} - 1/_{x}$, of x = 4, 5, 6

**Problem XXVI.**

Points A ($1/_{n}$) and B ($1/_{m}$) are given on the number line below. On the same number line, label the following points:

1) $\frac{1}{m} + \frac{1}{n}$

2) $\frac{1}{n} - \frac{1}{m}$




To solve these problems students can use a compass. They need to understand the principle of adding and subtracting numbers using a number line.

Word problems also come handy in teaching this concept.

**Problem XXVII.**

John played soccer for $9/_{10}$ of an hour. Then he played volleyball for $9/_{15}$ of an hour.

1. What game did he spend more time on?
2. How much time did he spend playing both games?

**Problem XXVIII.**

The length of a rectangle is $3/_{4}$m, the width is less than the length by $5/_{8}$m.

1. Find the width of the rectangle.
2. Find the perimeter of the rectangle.

**Problem XXIX.**

The first harvester can harvest an entire field in 6 days, the other harvester in 4 days.

1. What portion of the field can the two harvesters harvest in one day?
2. What portion of the field will remain unharvested after one day of work by both harvesters?

It is helpful to remind the students that the whole work on harvesting the field will be considered equal to 1.

**Problem XXX**

There are two motors of the same size. Each motor has its own tank. One motor consumes a full tank of gasoline in 18 hours, the other in 12 hours. What portion of the full tank will be consumed by both motors if the first will be working for 5 hours, and the other – for 7 hours?

**Problem XXXI.**

Jessica spent $1/3$ of an hour reading the first book. She spent $1/6$ of an hour more on reading the second book than on reading the first book. To read the third book she spent $7/12$ of an hour less than she spent on reading the first and the second book together. How much time did Jessica spend to read all three books?  Problems on multiplying and dividing fractions.

**Problem XXXII**

Find the product:

1) $\frac{1}{8} \times \frac{3}{4}$

2) $\frac{11}{12} \times \frac{8}{9}$

3) $(\frac{1}{2})2$

4) $(\frac{2}{3})3$

5) $\frac{1}{5} \times 0.3$

6) $0.8 \times \frac{5}{8}$

7) $\frac{3}{5} \times \frac{2}{7} \times \frac{5}{6}$

8) $\frac{7}{10} \times \frac{5}{49} \times \frac{2}{3}$

9) $\frac{6}{7} \times \left(\frac{11}{18} - \frac{5}{12}\right)$

10) $\left(\frac{5}{12} + \frac{3}{8}\right) \times \frac{12}{19}$

11) $\frac{2}{3} \times \frac{9}{16} - \frac{5}{24} \times \frac{2}{5} - \frac{1}{6}$

12) $(\frac{1}{2} - \frac{1}{4}) \times (\frac{5}{6} - \frac{2}{3})$

**Problem XXXIII.**

The mass of 1 liter of kerosene is $4/5$ kg. Find the mass of $3/4$ liter of kerosene, $1/2$ liter of kerosene, $2/5$ liter of kerosene.

Note: It is extremely important to teach that to find a fraction of a number we should multiply this number by the fraction. So, in this case students should multiply $4/5$ by $3/4$, and so on. This skill becomes absolutely essential in algebraic expressions when students should find, for example, $2/3$ of x. They should be able to write it down as $2/3$X x

**Problem XXXIV.**

The length of a side of a square is $7/8$ in. Find the area of the square.

Note: I prefer that students write down a geometric formula first, and then plug in the number. $A = s^2$ , so $A = (7/8)^2$ $in^2$ .

I will insist that they always specify the units in any measurement problem or word problem.

**Problem XXXV.**

Find the volume of a cube with a side length equal $3/4$ in.

This problem should be done in the same way as the previous one.

**Problem XXXVI.**

The velocity of a car is $3/4$km/min. Find the distance the car has travelled in $2/3$ min, in $1/6$ min. Express the velocity in km/hr. Have you ever been in a car going that fast?

