Curriculum Units by Fellows of the National Initiative
2023 Volume III: Transitions in the Conception of Number: From Whole Numbers to Rational Numbers to Algebra

Introduction

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The YNI seminar for 2023 "Transitions in the Conception of Number: From Whole Numbers to Rational Numbers to Algebra" was motivated by the observation that, as students progress in their study of mathematics, the role or nature of the numbers they deal with changes. When first introduced, numbers are counts, but when fractions are introduced, they need to be thought of as ratios: they tell you the size of a quantity of interest in relation to a reference quantity, which functions as a unit. And in symbolic algebra, numbers need to change again: they should be thought of as members of a system, and this system is governed by the Rules of Arithmetic (aka, Properties of the Operations). However, these necessary transitions are not explicitly recognized in the standard US math curriculum, and it is plausible that this omission contributes to the failure of many students to master fractions and/or algebra.

The seminar studied two main models to help students deal with fractions: length models, especially the number line; and area models, especially rectangles. The number line is very effective for picturing the system of fractions with a fixed denominator, for comparing fractions, for providing examples of equivalent fractions, and for understanding addition of fractions. It can be used to emphasize two key points about which many students develop misconceptions:

1. that when dividing a quantity into a given number of fractional parts, all the parts must be equal; and
2. that when the numerator of a fraction increases, the fraction increases, *but* when the denominator of a fraction increases, the fraction decreases.

The number line can also be used to promote the understanding of division as measurement: for numbers a and b, the quotient a/b describes how many (perhaps fractional) copies of b it takes to make a.

Area models are particularly valuable for understanding multiplication, which is a considerably less accessible operation than addition. They make visible the commutativity of multiplication, and they provide a visual way to understand the formula for multiplication of fractions. They can also be used to explain equivalence of fractions, and division.

For dealing with algebra, the seminar studied several collections of word problems. A key takeaway from this work was that there are large collections of word problems that can be solved using equations of the same general form, and symbolic algebra provides a single formula that can be used to find the solutions for all problems in the given collection. This provides convincing evidence for the usefulness of algebra. On the other hand, one can find problems that lead to equations of different types, and this provides a useful method for sorting problems according to their structural properties.

The seminar also studied some consequences of the Rules of Arithmetic. For example, the Associative Rule of Addition justifies writing expressions involving adding several addends without any parentheses. Also, the Commutative and Associative Laws for Addition together imply that there are many ways to compute the sum of several addends, by varying the order of summation, and selecting pairwise additions. The number of possibilities grows rapidly with the number of addends. For 5 addends, there are 1680 ways to compute the sum, and for 10 addends, there are over 170 billion ways. This flexibility is used, for example, in the standard algorithm for adding base 10 numbers. We also attended to the Inverse Rule of Multiplication, and noted that *1/a* is the multiplicative inverse, aka reciprocal, of *a*. Together with the Commutative and Associative Rules, this justifies the "Invert and Multiply" rule for division by a fraction, which is often considered mysterious.

The Fellows in the seminar included four elementary teachers and four high school teachers. Nearly all the units involved fractions. The units by Jessica Mason and Irene Jones use length models (number line or fraction strips) to help students visualize fractions, and to try to avoid the two misconceptions mentioned above. Lisa Yau wrote a unit that uses coins to help students master decimal notation, and to develop money sense while they are at it. She includes substantial attention to history, through the history of coinage in the US, and content about the people depicted on the coins. The fourth elementary unit, by Valerie Schwarz, focuses on word problems, and seeks to help students see patterns, or "schemas", in collections of problems, to promote broad problem-solving skills.

The units of the high school teachers have a more algebraic slant. They focus on dealing with word problems: translating them into equations, and then solving the resulting equations. They also testify to the poor job the US curriculum does with teaching fractions. The units of Irina Alekseeva and Ulises Reveles have as a main goal, improving their students' ability to deal with fractions, especially to understand how to compute with them, in order to solve the word problems where fractions appear in the data. Julie Skrzypczak's unit also pays substantial attention to fractions, toward the goal of helping her students understand and deal with proportional relationships. The fourth high school unit, by Kristina Kirby, also focuses on the analysis and solution of word problems, with somewhat less specific focus on fractions. Three of the units provide collections of word problems for use with the unit.