**Problem XXXVII.**

Find the quotient:

1) $\frac{3}{8} \div \frac{5}{7}$

2) $\frac{7}{8} \div 2$

3) $1 \div \frac{3}{11}$

4) $\frac{3}{16} \div \frac{5}{12}$

5) $\frac{m}{n} \div \frac{p}{k}$

6) $\frac{a}{b} \div c$

7) $p \div \frac{c}{n}$

**Problem XXXVIII.**

The area of a rectangle is $15/64$ sq m. Find the perimeter of the rectangle if its width is $3/8$ m.

**Problem XXXIX.**

The length and the width of a rectangle are equal to $3/4$ in and $1/6$ in respectively. Find the length of another rectangle, if its width is equal to $1/2$ in, and the area is equal to the area of the first rectangle.

**Problem XL.**

The mass of $4/5$ cu dm of pine wood is equal to $2/5$kg.

- What is the mass of 1 cu dm of pine wood?
- What is the volume of the pine wood bar if its mass is equal to 1 kg?

**Problem XLI.**

Find the velocity of a vehicle if it travelled 15 mi in $5/6$ hour; in $5/3$ hour.

## Classroom activities

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In this section I would like to explain how I plan to use the problems from the problem set mentioned above, to improve my students` skills on fraction notations as well as adding, subtracting, multiplying and dividing fractions.

**Lesson 1.**

Objective: Students will practice writing down fractions that represent the area of the shaded region on the diagram. Students will learn how to place fractions on the number line and compare them.

The problems in the unit are designed in a way to gradually increase the rigor. So, in the beginning I will review/teach my students how to use fraction notation and write down the answers to the questions that imply figuring out a fractional part of the whole in a real-life problem. Using a number line will be a big help, so I will build a number line, showing my students how to place fractions on a number line and compare fractions with the same numerator and with the same denominator. I plan to use problems I-XII for this lesson. I will decide what problems in this set to use for classwork, and will give similar ones for a homework assignment. Students can work on the other problems independently, in pairs or groups, followed by whole group discussion.

**Lesson 2.**

Objective: Students will learn how to add and subtract fractions with the same and different denominators.

First, I will teach/review with my students how to add and subtract fractions with the same denominator. From my previous experience, it is easy to understand, so my students will practice on this topic using arithmetic expressions, equations and word problems. Next, I will introduce my students to the operations on fractions with different denominators.  To start this conversation, I will draw students` attention to the fact that on the number line, equivalent fractions are represented by the same point. Students will learn how to reduce and expand fractions, followed by finding values of arithmetic expressions, solving equations and word problems that involve fractions with different denominators. I will select from problems XIII – XXXI for this practice. Again, depending on objectives, student skills and curricular timeframe, I will adjust the particular classwork and homework assignment.

**Lesson 3.**

Objectives: Students will learn how to multiply and divide fractions.

These are the most complicated operations on fractions. Multiplying fractions makes sense to most of the students in my classes, however dividing fractions traditionally presents a lot of issues. I will be patient teaching this skill. I will start with Problem XXXII and gradually progress through the set of problems. My teaching routine will be similar to the previous two lessons. Depending on my students` progress I may decide to spend more time teaching some particular skills and adjust classwork and homework assignments accordingly.

## Resources

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https://www.khanacademy.org/math/k-8-grades

Here students and teachers can find a great selection of videos to practice on comparing, adding, subtracting, multiplying and dividing fractions (3 rd-5th grades), as well as on ratios, rates and percentages (6th and 7th grades).

https://www.khanacademy.org/math/ap-calculus-ab/ab-applications-of-integration-new

A great selection of videos and practice problems on Applications of integration where high school students will come across manipulations with fractions.

## Appendix on Implementing District Standards

This curriculum unit is aligned with the Virginia Standards of Learning. Virginia did not adopt the Common Core standards; however, the Curriculum Framework clearly states the fraction skills that our students should learn during their school career. Below I have listed relevant standards. As you can see, starting from Grade Three, students should be able to name and write down a fraction represented by a model. They should also be competent in comparing, adding, subtracting, multiplying and dividing fractions. Thus, elementary and middle school teachers will be able to use problems from my unit in their classrooms. Operations with fractions is an essential part of solving problems in Geometry, Trigonometry, Algebra and Calculus. After students master the content within my unit, they will be better prepared for upper level mathematics, biology, chemistry and physics where they have to use fractions on daily basis. In my classroom the unit will help me remediate my students who study Geometry, Math Analysis and AP Calculus.

Implementing Mathematics Standards of Learning for Virginia Public Schools

### Grade Three

#### Number and Number Sense.

3.2 The student will

a. name and write fractions and mixed numbers represented by a model;
b. represent fractions and mixed numbers with models and symbols; and
c. compare fractions having like and unlike denominators, using words and symbols (>, <, =, or ≠), with models.

#### Computation and Estimation

3.5 The student will solve practical problems that involve addition and subtraction with proper fractions having like denominators of 12 or less.

#### Measurement and Geometry

3.7 The student will estimate and use U.S. Customary and metric units to measure

a. length to the nearest $1/2$ inch, inch, foot, yard, centimeter, and meter.

### Grade Four

#### Number and Number Sense

4.2 The student will

a. compare and order fractions and mixed numbers, with and without models;
b. represent equivalent fractions; and
c. identify the division statement that represents a fraction, with models and in context.

4.3 The student will

d. given a model, write the decimal and fraction equivalents.

4.4 The student will

b. add and subtract fractions and mixed numbers having like and unlike denominators; and
c. solve single-step practical problems involving addition and subtraction with fractions and mixed numbers.

## Grade Five

### Number and Number sense

5.2 The student will

a. represent and identify equivalencies among fractions and decimals, with and without models; and
b. compare and order fractions, mixed numbers, and/or decimals in a given set, from least to greatest and greatest to least.

5.6 The student will

a. solve single-step and multistep practical problems involving addition and subtraction with fractions and mixed numbers; and
b. solve single-step practical problems involving multiplication of a whole number, limited to 12 or less, and a proper fraction, with models.

## Grade Six

### Number and Number sense

6.1 The student will represent relationships between quantities using ratios, and will use appropriate notations, such as $a/b$, *a* to *b*, and *a*:*b*.

6.2 The student will

a. represent and determine equivalencies among fractions, mixed numbers, decimals, and percents.

### Computation and Estimation

6.5 The student will

a. multiply and divide fractions and mixed numbers;
b. solve single-step and multistep practical problems involving addition, subtraction, multiplication, and division of fractions and mixed numbers

## Grade Seven

### Computation and Estimation

7.3 The student will solve single-step and multistep practical problems, using proportional reasoning.

a. represent verbal quantitative situations algebraically; and

b. evaluate algebraic expressions for given replacement values of the variables.

Geometry

G.3 The student will solve problems involving symmetry and transformation. This will include

a. investigating and using formulas for determining distance, midpoint, and slope;
b. applying slope to verify and determine whether lines are parallel or perpendicular;

G.14 The student will apply the concepts of similarity to two- or three-dimensional geometric figures.  This will include

a. comparing ratios between lengths, perimeters, areas, and volumes of similar figures;
b. determining how changes in one or more dimensions of a figure affect area and/or volume of the figure; determining how changes in area and/or volume of a figure affect one or more dimensions of the figure.

**Trigonometry**

**Triangular and Circular Trigonometric Functions**

T.1  The student, given a point on the terminal side of an angle in standard position, or the value of the trigonometric function of the angle, will determine the sine, cosine, tangent, cotangent, secant, and cosecant of the angle.

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## Notes.

[1] Naum Vilenkin, *Mathematics 5 th grade*, 246 (translated from Russian by me).

[2] Roland Larson, *Calculus of a Single Variable*, 411

[3] Roger Howe, *A collection of Authentic Expository Mathematical Text*, 30.

[4] Naum Vilenkin, *Mathematics 5 th grade*, 203 (translated from Russian by me).

[5] Naum Vilenkin, *Mathematics 5 th grade*, 71 (translated from Russian by me).

[6] National Research Council 2001, *“Adding it up: Helping Children Learn  Mathematics”, 86*

[7] Naum Vilenkin, *Mathematics 5 th grade*, 207 (translated from Russian by me).